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ZARANTONELLO'S INEQUALITY AND THE ISOMETRIES OF THE M-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. The main purpose of this paper is to use Zarantonello's inequality (see reference [9]) in order to prove that any isometry of the Euclidean m-space is affine, and then describe all the isometries of the Euclidean m-space.

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1. Introduction and preliminaries

Let (X, d) be a metric space. The map $f : X \to X$ is called an *isometry* with respect to the metric d (or a *d-isometry*), if f is surjective and preserves the distances. That is for any points $x, y \in X$ the relation $d(f(x), f(y)) = d(x, y)$ holds. From this relation it follows that the map f is injective, hence it is bijective. Denote by $Iso_d(X)$ the set of all isometries of the metric space (X, d). It is clear that $(Iso_d(X), \circ)$ is a subgroup of $(S(X), \circ)$, where $S(X)$ denotes the group of all bijective transformations $f : X \to X$. We will call $(Iso_d(X), \circ)$ the *group of isometries* of the metric space (X, d). A general, important and complicated problem is to describe the group $(Iso_d(X), \circ)$. This problem was formulated in paper [1] for metric spaces with a metric that is not given by a norm and it is of great interest (see for instance [3]).

In this article we want to prove a standard result of Ulam concerning the group of isometries of the Euclidean space $\mathbb{R}^m$, using the so-called Zarantonello's inequality. In the space $\mathbb{R}^m$ we consider the Euclidean metric, which is defined by the inner product. The inner product of two vectors $x, y \in \mathbb{R}^m$ is given by

$$\langle x, y\rangle = x^1y^1 + x^2y^2 + \cdots + x^my^m.$$

The norm of vector x is given by

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^m)^2}.$$

The Euclidean metric is defined by

$$d(x,y) = \|x-y\| = \sqrt{(x^1-y^1)^2+(x^2-y^2)^2+\cdots+(x^m-y^m)^2}.$$

Here are few standard examples of isometries with respect to this metric.

Example 1. 1. If $A \in O(m)$ is an orthogonal matrix, i.e. $A^TA = I$, then the linear map $f_A : \mathbb{R}^m \to \mathbb{R}^m$, defined by $f_A(x) = Ax$ is an isometry.

2. If $b \in \mathbb{R}^m$, then *the translation of vector* b, $t_b : \mathbb{R}^m \to \mathbb{R}^m, t_b(x) = x + b$, is an isometry. Moreover, the inverse of t_b is t_{-b}.

From the well-known result of S.Mazur and S.Ulam (see the original reference [5]) we have:

Every isometry $f : E \to F$ between real normed spaces is affine. In this case an isometry is a surjective map satisfying for any $x, y \in E$ the relation $||f(x)-f(y)||_F = ||x-y||_E$. This result was proved by S.Mazur and S.Ulam in 1932. A simple proof was given by J.Vaisala [7], the proof is based on the ideas of A.Vogt [8], and it uses reflections in points. If we apply this result for the normed spaces $E = F = \mathbb{R}^m$ with the Euclidean norm $\|\cdot\|$, then it follows that any isometry is affine. Therefore, it is sufficient to see which are the affine maps that preserve the distances.

Our purpose is to use Zarantonello's inequality (see the reference [9]) in order to prove that any isometry of the Euclidean m-space is affine, and then describe all the isometries of the Euclidean m-space. In paper [2] we have used the same idea in the complex plane. In this respect we need few auxiliary results.

Theorem 1 (Lagrange's theorem). *Let n, m be positive integers, let $x_1, x_2, \ldots, x_n \in \mathbb{R}^m$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ be real numbers such that $\alpha_1 + \cdots + \alpha_n = 1$. Then for each $x \in \mathbb{R}^m$ the following relation holds:*

$$\sum_{k=1}^{n} \alpha_k \|x - x_k\|^2 = \|x - \alpha_1 x_1 - \cdots - \alpha_n x_n\|^2 + \sum_{k=1}^{n} \alpha_k \|x_k - \alpha_1 x_1 - \cdots - \alpha_n x_n\|^2.$$

Proof. Using the properties of the inner product, we get:

$$\sum_{k=1}^{n} \alpha_k \|x - x_k\|^2 = \sum_{k=1}^{n} \alpha_k \langle x - x_k, x - x_k\rangle = \sum_{k=1}^{n} \alpha_k (\|x\|^2 - 2\langle x, x_k\rangle + \|x_k\|^2)$$

$$= \|x\|^2 - 2\sum_{k=1}^{n} \alpha_k \langle x, x_k\rangle + \sum_{k=1}^{n} \alpha_k \|x_k\|^2. \tag{1}$$

On the other hand:

$$\|x-\alpha_1x_1-\cdots-\alpha_nx_n\|^2+\sum_{k=1}^{n}\alpha_k\|x_k-\alpha_1x_1-\cdots-\alpha_nx_n\|^2$$

$$=\langle x-\alpha_1x_1-\cdots-\alpha_nx_n,x-\alpha_1x_1-\cdots-\alpha_nx_n\rangle$$

$$+\sum_{k=1}^{n}\alpha_k\langle x_k-\alpha_1x_1-\cdots-\alpha_nx_n,x_k-\alpha_1x_1-\cdots-\alpha_nx_n\rangle$$

$$=\|x\|^2-2\sum_{k=1}^{n}\alpha_k\langle x,x_k\rangle+2\|\alpha_1x_1+\cdots+\alpha_nx_n\|^2+\sum_{k=1}^{n}\alpha_k\|x_k\|^2$$

$$-2\sum_{k=1}^{n}\alpha_k\langle x_k,\alpha_1x_1+\cdots+\alpha_nx_n\rangle=\|x\|^2-2\sum_{k=1}^{n}\alpha_k\langle x,x_k\rangle+\sum_{k=1}^{n}\alpha_k\|x_k\|^2. \qquad (2)$$

From (1) and (2) we obtain the desired relation. □

Theorem 2 (Stewart's theorem). *Let m be a positive integer, $x_1,x_2\in\mathbb{R}^m$ and $a\in\mathbb{R}$. Then for each $x\in\mathbb{R}^m$ the following relation holds:*

$$\|x-ax_1-(1-a)x_2\|^2=a\|x-x_1\|^2+(1-a)\|x-x_2\|^2-a(1-a)\|x_1-x_2\|^2.$$

Proof. Using the properties of the inner product, the left hand side of the desired relation can be written as:

$$\|x-ax_1-(1-a)x_2\|^2=\|x+ax-ax-ax_1-(1-a)x_2\|^2$$

$$=\|a(x-x_1)+(1-a)(x-x_2)\|^2$$

$$=\langle a(x-x_1)+(1-a)(x-x_2),a(x-x_1)+(1-a)(x-x_2)\rangle$$

$$=a^2\|x-x_1\|^2+(1-a)^2\|x-x_2\|^2$$

$$+2a(1-a)\langle x-x_1,x-x_2\rangle=a^2\|x-x_1\|^2+(1-a)^2\|x-x_2\|^2$$

$$+2a(1-a)(\|x\|^2-\langle x,x_1\rangle-\langle x,x_2\rangle+\langle x_1,x_2\rangle).$$

We get:

$$\|x-ax_1-(1-a)x_2\|^2=a^2\|x-x_1\|^2+(1-a)^2\|x-x_2\|^2+a(1-a)(\|x-x_1\|^2+\|x-x_2\|^2$$

$$-\|x_1\|^2-\|x_2\|^2+2\langle x_1,x_2\rangle)=a\|x-x_1\|^2+(1-a)\|x-x_2\|^2-a(1-a)\|x_1-x_2\|^2.$$

□

Stewart's theorem can be extended in the following way:

Theorem 3. *Let n, m be positive integers, $n \geq 2$, $m \geq 1$, $x_1, x_2, \ldots, x_n \in \mathbb{R}^m$, $a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that $a_1 + \cdots + a_n = 1$. Then for each $x \in \mathbb{R}^m$ the following relation holds:*

$$\left\| x - \sum_{k=1}^{n} a_k x_k \right\|^2 = \sum_{k=1}^{n} a_k \|x - x_k\|^2 - \sum_{1 \leq k < l \leq n} a_k a_l \|x_k - x_l\|^2.$$

Proof. The relation we want to prove is equivalent to:

$$\langle x - \sum_{k=1}^{n} a_k x_k, x - \sum_{k=1}^{n} a_k x_k \rangle = \sum_{k=1}^{n} a_k \langle x - x_k, x - x_k \rangle - \sum_{1 \leq k < l \leq n} a_k a_l \langle x_k - x_l, x_k - x_l \rangle$$

and we get

$$\|x\|^2 - 2 \sum_{k=1}^{n} a_k \langle x, x_k \rangle + \sum_{1 \leq k, l \leq n} a_k a_l \langle x_k, x_l \rangle =$$

$$\sum_{k=1}^{n} a_k (\|x\|^2 - 2\langle x, x_k \rangle + \|x_k\|^2) - \sum_{1 \leq k < l \leq n} a_k a_l (\|x_k\|^2 - 2\langle x_k, x_l \rangle + \|x_l\|^2).$$

It suffices to prove that

$$\sum_{1 \leq k < l \leq n} a_k a_l (\|x_k\|^2 + \|x_l\|^2) + \sum_{k=1}^{n} a_k^2 \|x_k\|^2 = \sum_{k=1}^{n} a_k \|x_k\|^2,$$

which is equivalent to

$$\sum_{k=1}^{n} a_k (1 - a_k) \|x_k\|^2 = \sum_{1 \leq k < l \leq n} a_k a_l (\|x_k\|^2 + \|x_l\|^2),$$

and we get

$$\sum_{k=1}^{n} a_k (a_1 + \cdots + a_{k-1} + a_{k+1} + \cdots + a_n) \|x_k\|^2 = \sum_{1 \leq k < l \leq n} a_k a_l (\|x_k\|^2 + \|x_l\|^2),$$

which is true and the desired result is proved. □

Theorem 3 has some interesting consequences.

Corollary 1. *If n, m, are positive integers, $n \geq 2,\ m \geq 1,\ x_1, x_2, \ldots, x_n \in \mathbb{R}^m$, then for each $x \in \mathbb{R}^m$ the following relation holds:*

$$\left\| x - \frac{x_1 + x_2 + \cdots + x_n}{n} \right\|^2 = \frac{1}{n} \sum_{k=1}^{n} \|x - x_k\|^2 - \frac{1}{n^2} \sum_{1 \leq k < l \leq n} \|x_k - x_l\|^2.$$

Proof. In Theorem 3 we take $a_1 = a_2 = \cdots = a_n = \dfrac{1}{n}$. □

Corollary 2. *If n, m are positive integers, $n \geq 2,\ m \geq 1,\ x_1, x_2, \ldots, x_n \in \mathbb{R}^m$, then the following relation holds:*

$$\sum_{k=1}^{n} \left\| x_k - \frac{x_1 + x_2 + \cdots + x_n}{n} \right\|^2 = \sum_{k=1}^{n} \|x_k\|^2 - n \left\| \frac{x_1 + x_2 + \cdots + x_n}{n} \right\|^2.$$

Proof. In Corollary 1 we take $x = 0$, then $x = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$ and we obtain the desired relation. □

Corollary 3. *If n, m are positive integers, $n \geq 2,\ m \geq 1,\ x_1, x_2, \ldots, x_n \in \mathbb{R}^m$, then the following relation holds:*

$$\sum_{1 \leq k < l \leq n} \|x_k - x_l\|^2 = n \sum_{k=1}^{n} \|x_k\|^2 - n^2 \left\| \frac{1}{n} \sum_{k=1}^{n} x_k \right\|^2.$$

Proof. In Corollary 1 we take $x = 0$. □

Corollary 4. *If n, m are positive integers, $n \geq 2,\ m \geq 1,\ x_1, x_2, \ldots, x_n \in \mathbb{R}^m$, $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbb{R}$ such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n = 1$, then the following relation holds:*

$$\|b_1 x_1 + \cdots + b_n x_n - a_1 x_1 - \cdots - a_n x_n\|^2 = - \sum_{1 \leq k < l \leq n} (a_k - b_k)(a_l - b_l) \|x_k - x_l\|^2.$$

Proof. By applying Theorem 3, we get:

$$\|b_1 x_1 + \cdots + b_n x_n - a_1 x_1 - \cdots - a_n x_n\|^2 = a_1 \|x_1 - b_1 x_1 - \cdots - b_n x_n\|^2 + \cdots +$$

$$+ a_n \|x_n - b_1 x_1 - \cdots - b_n x_n\|^2 - \sum_{1 \leq k < l \leq n} a_k a_l \|x_k - x_l\|^2 = a_1 (b_1 \|x_1 - x_1\|^2 + \cdots +$$

$$+ b_n \|x_1 - x_n\|^2 - \sum_{1 \leq k < l \leq n} b_k b_l \|x_k - x_l\|^2) + \cdots + a_n (b_1 \|x_n - x_1\|^2 + \cdots +$$

$$+b_n\|x_n - x_n\|^2 - \sum_{1\le k<l\le n} b_k b_l \|x_k - x_l\|^2) - \sum_{1\le k<l\le n} a_k a_l \|x_k - x_l\|^2.$$

Since $a_1 + a_2 + \cdots + a_n = 1$, we obtain:

$$\|b_1x_1+\cdots+b_nx_n-a_1x_1-\cdots-a_nx_n\|^2 = \sum_{1\le k,l\le n} a_k b_l\|x_k-x_l\|^2 - \sum_{1\le k<l\le n} a_k a_l\|x_k-x_l\|^2$$

$$- \sum_{1\le k<l\le n} b_k b_l \|x_k - x_l\|^2 = - \sum_{1\le k<l\le n} (a_k - b_k)(a_l - b_l)\|x_k - x_l\|^2.$$

□

2. Zarantonello's inequality and the group of isometries

In this section we will use the previous results in order to prove the so-called Zarantonello's inequality in the Euclidean m-space. This inequality was proved in [9] for mappings in Hilbert spaces. It will give us an useful instrument to describe the isometries of the space $\mathbb{R}^m$.

Theorem 4 (Zarantonello's inequality in $\mathbb{R}^m$). *Let m be a positive integer and let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a function such that $\|f(x)-f(y)\| \le \|x-y\|$, for each $x, y \in \mathbb{R}^m$. Then for each positive integer n, $n \ge 2$ and for any real numbers $a_1, a_2, \ldots, a_n \ge 0$ such that $a_1 + a_2 + \cdots + a_n = 1$ and $x_1, x_2, \ldots, x_n \in \mathbb{R}^m$ the following relation holds:*

$$\|f(a_1x_1 + \cdots + a_nx_n) - a_1f(x_1) - \cdots - a_nf(x_n)\|^2$$
$$\le \sum_{1\le k<l\le n} a_k a_l(\|x_k - x_l\|^2 - \|f(x_k) - f(x_l)\|^2).$$

Proof. By applying Theorem 3, we get:

$$\|f(a_1x_1 + \cdots + a_nx_n) - a_1f(x_1) - \cdots - a_nf(x_n)\|^2$$
$$= a_1\|f(a_1x_1 + \cdots + a_nx_n) - f(x_1)\|^2 + \cdots +$$
$$+a_n\|f(a_1x_1 + \cdots + a_nx_n) - f(x_n)\|^2 - \sum_{1\le k<l\le n} a_k a_l\|f(x_k) - f(x_l)\|^2. \quad (3)$$

Using the contraction condition for f, we get:

$$a_1\|f(a_1x_1 + \cdots + a_nx_n) - f(x_1)\|^2 + \cdots + a_n\|f(a_1x_1 + \cdots + a_nx_n) - f(x_n)\|^2$$
$$\le a_1\|x_1 - a_1x_1 - \cdots - a_nx_n\|^2 + \cdots + a_n\|x_n - a_1x_1 - \cdots - a_nx_n\|^2. \quad (4)$$

Taking in Theorem 3 $x = a_1x_1 + \cdots + a_nx_n$, we obtain:

$$a_1\|x_1-a_1x_1-\cdots-a_nx_n\|^2+\cdots+a_n\|x_n-a_1x_1-\cdots-a_nx_n\|^2 = \sum_{1\le k<l\le n} a_k a_l\|x_k-x_l\|^2,$$

and together with (3) and (4) we obtain the desired inequality. □

Let S be a subset of the Euclidean m-space. The map $f : \mathbb{R}^m \to \mathbb{R}^m$ is called a *S-isometry* if it preserves distances for the points in S, i.e. $\|f(x) - f(y)\| = \|x - y\|$, for each $x, y \in S$.

Corollary 5. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a function such that*

$$\|f(x) - f(y)\| \leq \|x - y\|,$$

for each $x, y \in \mathbb{R}^m$, and let $x_1, x_2, \dots, x_n \in \mathbb{R}^m$ be given points. If $\|f(x_k) - f(x_l)\| = \|x_k - x_l\|$, for $k, l = 1, 2, ..., n$, $k \neq l$, then for any real numbers $a_1, a_2, \dots, a_n \geq 0$ with $a_1 + a_2 + \cdots + a_m = 1$, we have:

$$f(a_1 x_1 + \cdots + a_n x_n) = a_1 f(x_1) + \cdots + a_n f(x_n).$$

Proof. Indeed, we have $\|x_k - x_l\|^2 - \|f(x_k) - f(x_l)\|^2 = 0$ and $a_k a_l \geq 0$ for $k, l = 1, 2, ..., n$, $k \neq l$. From Zarantonello's inequality it follows $\|f(a_1 x_1 + \cdots + a_n x_n) - a_1 f(x_1) - \cdots - a_n f(x_n)\|^2 = 0$, hence the conclusion. □

The result contained in the previous Corollary shows that any function $f : \mathbb{R}^m \to \mathbb{R}^m$ satisfying the contraction condition $\|f(x) - f(y)\| \leq \|x - y\|$, for any $x, y \in \mathbb{R}^m$ and preserving all distances in the set $S = \{x_1, x_2, \dots, x_m\}$, i.e. f is a S-isometry, is affine on the convex envelope of this set.

Remark 1. If $f : \mathbb{R}^m \to \mathbb{R}^m$ is an isometry, i.e. $\|f(x) - f(y)\| = \|x - y\|$, for each $x, y \in \mathbb{R}^m$, then for each positive integer n, $n \geq 2$ and for each $a_1, a_2, \dots, a_n \geq 0$ such that $a_1 + a_2 + \cdots + a_n = 1$ and $x_1, x_2, \dots, x_n \in \mathbb{R}^m$ the following relation holds:

$$f(a_1 x_1 + \cdots + a_n x_n) = a_1 f(x_1) + \cdots + a_n f(x_n).$$

Lemma 1. If $f : \mathbb{R}^m \to \mathbb{R}^m$ is an isometry, i.e. $\|f(x) - f(y)\| = \|x - y\|$, for each $x, y \in \mathbb{R}^m$, then the function $g : \mathbb{R}^m \to \mathbb{R}^m$, defined by $g(x) = f(x) - f(0_m)$ is additive, i.e. $g(x + y) = g(x) + g(y)$, for each $x, y \in \mathbb{R}^m$.

Proof. We note that g is also an isometry and by applying the previous remark, we get:

$$g\left(\frac{x+y}{2}\right) = \frac{g(x) + g(y)}{2},$$

for each $x, y \in \mathbb{R}^m$. Since $g(0_m) = 0_m$, we obtain $g\left(\frac{x}{2}\right) = \frac{g(x)}{2}$, for each $x \in \mathbb{R}^m$. Consequently:

$$g(x + y) = 2g\left(\frac{x+y}{2}\right) = 2\frac{g(x) + g(y)}{2} = g(x) + g(y),$$

for each $x, y \in \mathbb{R}^m$. □

The next result gives the general form of the isometries of the space $\mathbb{R}^m$ with the Euclidean metric.

Theorem 5. *Let m be a positive integer. A function $f : \mathbb{R}^m \to \mathbb{R}^m$ is an isometry if and only if there exists an orthogonal matrix $A \in \mathcal{M}_m(\mathbb{R})$ and there exists $x_0 \in \mathbb{R}^m$ such that $f(x) = Ax + x_0$, for each $x \in \mathbb{R}^m$.*

Proof. Let $f : \mathbb{R}^m \to \mathbb{R}^m, f(x) = Ax + x_0$, where $A \in \mathcal{M}_m(\mathbb{R})$ is an orthogonal matrix and $x_0 \in \mathbb{R}^m$. We have:

$$\|f(x) - f(y)\|^2 = \|Ax - Ay\|^2 = \langle A(x-y), A(x-y)\rangle$$

$$= (x-y)^T A^T A(x-y) = (x-y)^T(x-y) = \langle x-y, x-y\rangle = \|x-y\|^2.$$

Thus f is an isometry.

Conversely, let us define $g : \mathbb{R}^m \to \mathbb{R}^m$, $g(x) = f(x) - f(0_m)$. From the previous remark, g is additive, i.e. $g(x+y) = g(x)+g(y)$, for each $x, y \in \mathbb{R}^m$. g is a Lipschitz function of constant 1, so g is continuous. Using the fact that g is continuous, it is easy to prove that $g(t \cdot x) = t \cdot g(x)$, for each $t \in \mathbb{R}$ and $x \in \mathbb{R}^m$. We have obtained that g is linear, so $g(x) = Ax$, where $A \in \mathcal{M}_m(\mathbb{R})$, for each $x \in \mathbb{R}^m$. Then $f(x) = g(x) + f(0_m)$, and we get that $f(x) = Ax + x_0$, where $x_0 = f(0_m)$. Thus it suffices to prove that A is an orthogonal matrix.

First step. We will prove that the map g preserves the inner product.

For each $x, y \in \mathbb{R}^m$, we have:

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \tag{5}$$

Also, we have:

$$\|g(x) - g(y)\|^2 = \|g(x)\|^2 - 2\langle g(x), g(y)\rangle + \|g(y)\|^2. \tag{6}$$

Using the fact that f is an isometry, we obtain that g is also an isometry, thus $\|x - y\| = \|g(x) - g(y)\|$, for each $x, y \in \mathbb{R}^m$. Taking $y = 0_m$, we obtain that $\|x\| = \|x - 0_m\| = \|g(x) - g(0_m)\| = \|g(x) - 0_m\| = \|g(x)\|$, for each $x \in \mathbb{R}^m$. Using (5) and (6), we get that $\langle x, y\rangle = \langle g(x), g(y)\rangle$, for each $x, y \in \mathbb{R}^m$.

Second step. We will prove that if $\{e_1, \ldots, e_m\}$ is the canonical base of the space $\mathbb{R}^m$, then $\{g(e_1), \ldots, g(e_m)\}$ is an orthonormal base of the space $\mathbb{R}^m$.

We know that $e_1, \ldots, e_m$ are orthonormal vectors. Using the first step, we obtain that $g(e_1), \ldots, g(e_m)$ are also orthonormal vectors. Since orthonormal vectors are linearly independent and dim $\mathbb{R}^m = m$, we obtain that $\{g(e_1), \ldots, g(e_m)\}$ is an orthormal base of the space $\mathbb{R}^m$.

Third step. We now prove that matrix A satisfies $A^T A = I$, thus it is an orthogonal matrix.

Indeed, we have:

$$g(e_i)^T g(e_j) = \langle g(e_i), g(e_j)\rangle = \langle e_i, e_j\rangle = \delta_{ij}.$$

If $A = (a_{ij})_{1\le i\le j\le m}$, we get

$$\sum_{k=1}^{m} a_{ki}a_{kj} = \delta_{ij},$$

for each $i, j = 1, \ldots, m$, which means that matrix A is orthogonal. □

Corollary 6. *The isometry group $Iso(\mathbb{R}^m)$ is isomorphic to $O_m(\mathbb{R}) \cdot T(m)$, where $T(m)$ is the translations group of the space $\mathbb{R}^m$, $O_m(\mathbb{R})$ denotes the orthogonal group of the space $\mathbb{R}^m$, and " $\cdot$ " is the semi-direct product of groups.*

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