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ON HARMONIC UNIVALENT FUNCTION DEFINED BY GENERALIZED SĂLĂGEAN DERIVATIVES

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Abstract. In the present paper and by making use of the generalized Sălăgean derivatives, we have introduce and study a class of analytic function and prove the coefficient conditions, distortion bounds for the function in our class and some other interesting properties are also investigated.

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1. Introduction

Let H be denote the family of all complex-valued, harmonic orientation-preserving, univalent function f in the open unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f'(0) - 1 = 0$. Each $f \in H$ can be expressed as $f = h + g$, where h and g are the analytic and the co-analytic part of f, respectively. Then for $f = h + g \in H$ we can write the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{0.1}$$

Firstly, Clunie and Sheil-Small[2] studied the class H together with some geometric subclasses and obtained some coefficient bounds. Since then, there has been several articles related to H and it is subclasses (see eg.[1],[2],[3],[4]).

The differential operator D^k was introduced by[5], and generalized by[1]. Jahangiri and et. al. [3] defined the modified Sălăgean operator of f as

$$D^k f(z) = D^k h(z) + (-1)^k D^k g(z), \tag{0.2}$$

where

$$D^k h(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n,$$

$$D^k g(z) = \sum_{n=1}^{\infty} n^k b_n z^n, \quad k \in \mathbb{N}_0 = \{0, 1, 2, ...\}.$$

Here, we define the modified generalized Sălăgean operator of f as

$$D^k_\lambda f(z) = D^k_\lambda h(z) + (-1)^k \overline{D^k_\lambda g(z)}, \tag{0.3}$$

where

$$D^k_\lambda h(z) = z + \sum_{n=2}^{\infty} (1 + (n-1)\lambda)^k a_n z^n, \quad D^k_\lambda g(z) = \sum_{n=1}^{\infty} (1 + (n-1)\lambda)^k b_n z^n.$$

For $0 \leq \alpha < 1, k \in \mathbb{N}, \lambda \geq 0$ and $z \in U$, let $H(k, \alpha)$ denote the family of harmonic functions f of the form (0.1)such that

$$\Re\left(\frac{D^k_\lambda f(z)}{D^{k+1}_\lambda f(z)}\right) > \alpha, \tag{0.4}$$

where D^k_λ is defined by (0.3).

We denote the subclass $H^-(k, \alpha)$ consists of harmonic functions $f_k = h + \overline{g_k}$ in $H^-(k, \alpha)$ so that h and g_k are of the form

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad and \quad g_k(z) = (-1)^{k-1} \sum_{n=2}^{\infty} b_n z^n,$$

where $a_n, b_n \geq 0, |b_n| < 1$.

For the harmonic function f of the form (0.1) with $b_1 = 0$, Avci and Zlotkiewich in [1] show that if $\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq 1$, then $f \in SH(0)$, where $SH(0) = H^-(0, 0)$.

If $\sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \leq 1$, then $f \in KH(0)$, where $KH(0) = H^-(1, 0)$.

For harmonic functions f of the form (0.4) with $k = 0$, Jahangiri[3] showed that $f \in SH(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} (n - \alpha)|a_n| + \sum_{n=1}^{\infty} (n + \alpha)|b_n| \leq 1 - \alpha$$

and $f \in H^{-}(1,\alpha)$ if and only if

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| + \sum_{n=1}^{\infty} n(n+\alpha)|b_n| \leq 1-\alpha.$$

2.Coefficient Conditions

First we state and prove a sufficient coefficient condition for the class $H(k,\alpha)$.

Theorem 1. *Let $f = h + \overline{g}$ be given by (0.1). If*

$$\sum_{n=1}^{\infty} \{\Psi(k,n,\alpha)|a_n| + \Theta(k,n,\alpha)|b_n|\} \leq 2, \tag{0.5}$$

where

$$\Psi(k,n,\alpha) = \frac{(1+(n-1)\lambda)^k - \alpha(1+(n-1)\lambda)^{k+1}}{1-\alpha},$$

$$\Theta(k,n,\alpha) = \frac{(1+(n-1)\lambda)^k + \alpha(1+(n-1)\lambda)^{k+1}}{1-\alpha},$$

$$a_1 = 1, \quad 0 \leq \alpha \leq 1, \quad k \in \mathbb{N},$$

then f is sense preserving in U and $f \in H(k,\alpha)$.

Proof. According to (0.2) and (0.3) it is sufficient to show that

$$\Re\Big(\frac{D_\lambda^k f(z) - \alpha D_\lambda^{k+1} f(z)}{D_\lambda^{k+1} f(z)}\Big) \geq 0.$$

If $r = 0$, obvious.
Now, if $0 < r < 1$, we have

$$\Re\Big(\frac{D_\lambda^k f(z) - \alpha D_\lambda^{k+1} f(z)}{D_\lambda^{k+1} f(z)}\Big) =$$

$$\Re\Big[\frac{z(1-\alpha) + \sum_{n=2}^{\infty}[(1+(n-1)\lambda)^k - \alpha(1+(n-1)\lambda)^{k+1}]a_n z^n}{z + \sum_{n=2}^{\infty}(1+(n-1)\lambda)^{k+1} a_n z^n + (-1)^{k+1}\sum_{n=1}^{\infty}(1+(n-1)\lambda)^{k+1}\overline{b_n z^n}} +$$

$$+\frac{(-1)^k\sum_{n=1}^{\infty}[(1+(n-1)\lambda)^k+\alpha(1+(n-1)\lambda)^{k+1}]\overline{b_n}\overline{z}^n}{z+\sum_{n=2}^{\infty}(1+(n-1)\lambda)^{k+1}a_nz^n+(-1)^{k+1}\sum_{n=1}^{\infty}(1+(n-1)\lambda)^{k+1}\overline{b_n}\overline{z^n}}] =$$

$$\Re\{\frac{(1-\alpha)+\sum_{n=2}^{\infty}[(1+(n-1)\lambda)^k-\alpha(1+(n-1)\lambda)^{k+1}]a_nz^{n-1}}{1+\sum_{n=2}^{\infty}(1+(n-1)\lambda)^{k+1}a_nz^{n-1}+(-1)^{k+1}\sum_{n=1}^{\infty}(1+(n-1)\lambda)^{k+1}\overline{b_n}\overline{z}^{n-1}}$$

$$+\frac{(-1)^k\sum_{n=1}^{\infty}[(1+(n-1)\lambda)^k+\alpha(1+(n-1)\lambda)^{k+1}]\overline{b_n}\overline{z}^{n-1}}{1+\sum_{n=2}^{\infty}(1+(n-1)\lambda)^{k+1}a_nz^{n-1}+(-1)^{k+1}\sum_{n=1}^{\infty}(1+(n-1)\lambda)^{k+1}\overline{b_n}\overline{z}^{n-1}}\} =$$

$$=\Re[\frac{(1-\alpha)+A(z)}{1+B(z)}].$$

For $z = re^{i\Theta}$ we have

$$A(re^{i\Theta})=\sum_{n=2}^{\infty}[(1+(n-1)\lambda)^k-\alpha(1+(n-1)\lambda)^{k+1}]a_nr^{n-1}e^{(n-1)\theta i}+$$

$$+(-1)^k\sum_{n=1}^{\infty}[(1+(n-1)\lambda)^k+\alpha(1+(n-1)\lambda)^{k+1}]\overline{b_n}r^{n-1}e^{-(n+1)\theta i};$$

$$B(re^{i\Theta}) =$$

$$\sum_{n=2}^{\infty}(1+(n-1)\lambda)^{k+1}a_nr^{n-1}e^{(n-1)\theta i}+(-1)^{k+1}\sum_{n=1}^{\infty}(1+(n-1)\lambda)^{k+1}\overline{b_n}r^{n-1}e^{-(n+1)\Theta i}.$$

Setting

$$\frac{(1-\alpha)+A(z)}{1+B(z)}=(1-\alpha)\frac{1+w(z)}{1-w(z)}.$$

Hence we want to show that $|w(z)| \leq 1$.

By (0.5), we can write

$$|w(z)| = |\frac{A(z)-(1-\alpha)B(z)}{A(z)+(1-\alpha)B(z)+2(1-\alpha)}| =$$

$$|\frac{\sum_{n=2}^{\infty}[(1+(n-1)\lambda)^k-(1+(n-1)\lambda)^{k+1}]a_n r^{n-1}e^{(n-1)\Theta i}}{2(1-\alpha)+\sum_{n=2}^{\infty}C(k,n,\alpha)a_n r^{n-1}e^{(n-1)\Theta i}+(-1)^k\sum_{n=1}^{\infty}D(k,n,\alpha)\overline{b_n}r^{n-1}e^{-(n+1)\Theta i}}+$$

$$\frac{(-1)^k\sum_{n=1}^{\infty}[(1+(n-1)\lambda)^k+(1+(n-1)\lambda)^{k+1}]\overline{b_n}r^{n-1}e^{-(n+1)\Theta i}}{2(1-\alpha)+\sum_{n=2}^{\infty}C(k,n,\alpha)a_n r^{n-1}e^{(n-1)\Theta i}+(-1)^k\sum_{n=1}^{\infty}D(k,n,\alpha)\overline{b_n}r^{n-1}e^{-(n+1)\Theta i}}|$$

$$\leq\frac{\sum_{n=1}^{\infty}[(1+(n-1)\lambda)^k-(1+(n-1)\lambda)^{k+1}]|a_n|r^{n-1}}{4(1-\alpha)-\sum_{n=1}^{\infty}[C(k,n,\alpha)|a_n|+D(k,n,\alpha)|b_n|r^{n-1}]}+$$

$$+\frac{\sum_{n=1}^{\infty}[(1+(n-1)\lambda)^k+(1+(n-1)\lambda)^{k+1}]|b_n|r^{n-1}}{4(1-\alpha)-\sum_{n=1}^{\infty}[C(k,n,\alpha)|a_n|+D(k,n,\alpha)|b_n|]r^{n-1}}<$$

$$<\frac{\sum_{n=1}^{\infty}[(1+(n-1)\lambda)^k-(1+(n-1)\lambda)^{k+1}]|a_n|}{4(1-\alpha)-\sum_{n=1}^{\infty}[C(k,n,\alpha)|a_n|+D(k,n,\alpha)|b_n|]}+$$

$$+\frac{\sum_{n=1}^{\infty}[(1+(n-1)\lambda)^k+(1+(n-1)\lambda)^{k+1}]|b_n|}{4(1-\alpha)-\sum_{n=1}^{\infty}[C(k,n,\alpha)|a_n|+D(k,n,\alpha)|b_n|]}<1,$$

where

$$C(k,n,\alpha)=(1+(n-1)\lambda)^k+(1-2\alpha)(1+(n-1)\lambda)^{k+1}$$

and

$$D(k,n,\alpha)=(1+(n-1)\lambda)^k+(-1)(1-2\alpha)(1+(n-1)\lambda)^{k+1}.$$

For sharpness, we take the functions

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1}{\Psi(k,n,\alpha)} x_n z^n + \sum_{n=1}^{\infty} \frac{1}{\Theta(k,n,\alpha)} \overline{y_n z^n}, \qquad (0.6)$$

where $k \in \mathbb{N}$ and $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$. The functions of the form (0.6) are in $H(k,\alpha)$ because

$$\sum_{n=2}^{\infty} \{\Psi(k,n,\alpha)|a_n| + \Theta(k,n,\alpha)|b_n|\} =$$

$$= 1 + \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 2.$$

Theorem 2. *Let $f_n = h + \overline{g_n}$ be given by (0.4). Then $f_n \in H^-(k,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \{\Psi(k,n,\alpha)a_n + \Theta(k,n,\alpha)b_n\} \le 2, \qquad (0.7)$$

where $a_1 = 1, 0 \le \alpha < 1, k \in \mathbb{N}$. *Proof.* Since $H^-(k,\alpha) \subset H(k,\alpha)$ then the "if" part of theorem follows from Theorem No.1. For "only if " part, we need to prove it:

for f_k of the form (0.4), we note the condition

$$\Re\Big[\frac{D_\lambda^k f_k(z)}{D_\lambda^{k+1} f_k(z)}\Big] > \alpha$$

is equivalent to

$$\Re\Big\{\frac{(1-\alpha)z - \sum_{n=2}^{\infty}[(1+(n-1)\lambda)^k - \alpha(1+(n-1)\lambda)^{k+1}]a_n z^n}{z - \sum_{n=2}^{\infty}(1+(n-1)\lambda)^{k+1} a_n z^n + (-1)^{2k}\sum_{n=1}^{\infty}(1+(n-1)\lambda)^{k+1} b_n \overline{z}^n}\Big\} + \qquad (0.8)$$

$$\Re\Big\{\frac{(-1)^{2k-1}\sum_{n=1}^{\infty}[(1+(n-1)\lambda)^k + \alpha(1+(n-1)\lambda)^{k+1}]b_n \overline{z}^n}{z - \sum_{n=2}^{\infty}(1+(n-1)\lambda)^{k+1} a_n z^n + (-1)^{2k}\sum_{n=1}^{\infty}(1+(n-1)\lambda)^{k+1} b_n \overline{z}^n}\Big\} \ge 0.$$

The above required condition (0.8) must hold for all values of $z \in U$. Upon choosing the values of z on the positive real axis , where $0 \leq z = r < 1$, we must have

$$\frac{(1-\alpha) - \sum_{n=2}^{\infty}[(1+(n-1)\lambda)^k - \alpha(1+(n-1)\lambda)^{k+1}]a_n r^{n-1}}{1 - \sum_{n=2}^{\infty}(1+(n-1)\lambda)^{k+1}a_n r^{n-1} + \sum_{n=1}^{\infty}(1+(n-1)\lambda)^{k+1}b_n r^{n-1}} + \tag{0.9}$$

$$+ \frac{-\sum_{n=1}^{\infty}[(1+(n-1)\lambda)^k + \alpha(1+(n-1)\lambda)^{k+1}]b_n r^{n-1}}{1 - \sum_{n=2}^{\infty}(1+(n-1)\lambda)^{k+1}a_n r^{n-1} + \sum_{n=1}^{\infty}(1+(n-1)\lambda)^{k+1}b_n r^{n-1}} \geq 0.$$

If (0.7) does not hold consequently also (0.9) does not hold and so that the numerator in (0.9) is negative. This is not possible and the proof is complete.

Next we determine the extreme points of the closed convex hull of $H^-(k,\alpha)$ denoted by clco $H^-(k,\alpha)$.

Theorem 3. *Let f_k be given by(0.4). Then $f_k \in H^-(k,\alpha)$ if and only if*

$$f_k(z) = \sum_{n=1}^{\infty}[x_n h_n(z) + y_n g_{k_n}(z)],$$

where

$$h_1(z) = z, \quad h_n(z) = z - \frac{1}{\Psi(k,n,\alpha)}z^n, \qquad n = 2, 3, ...$$

and

$$g_{k_n}(z) = z + (-1)^{k-1}\frac{1}{\Theta(k,n,\alpha)}\overline{z}^n, \quad n = 1, 2, ...$$

$$x_n \geq 0, \quad y_n \geq 0, \quad x_1 = 1 - \sum_{n=2}^{\infty} x_n - \sum_{n=1}^{\infty} y_n.$$

In particular, the extreme points of $H^-(k,\alpha)$ are $\{h_n\}$ and $\{g_{k_n}\}$.

Proof. Suppose f_k can be expressed as following

$$f_k(z) = \sum_{n=1}^{\infty}[x_n h_n(z) + y_n g_{k_n}(z)] =$$

$$= \sum_{n=1}^{\infty}(x_n + y_n)z - \sum_{n=2}^{\infty}\frac{1}{\Psi(k,n,\alpha)}x_n z^n + (-1)^{k-1}\sum_{n=1}^{\infty}\frac{1}{\Theta(k,n,\alpha)}y_n\overline{z}^n.$$

Then

$$\sum_{n=2}^{\infty}\Psi(k,n,\alpha)\big(\frac{1}{\Psi(k,n,\alpha)}x_n\big) + \sum_{n=1}^{\infty}\Theta(k,n,\alpha)\big(\frac{1}{\Theta(k,n,\alpha)}y_n\big) =$$

$$= \sum_{n=2}^{\infty}x_n + \sum_{n=1}^{\infty}y_n = 1 - x_1 \leq 1,$$

therefore $f_k(z) \in clcoH^-(k,\alpha)$.

Conversely, let $f_k(z) \in clcoH^-(k,\alpha)$. Letting

$$x_1 = 1 - \sum_{n=2}^{\infty}x_n - \sum_{n=1}^{\infty}y_n.$$

Let $x_n = \Psi(k,n,\alpha)a_n$ and $y_n = \Theta(k,n,\alpha)b_n, n = 2,3,...$ We get the required representation, since

$$f_k(z) = z - \sum_{n=2}^{\infty}a_n z^n + (-1)^{k-1}\sum_{n=1}^{\infty}b_n\overline{z}^n =$$

$$= z - \sum_{n=2}^{\infty}\frac{1}{\Psi(k,n,\alpha)}x_n z^n + (-1)^{k-1}\sum_{n=1}^{\infty}\frac{1}{\Theta(k,n,\alpha)}y_n\overline{z}^n =$$

$$= z - \sum_{n=2}^{\infty}(z - h_n(z))x_n - \sum_{n=1}^{\infty}(z - g_{k_n}(z))y_n =$$

$$= [1 - \sum_{n=2}^{\infty}x_n - \sum_{n=1}^{\infty}y_n]z + \sum_{n=2}^{\infty}x_n h_n(z) + \sum_{n=1}^{\infty}y_n g_{k_n}(z) =$$

$$= \sum_{n=1}^{\infty}[x_n h_n(z) + y_n g_{k_n}(z)].$$

Theorem 4. *Let for $j = 1,2,...,m, f_j \in H^-(k,\alpha)$, then the function $F(z) = \sum_{j=1}^{m}d_j f_j(z)$ also belongs to $H^-(k,\alpha)$, where*

$$f_j(z) = z - \sum_{n=2}^{\infty}a_{n,j}z^n + \sum_{n=1}^{\infty}b_{n,j}\overline{z}^n, j = 1,2,...,m, \quad \sum_{j=1}^{m}d_j = 1.$$

Proof. For every $j \in \{1, 2, ..., m\}$ we have

$$\sum_{n=1}^{\infty}[\Psi(k,n,\alpha)a_{n,j} + \Theta(k,n,\alpha)b_{n,j}] \leq 2, \tag{0.10}$$

and we can write

$$F(z) = \sum_{j=1}^{\infty} d_j f_j(z) = z - \sum_{n=2}^{\infty}(\sum_{j=1}^{m} d_j a_{n,j}) + \sum_{n=1}^{\infty}(\sum_{j=1}^{m} d_j b_{n,j}).$$

Now, by making use of (0.10) we can write

$$\sum_{n=1}^{\infty}[\Psi(k,n,\alpha)(\sum_{j=1}^{m} d_j a_{n,j}) + \Theta(k,n,\alpha)(\sum_{j=1}^{m} d_j b_{n,j})] =$$

$$= \sum_{j=1}^{m}[\sum_{n=1}^{\infty}\left(\Psi(k,n,\alpha)a_{n,j} + \Theta(k,n,\alpha)b_{n_j}\right)]d_j \leq 2,$$

and this give the result.

The following theorem gives the distortion bounds for functions in $H^{-}(k,\alpha)$ which yields a covering results for this class.

Theorem 5. *Let $f_n \in H^{-}(k,\alpha)$. Then for $|z| = r < 1$ we have*

$$|f_k(z)| \leq (1 + b_1)r + [\varphi(k,n,\alpha) - \Omega(k,n,\alpha)b_1]r^2$$

and

$$|f_k(z)| \geq (1 - b_1)r - [\varphi(k,n,\alpha) - \Omega(k,n,\alpha)b_1]r^2$$

where

$$\varphi(k,n,\alpha) = \frac{1-\alpha}{(1+\lambda)^k - \alpha(1+\lambda)^{k+1}}$$

and

$$\Omega(k,n,\alpha) = \frac{1+\alpha}{(1+\lambda)^k - \alpha(1+\lambda)^{k+1}}.$$

The above results are sharp for the functions

$$f_k(z) = z + b_1\overline{z} + [\varphi(k,n,\alpha) - \Omega(k,n,\alpha)b_1]\overline{z}^2, \quad 0 \leq b_1 < \frac{1-\alpha}{1+\alpha}, \quad z = r$$

$f_k(z) = z - b_1\overline{z} - [\varphi(k,n,\alpha) - \Omega(k,n,\alpha)b_1]\overline{z}^2, \quad \frac{1-\alpha}{1+\alpha} < b_1 < 1, \quad z = r$ *respectively.*

Proof. We have

$$|f_k(z)| = |z - \sum_{n=2}^{\infty} a_n z^n + (-1)^{k-1} \sum_{n=1}^{\infty} b_n \overline{z}^n| \leq$$

$$\leq r + \sum_{n=2}^{\infty} a_n r^n + \sum_{n=1}^{\infty} b_n r^n = r + b_1 r + \sum_{n=2}^{\infty} (a_n + b_n) r^n \leq$$

$$\leq r + b_1 r + \sum_{n=2}^{\infty} (a_n + b_n) r^2 =$$

$$= (1 + b_1) r + \varphi(k, n, \alpha) \sum_{n=2}^{\infty} \frac{1}{\varphi(k, n, \alpha)} (a_n + b_n) r^2 \leq$$

$$\leq (1 + b_1) r + \varphi(k, n, \alpha) r^2 [\sum_{n=2}^{\infty} \Psi(k, n, \alpha) a_n + \Theta(k, n, \alpha) b_n] \leq$$

$$\leq (1 + b_1) r + [\varphi(k, n, \alpha) - \Omega(k, n, \alpha) b_1] r^2.$$

The proof for the left hand side of inequality can be done using similar arguments.

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