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ON L^1-CONVERGENCE OF THE $R-$TH DERIVATIVE OF COSINE SERIES WITH SEMI-CONVEX COEFFICIENTS

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Abstract. We study L^1-convergence of $r-th$ derivative of modified sine sums introduced by K. Kaur [2] and deduce some corollaries.

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1. Introduction and Preliminaries

Let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos kx \qquad (1)$$

be cosine trigonometric series with its partial sums denoted by $S_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n} a_k \cos kx$, and let $f(x)=\lim_{n\to\infty} S_n(x)$.

For convenience, in the following of this paper we shall assume that $a_0=0$.

A sequence (a_n) is said to be semi-convex, or briefly $(a_n)\in(SC)$, if $a_n\to 0$ as $n\to\infty$, and

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1}+\Delta^2 a_n|<\infty,$$

where $\Delta^2 a_k=a_k-2a_{k+1}+a_{k+2}$.

We shall generalize the class of sequences (SC) in the following manner:

A sequence (a_n) is said to be semi-convex of order r, $(r=0,1,\dots)$ or $(a_n)\in(SC)^r$, if $a_n\to 0$ as $n\to\infty$, and

$$\sum_{n=1}^{\infty} n^{r+1}|\Delta^2 a_{n-1}+\Delta^2 a_n|<\infty.$$

It is clear that $(SC)\subset(SC)^r$, $(r=1,2,\dots)$ and $(SC)\equiv(SC)^0$.

R. Bala and B. Ram [1] have proved that for series (1) with semi-convex null coefficients the following theorem holds true.

Theorem A. *If (a_n) is a semi-convex null sequence, then for the convergence of the cosine series (1) in the metric L^1, it is necessary and sufficient that $a_{n-1}\log n = o(1), n\to\infty$.*

Later on, K. Kaur [2] introduced new modified sine sums as

$$K_n(x)=\frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_{j-1}-\Delta a_{j+1})\sin kx,$$

where $\Delta a_j = a_j - a_{j+1}$, and studied the L^1-convergence of this modified sine sum with semi-convex coefficients proving the following theorem.

Theorem B. *Let (a_n) be a semi-convex null sequence, then $K_n(x)$ converges to $f(x)$ in L^1-norm.*

The main goal of the present work is to study the L^1-convergence of $r-th$ derivative of these new modified sine sums with semi-convex null coefficients of order r and to deduce the sufficient condition of Theorem A and Theorem B as corollaries.

As usually with $D_n(x)$ and $\tilde{D}_n(x)$ we shall denote the Dirichlet and its conjugate kernels defined by

$$D_n(x)=\frac{1}{2}+\sum_{k=1}^{n}\cos kx,\quad \tilde{D}_n(x)=\sum_{k=1}^{n}\sin kx.$$

Everywhere in this paper the constants in the O-expression denote positive constants and they may be different in different relations.

To prove the main results we need the following lemmas:

Lemma 1. ([3]) *For the r-th derivatives of the Dirichlet's kernels $D_n(x)$ and $\tilde{D}_n(x)$ the following estimates hold*

(1) $\|D_n^{(r)}(x)\|_{L^1}=\frac{4}{\pi}n^r\log n+O(n^r)$, $r=0,1,2,\dots$

(2) $\|\tilde{D}_n^{(r)}(x)\|_{L^1}=O\left(n^r\log n\right)$, $r=0,1,2,\dots$

Lemma 2. *If $x \in [\epsilon, \pi-\epsilon], \epsilon \in (0,\pi)$ and $m \in N$, then the following estimate holds*

$$\left|\left(\frac{\tilde{D}_m(x)}{2\sin x}\right)^{(r)}\right| = O_{r,\epsilon}\left(m^{r+1}\right), \quad (r = 0, 1, 2 \dots)$$

where $O_{r,\epsilon}$ depends only on r and ϵ.

Proof. By Leibniz formula we have

$$\begin{aligned}\left(\frac{\tilde{D}_m(x)}{2\sin x}\right)^{(r)} &= \sum_{i=0}^{r}\binom{r}{i}\left(\frac{1}{2\sin x}\right)^{(r-i)}\left(\tilde{D}_m(x)\right)^{(i)} \\ &= \sum_{i=0}^{r}\binom{r}{i}\left(\frac{1}{2\sin x}\right)^{(r-i)}\sum_{j=1}^{m} j^i \sin\left(jx+\frac{i\pi}{2}\right) \\ &= O(1)m^{r+1}\sum_{i=0}^{r}\binom{r}{i}\left(\frac{1}{2\sin x}\right)^{(r-i)}. \qquad (2)\end{aligned}$$

We shall prove by mathematical induction the equality $\left(\frac{1}{2\sin x}\right)^{(\tau)} = \frac{P_\tau(\cos x)}{\sin^{\tau+1} x}$, where P_τ is a cosine polynomial of degree τ.

Namely, we have $\left(\frac{1}{2\sin x}\right)' = \frac{(-1/2)\cos x}{\sin^2 x} = \frac{P_1(\cos x)}{\sin^2 x}$, so that for $\tau = 1$ the above equality is true.

Assume that the equality $F(x) := \left(\frac{1}{2\sin x}\right)^{(\tau)} = \frac{P_\tau(\cos x)}{\sin^{\tau+1} x}$ holds. For the $(\tau+1)-th$ derivative of $\frac{1}{2\sin x}$ we get

$$\begin{aligned} & F'(x) := \\ = & \frac{\left(-\sin^{\tau+2} x\right)P'_\tau(\cos x) - (r+1)P_\tau(\cos x)\sin^\tau x\cos x}{\sin^{2\tau+2} x} \\ = & \frac{(-\sin^2 x)P'_\tau(\cos x) - (\tau+1)P_\tau(\cos x)\cos x}{\sin^{\tau+2} x} \\ = & \frac{(\cos^2 x - 1)P'_\tau(\cos x) - (\tau+1)P_\tau(\cos x)\cos x}{\sin^{\tau+2} x} \\ = & \frac{P_{\tau+1}(\cos x)}{\sin^{\tau+2} x}, \qquad (3)\end{aligned}$$

where $P_{\tau+1}(\cos x)$, is a cosine polynomial of the degree $\tau+1$.

Therefore for $x \in [\epsilon, \pi-\epsilon]$, $\epsilon > 0$, from (2) dhe (3) we obtain

$$\left|\left(\frac{\tilde{D}_m(x)}{2\sin x}\right)^{(r)}\right| = O(1)m^{r+1}\sum_{i=0}^{r}\binom{r}{i}\frac{|P_{r-i}(\cos x)|}{\sin^{r-i+1} x} = O_{r,\epsilon}\left(m^{r+1}\right).$$

2.Main Results

At the begining we prove the following result:

Theorem 1. *Let (a_n) be a semi-convex null sequence of order r, then $K_n^{(r)}(x)$ converges to $g^{(r)}(x)$ in L^1-norm.*

Proof. We have

$$\begin{aligned} K_n(x) &= \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_{j-1}-\Delta a_{j+1})\sin kx \\ &= \frac{1}{2\sin x}\left[\sum_{k=1}^{n}(a_{k-1}-a_{k+1})\sin kx-(a_n-a_{n+2})\,\tilde{D}_n(x)\right]. \end{aligned}$$

Applying Abel's transformation (see [4], p. 17), we get

$$\begin{aligned} K_n(x) &= \frac{1}{2\sin x}\sum_{k=1}^{n}(\Delta a_{k-1}-\Delta a_{k+1})\,\tilde{D}_k(x) \\ &= \frac{1}{2\sin x}\sum_{k=1}^{n}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)\tilde{D}_k(x). \end{aligned}$$

Therefore,

$$K_n^{(r)}(x) = \sum_{k=1}^{n}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)\left(\frac{\tilde{D}_k(x)}{2\sin x}\right)^{(r)}. \tag{4}$$

On the other side we have

$$S_n(x) = \frac{1}{\sin x}\sum_{k=1}^{n}a_k\cos kx\sin x = \frac{1}{2\sin x}\sum_{k=1}^{n}a_k\left[\sin(k+1)x-\sin(k-1)x\right]$$

$$= \frac{1}{2\sin x}\sum_{k=1}^{n}(a_{k-1}-a_{k+1})\sin kx + a_{n+1}\frac{\sin nx}{2\sin x}+a_n\frac{\sin(n+1)x}{2\sin x}$$

$$= \frac{1}{2\sin x}\sum_{k=1}^{n}(\Delta a_{k-1}+\Delta a_k)\sin kx + a_{n+1}\frac{\sin nx}{2\sin x}+a_n\frac{\sin(n+1)x}{2\sin x}. \tag{5}$$

Applying Abel's transformation to the equality (5) we get

$$\begin{aligned} S_n(x) &= \sum_{k=1}^{n} \left(\Delta^2 a_{k-1} + \Delta^2 a_k\right) \frac{\tilde{D}_k(x)}{2\sin x} \\ &\qquad + (a_n - a_{n+2}) \frac{\tilde{D}_n(x)}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x}. \end{aligned}$$

Thus

$$\begin{aligned} S_n^{(r)}(x) &= \sum_{k=1}^{n} \left(\Delta^2 a_{k-1} + \Delta^2 a_k\right) \left(\frac{\tilde{D}_k(x)}{2\sin x}\right)^{(r)} + (a_n - a_{n+2}) \left(\frac{\tilde{D}_n(x)}{2\sin x}\right)^{(r)} \\ &\quad + a_{n+1} \left(\frac{\sin nx}{2\sin x}\right)^{(r)} + a_n \left(\frac{\sin(n+1)x}{2\sin x}\right)^{(r)}. \qquad (6) \end{aligned}$$

By Lemma 1 and since (a_n) is semi-convex null sequence of order r, we have

$$\begin{aligned} &\left| (a_n - a_{n+2}) \left(\frac{\tilde{D}_n(x)}{2\sin x}\right)^{(r)} \right| \\ &= O_{r,\epsilon}\left(\left|(n+1)^{r+1}(a_n - a_{n+2})\right|\right) = O_{r,\epsilon}\left(\left|(n+1)^{r+1}\sum_{k=n}^{\infty}(\Delta a_k - \Delta a_{k+2})\right|\right) \\ &= O_{r,\epsilon}\left(\left|(n+1)^{r+1}\sum_{k=n+1}^{\infty}(\Delta a_{k-1} - \Delta a_{k+1})\right|\right) \\ &= O_{r,\epsilon}\left(\sum_{k=n+1}^{\infty} k^{r+1}|\Delta^2 a_{k-1} + \Delta^2 a_k|\right) = o(1), n \to \infty. \qquad (7) \end{aligned}$$

Also after some elementary calculations and by virtue of Lemma 2 we obtain

$$\begin{aligned} & a_{n+1}\left(\frac{\sin nx}{2\sin x}\right)^{(r)} + a_n \left(\frac{\sin(n+1)x}{2\sin x}\right)^{(r)} = \\ &= a_{n+1}\left[\left(\frac{\tilde{D}_n(x)}{2\sin x}\right)^{(r)} - \left(\frac{\tilde{D}_{n-1}(x)}{2\sin x}\right)^{(r)}\right] + a_n \left[\left(\frac{\tilde{D}_{n+1}(x)}{2\sin x}\right)^{(r)} - \left(\frac{\tilde{D}_n(x)}{2\sin x}\right)^{(r)}\right] \\ &= a_{n+1} O_{r,\epsilon}\left(n^{r+1} + (n-1)^{r+1}\right) + a_n O_{r,\epsilon}\left((n+1)^{r+1} + n^{r+1}\right) \\ &= O_{r,\epsilon}\left((n+1)^{r+1}(a_n + a_{n+1})\right) \end{aligned}$$

$$= O_{r,\epsilon}\left((n+1)^{r+1}\left[(a_n - a_{n+2}) + (a_{n+1} - a_{n+3}) + (a_{n+2} - a_{n+4}) + \cdots\right]\right)$$

$$= O_{r,\epsilon}\left(\sum_{k=n+1}^{\infty} k^{r+1}|\Delta^2 a_{k-1} + \Delta^2 a_k| + \sum_{k=n+2}^{\infty} k^{r+1}|\Delta^2 a_{k-1} + \Delta^2 a_k| + \cdots\right) = o(1), n \to \infty. \tag{8}$$

Because of (7) and (8), when we pass on limit as $n \to \infty$ to (4) and (6) we get

$$\begin{aligned} g^{(r)}(x) &= \lim_{n\to\infty} S_n^{(r)}(x) \\ &= \lim_{n\to\infty} K_n^{(r)}(x) = \sum_{k=1}^{\infty}\left(\Delta^2 a_{k-1} + \Delta^2 a_k\right)\left(\frac{\tilde{D}_k(x)}{2\sin x}\right)^{(r)}. \end{aligned} \tag{9}$$

Using Lemma 2, from (4) and (9) we have

$$\begin{aligned} &\int_{-\pi}^{\pi}\left|g^{(r)}(x) - K_n^{(r)}(x)\right| dx \\ =\ & 2\int_0^{\pi}\sum_{k=n+1}^{\infty}\left|\Delta^2 a_{k-1} + \Delta^2 a_k\right|\left|\left(\frac{\tilde{D}_k(x)}{2\sin x}\right)^{(r)}\right| dx \\ =\ & O_{r,\epsilon}\left(\sum_{k=n+1}^{\infty} k^{r+1}\left|\Delta^2 a_{k-1} + \Delta^2 a_k\right|\right) = o(1), n \to \infty, \end{aligned}$$

which fully proves the Theorem 1.

Remark 1. If we replace $r = 0$ in Theorem 1 we get Theorem B.

Corollary 1. *Let $(a_n) \in (SC)^r$, then the sufficient condition for L^1-convergence of the $r-th$ derivative of the series* (1) *is* $n^r a_n \log n = o(1)$, as $n \to \infty$.

Proof. We have

$$\left\|g^{(r)}(x) - S_n^{(r)}(x)\right\| \leq \left\|g^{(r)}(x) - K_n^{(r)}(x)\right\| + \left\|K_n^{(r)}(x) - S_n^{(r)}(x)\right\|$$

$$\begin{aligned} &= o(1) + \left\| (a_n - a_{n+2}) \left(\frac{\tilde{D}_n(x)}{2\sin x} \right)^{(r)} \right\| \\ &\quad + \left\| a_{n+1} \left(\frac{\sin nx}{2\sin x} \right)^{(r)} + a_n \left(\frac{\sin (n+1)x}{2\sin x} \right)^{(r)} \right\| \quad \text{(by Theorem 1)} \\ &\leq o(1) + a_n \left\| \tilde{D}_n^{(r)}(x) \right\| \quad \text{(by (7))} \\ &= o(1) + O\left(n^r a_n \log n\right) = o(1). \quad \text{(by Lemma 1)} \end{aligned}$$

Remark 2. If we replace $r = 0$ in Corollary 1 we get the sufficient condition of Theorem A.

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