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A FINAL VALUE PROBLEM FOR HEAT EQUATION: REGULARIZATION BY TRUNCATION METHOD AND NEW ERROR ESTIMATES

Dang Duc Trong, Pham Hoang Quan and Nguyen Huy Tuan

Abstract. We introduce the truncation method for solving a backward heat conduction problem. For this method, we give the stability analysis with new error estimates. Meanwhile, we investigate the roles of regularization parameters in these two methods. These estimates prove that our method is effective.

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1. Introduction

In recently, in [16],we consider the problem of finding the temperature $u(x,t)$, $(x,t) \in (0,\pi) \times [0,T]$, such that

$$u_t - u_{xx} = f(x,t), (x,t) \in (0,\pi) \times (0,T)$$
$$u(0,t) = u(\pi,t) = 0, t \in (0,T)$$
$$u(x,T) = g(x), (x,t) \in (0,\pi) \times (0,T)$$

where $g(x), f(x,z)$ are given. The problem is called the backward heat problem, the backward Cauchy problem, or the final value problem. As is known, the nonhomogeneous problem is severely ill-posed; i.e., solutions do not always exist, and in the case of existence, these do not depend continuously on the given data. Physically, g can only be measured, there will be measurement errors, and we would actually have as data some function $g^\epsilon \in L^2(0,\pi)$, for which $\|g^\epsilon - g\| \le \epsilon$ where the constant $\epsilon > 0$ represents a bound on the measurement error, $\|.\|$denotes the L^2-norm. Under the condition $\|u_{xx}(.,t)\| < \infty$, we obtained an following error estimate

$$\|u(.,t) - U^\epsilon(.,t)\| \le \frac{C}{1 + \ln(T/\epsilon)} \tag{1}$$

where

$$U^\epsilon(x,t) = \sum_{p=1}^{\infty} \left(\frac{e^{-tp^2}}{\epsilon p^2 + e^{-Tp^2}} \varphi_p - \int_t^T \frac{e^{(s-t-T)p^2}}{\epsilon p^2 + e^{-Tp^2}} f_p(s) ds \right) \sin px, \tag{2}$$

and

$$f_p(t) = \frac{2}{\pi} < f(x,t), \sin(px) >= \frac{2}{\pi} \int_0^\pi f(x,t) \sin(px) dx\,, \tag{3}$$

$$g_p = \frac{2}{\pi} < g(x), \sin(px) >= \frac{2}{\pi} \int_0^\pi g(x) \sin(px) dx. \tag{4}$$

and $< .,. >$ is the inner product in $L^2(0,\pi)$.

For the nonhomogeneous and nonlinear case, we refer the reader to some recent works of Feng Xiao-Li [7] , D.D.Trong and his group [14,15,16,17,18]. All of above papers gave the error estimates which is the logarithmic form. For easy to the reader, we recall some convergence rates on these works in Remark 3, page 10 of this paper.

In our knowledge, so far there are many papers on the backward heat equation, but theoretically the error estimates of most regularization methods in the literature are Holder type on $[0,T]$, i.e.,the approximate solution v^ϵ and the exact solution u satisfy

$$\|u(.,t) - v^\epsilon(.,t)\| \le C\epsilon^p, \quad p > 0. \tag{5}$$

where C is the constant depend on u, p is a constant is not depend on t, u and ϵ is the noise level on final data $u(x,T)$. The major object of this paper is to provide truncation method to established the Holder estimates such as 5. These orders is optimal order of backward heat as we know.

2.Regularization by truncation method and error estimates.

Suppose the Problem 1 has an exact solution $u \in C([0,T]; H_0^1(0,\pi)) \cap C^1((0,T); L^2(0,\pi))$, then u can be formulated in the frequency domain

$$u(x,t) = \sum_{m=1}^{\infty} (e^{-(t-T)m^2} g_m - \int_t^T e^{-(t-s)m^2} f_m(s) ds) \sin(mx) \tag{6}$$

where

$$f_m(t) = \frac{2}{\pi} \int_0^\pi f(x,t) \sin(mx) dx, \quad g_m = \frac{2}{\pi} \int_0^\pi g(x) \sin(mx) dx$$

and $< .,. >$ is the inner product in $L^2((0,\pi))$.

Throughout this paper, we suppose that $f(x,t) \in L^2(0,T); L^2(0,\pi))$ and $g(x) \in L^2(0,\pi)$.

Let $0 = q < \infty$. By $H^q(0,\pi)$ we denote the space of all functions $g \in L^2(0,\pi)$ with the property

$$\sum_{m=1}^{\infty} (1+m^2)^q |g_m|^2 < \infty,$$

where $g_m = \frac{2}{\pi}\int_0^\pi g(x)\sin(mx)dx$. This method is based on ideas of paper [8]. However, in that that paper, the Holder estimates is not investigated. In this section, we give a truncation method and obtain the convergence estimates under an a-priori assumption for the exact solution. The stability order of Holder type is also established.

From 6, we note that $e^{(T-t)m^2}$ tends to infinity as m tends to infinity, then in order to guarantee the convergence of solution u given by 6, the coefficient $< u, \sin mx >$ must decay rapidly. Usually such a decay is not likely to occur for the measured data g^ϵ. Therefore, a natural way to obtain a stable approximation solution u is to eliminate the high frequencies and consider the solution u for $m < N$, where N is a positive integer. We define the truncation regularized solution as follows

$$u_N^\epsilon(x,t) = \sum_{m=1}^{N} \left(e^{-(t-T)m^2} g_m - \int_t^T e^{-(t-s)m^2} f_m(s)ds \right) \sin(mx) \tag{7}$$

and

$$v_N^\epsilon(x,t) = \sum_{m=1}^{N} \left(e^{-(t-T)m^2} g_m^\epsilon - \int_t^T e^{-(t-s)m^2} f_m(s)ds \right) \sin(mx) \tag{8}$$

where the positive integer N plays the role of the regularization parameter.

Lemma 1. *The problem (1) has a unique solution u if and only if*

$$\sum_{m=1}^{\infty} \left(e^{Tm^2} g_m - \int_0^T e^{sm^2} f_m(s)ds \right)^2 < \infty. \tag{9}$$

Proof. Suppose the Problem (1) has an exact solution $u \in C([0,T]; H_0^1(0,\pi)) \cap C^1((0,T); L^2(0,\pi))$, then u can be formulated in the frequency domain

$$u(x,t) = \sum_{m=1}^{\infty} (e^{-(t-T)m^2} g_m - \int_t^T e^{-(t-s)m^2} f_m(s)ds) \sin(mx). \tag{10}$$

This implies that

$$u_m(0) = e^{Tm^2} g_m - \int_0^T e^{sm^2} f_m(s)ds. \tag{11}$$

Then

$$\|u(.,0)\|^2 = \sum_{m=1}^{\infty} \left(e^{Tm^2} g_m - \int_0^T e^{sm^2} f_m(s)ds \right)^2 < \infty.$$

If we get 9, then define $v(x)$ be as the function

$$v(x) = \sum_{m=1}^{\infty} \left(e^{Tm^2} g_m - \int_0^T e^{sm^2} f_m(s)ds \right) \sin mx \in L^2(0,\pi).$$

Consider the problem

$$\{u_t - u_{xx} = f(x,t), u(0,t) = u(\pi,t) = 0, \quad t \in (0,T) u(x,0) = v(x), \quad x \in (0,\pi) \tag{12}$$

It is clear to see that 12 is the direct problem so it has a unique solution u. We have

$$u(x,t) = \sum_{m=1}^{\infty} \left(e^{-tm^2} < v(x), \sin mx > + \int_0^t e^{(s-t)m^2} f_m(s)ds \right) \sin mx \tag{13}$$

Let $t = T$ in 14, we have

$$\begin{aligned} u(x,T) &= \sum_{m=1}^{\infty} \left(e^{-Tm^2} \left(e^{Tm^2} g_m - \int_0^T e^{sm^2} f_m(s)ds \right) + \int_0^T e^{(s-T)m^2} f_m(s)ds \right) \sin mx \\ &= \sum_{m=1}^{\infty} g_m \sin mx = g(x). \end{aligned}$$

Hence, u is the unique solution of (1).

Theorem 1 *The solution u_N^ϵ given in 7 depends continuously on g in $C([0,T]; L^2(0,\pi))$. Furthermore, we have*

$$\|v_N^\epsilon(x,t) - u_N^\epsilon(x,t)\| \le e^{(T-t)N^2} \epsilon.$$

Proof. Let u_N^ϵ and w_N^ϵ be two solutions of 7 corresponding to the final values g and h. From 7, we have

$$u_N^\epsilon(x,t) = \sum_{m=1}^{N} \left(e^{-(t-T)m^2} g_m - \int_t^T e^{-(t-s)m^2} f_m(s)ds \right) \sin(mx) \quad 0 \le t \le T, \quad (14)$$

$$w_N^\epsilon(x,t) = \sum_{m=1}^{N} \left(e^{-(t-T)m^2} h_m - \int_t^T e^{-(t-s)m^2} f_m(s)ds \right) \sin(mx) \quad 0 \le t \le T, \quad (15)$$

where

$$g_m = \frac{2}{\pi}\int_0^\pi g(x)\sin(mx)dx, \quad h_m = \frac{2}{\pi}\int_0^\pi h(x)\sin(mx)dx.$$

This follows that

$$\begin{aligned} \|u_N^\epsilon(.,t) - w_N^\epsilon(.,t)\|^2 &= \frac{\pi}{2}\sum_{m=1}^{N} \left| e^{(T-t)m^2}(g_m - h_m)\right|^2 \\ &\le \frac{\pi}{2} e^{2(T-t)N^2} \sum_{m=1}^{\infty} |g_m - h_m|^2 \\ &= e^{2(T-t)N^2}\|g-h\|^2. \end{aligned} \quad (16)$$

Hence

$$\|u_N^\epsilon(.,t) - w_N^\epsilon(.,t)\| \le e^{(T-t)N^2}\|g-h\|. \quad (17)$$

This completes the proof theorem.
Since 17 and the condition $\|g^\epsilon - g\| \le \epsilon$, we have

$$\|v_N^\epsilon(x,t) - u_N^\epsilon(x,t)\| \le e^{(T-t)N^2}\epsilon. \quad (18)$$

Theorem 2. *Assume that there exists the positive numbers A_1, A_2 such that $\|u(.,0)\|^2 \le A_1$ and*

$$\frac{\pi}{2}\sum_{m=1}^{\infty}\int_0^T e^{2sm^2} f_m^2(s)ds < A_2.$$

Let us $N = [p]$ where $[.]$ denotes the largest integer part of a real number with

$$p = \sqrt{\frac{1}{T}\ln(\frac{1}{\epsilon})},$$

then the following convergence estimate holds for every $t \in [0,T]$

$$\|v_N^\epsilon(x,t) - u(x,t)\| \le \left(\sqrt{2A_1 + 2\pi T A_2} + 1\right)\epsilon^{\frac{t}{T}}. \tag{19}$$

Proof. Since 7, we have

$$\begin{aligned}
u(x,t) - u_N^\epsilon(x,t) &= \sum_{m=N}^{\infty}\left(e^{-(t-T)m^2}g_m - \int\limits_t^T e^{-(t-s)m^2} f_m(s)ds\right)\sin(mx)\\
&= \sum_{m=N}^{\infty} < u(x,t), \sin mx > \sin mx.
\end{aligned}$$

Thus, using the inequality $(a+b)^2 \le 2(a^2+b^2)$ and Holder inequality, we have

$$\begin{aligned}
\|u(.,t) - u_N^\epsilon(.,t)\|^2 &= \frac{\pi}{2}\sum_{m=N}^{\infty}\left(e^{-(t-T)m^2}g_m - \int\limits_t^T e^{-(t-s)m^2} f_m(s)ds\right)^2\\
&= \frac{\pi}{2}\sum_{m=N}^{\infty} e^{-2tm^2}\left(u_m(0) + \int_0^t e^{sm^2} g_m(s)ds\right)^2\\
&\le \pi\sum_{m=N}^{\infty} e^{-2tm^2}u_m^2(0) + \pi T\sum_{m=N}^{\infty}\int_0^T e^{-2tm^2}e^{2sm^2} f_m^2(s)ds\\
&\le 2e^{-2tN^2}\left(\|u(.,0)\|^2 + \pi T\sum_{m=1}^{\infty}\int_0^T e^{2sm^2} f_m^2(s)ds\right)\\
&\le 2e^{-2tN^2}(A_1 + \pi T A_2).
\end{aligned} \tag{20}$$

Combining 18 and 20then

$$\begin{aligned}
\|v_N^\epsilon(x,t) - u(x,t)\| &\le \|v_N^\epsilon(.,t) - u_N(.,t)\| + u_N(.,t) - u(.,t)\|\\
&\le e^{-tN^2}\sqrt{2A_1 + 2\pi T A_2} + e^{(T-t)N^2}\epsilon.
\end{aligned}$$

From

$$N = \sqrt{\frac{1}{T}\ln(\frac{1}{\epsilon})}$$

then the following convergence estimate holds

$$\|v_N^\epsilon(x,t) - u(x,t)\| \le \epsilon^{\frac{t}{T}}\left(\sqrt{2A_1 + 2\pi T A_2} + 1\right).$$

Remark 1. *1. If $f(.,t) = 0$, the error 19 is the same order as [3].*

2. From Theorem 2, we find that v_N^ϵ is an approximation of exact solution u. The approximation error depends continuously on the measurement error for fixed $0 < t \leq T$. However, as $t \to 0$ the accuracy of regularized solution becomes progressively lower. This is a common thing in the theory of ill-posed problems, if we do not have additional conditions on the smoothness of the solution. To retain the continuous dependence of the solution at $t = 0$, we introduce a stronger a priori assumption.

Theorem 3. *Assume that there exists the positive numbers q, A_3 such that*

$$\|u(.,t)\|_{H^q(0,\pi)} < A_3^2.$$

Let us $N = [p]$ where $[.]$ denotes the largest integer part of a real number with

$$p = \sqrt{\frac{1}{T+k}\ln(\frac{1}{\epsilon})}$$

for $k > 0$. Then the following convergence estimate holds

$$\|v_N^\epsilon(x,t) - u(x,t)\| \leq A_3\left(\frac{1}{T+k}\ln(\frac{1}{\epsilon}\right)^{-\frac{q}{2}} + \epsilon^{\frac{t+k}{T+k}}. \tag{21}$$

for every $t \in [0,T]$.

Proof. Since 7, we have

$$\begin{aligned} u(x,t) - u_N^\epsilon(x,t) &= \sum_{m=N}^{\infty}\left(e^{-(t-T)m^2}g_m - \int_t^T e^{-(t-s)m^2}g_m(s)ds\right)\sin(mx) \\ &= \sum_{m=N}^{\infty} < u(x,t), \sin mx > \sin mx. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|u(.,t) - u_N^\epsilon(.,t)\|^2 &= \frac{\pi}{2}\sum_{m=N}^{\infty} m^{-2q}m^{2q}u_m^2(t) \\ &\leq N^{-2q}\frac{\pi}{2}\sum_{m=1}^{\infty} m^{2q}u_m^2(t) \\ &\leq N^{-2q}\frac{\pi}{2}\sum_{m=1}^{\infty}(1+m^2)^q u_m^2(t) \\ &\leq N^{-2q}\frac{\pi}{2}A_3^2. \end{aligned} \tag{22}$$

Combining 18 and 27 then

$$\begin{aligned}\|v_N^\epsilon(x,t)-u(x,t)\| &\le \|v_N^\epsilon(.,t)-u_N(.,t)\|+u_N(.,t)-u(.,t)\| \\ &\le N^{-q}A_3+e^{(T-t)N^2}\epsilon.\end{aligned}$$

From

$$N=\sqrt{\frac{1}{T+k}\ln(\frac{1}{\epsilon})}$$

then the following convergence estimate holds

$$\|v_N^\epsilon(x,t)-u(x,t)\|\le\left(\frac{1}{T+k}\ln(\frac{1}{\epsilon}\right)^{-\frac{q}{2}}A_3+\epsilon^{\frac{t+k}{T+k}}.$$

Remark 2. *1. Denche and Bessila in [5], Trong and his group [15,16,18] gave the error estimates in the form*

$$\|v^\epsilon(.,t)-u(.,t)\|\le\frac{C_1}{1+\ln\frac{T}{\epsilon}}. \tag{23}$$

In recently, Chu-Li Fu and his coauthors [4,7,8] gave the error estimates as follows

$$\|\|v^{\epsilon,\delta}(.)-u(.)\le\frac{\delta}{2\sqrt{\epsilon}}+\max\{\left(\frac{4T}{\ln\frac{1}{\epsilon}}\right)^{\frac{p}{2}},\epsilon^{\frac{1}{2}}\}. \tag{24}$$

If $q=2$, the error 21 is the same order as these above results.

2.Since 21, the first term of the right hand side of 21 is the logarithmic form, and the second term is a power, so the order of 21 is also logarithmic order. Suppose that $E_\epsilon=\|v^\epsilon-u\|$ be the error of the exact solution and the approximate solution. In most of results concerning the backward heat, then optimal error between is of the logarithmic form. It means that

$$E_\epsilon\le C\left(\ln\frac{T}{\epsilon}\right)^{-q}$$

where $q>0$.

The error order of logarithmic form is investigated in many recent papers, such as [3,4,5,7,8,14,15,16,17,18]. This often occurs in the boundary error estimate for ill-posed problems. To retain the Holder order in $[0,T]$, we introduce the following Theorem with different priori assumption.

Theorem 4. *Assume that there exists the positive numbers β, A_4 such that*

$$\frac{\pi}{2}\sum_{m=1}^{\infty} e^{2\beta m^2} u_m^2(t) < A_4^2. \tag{25}$$

Let us $N = [p]$ where $[.]$ denotes the largest integer part of a real number with

$$p = \sqrt{\frac{1}{T+\beta}\ln(\frac{1}{\epsilon})}$$

then the following convergence estimate holds

$$\|v_N^\epsilon(x,t) - u(x,t)\| \le \left(A_4 + \epsilon^{\frac{t}{T+\beta}}\right)\epsilon^{\frac{\beta}{T+\beta}}. \tag{26}$$

for every $t \in [0,T]$.

Proof. Since 7, we have

$$\begin{aligned} u(x,t) - u_N^\epsilon(x,t) &= \sum_{m=N}^{\infty}\left(e^{-(t-T)m^2} g_m - \int_t^T e^{-(t-s)m^2} f_m(s)ds\right)\sin(mx) \\ &= \sum_{m=N}^{\infty} < u(x,t), \sin mx > \sin mx. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|u(.,t) - u_N^\epsilon(.,t)\|^2 &= \frac{\pi}{2}\sum_{m=N}^{\infty} e^{-2\beta m^2} e^{2\beta m^2} u_m^2(t) \\ &\le \frac{\pi}{2} e^{-2\beta N^2}\sum_{m=N}^{\infty} e^{2\beta m^2} u_m^2(t) \\ &\le e^{-2\beta N^2}\frac{\pi}{2}\sum_{m=1}^{\infty} e^{2\beta m^2} u_m^2(t) \le e^{-2\beta N^2} A_4^2. \end{aligned} \tag{27}$$

Combining 18 and 27, we get

$$\begin{aligned} \|v_N^\epsilon(x,t) - u(x,t)\| &\le \|v_N^\epsilon(.,t) - u_N(.,t)\| + u_N(.,t) - u(.,t)\| \\ &\le e^{-\beta N^2} A_4 + e^{(T-t)N^2}\epsilon. \end{aligned}$$

From

$$N = \left[\sqrt{\frac{1}{T+\beta}\ln(\frac{1}{\epsilon})}\right]$$

then the following convergence estimate holds

$$\|v_N^\epsilon(x,t) - u(x,t)\| \le \epsilon^{\frac{\beta}{T+\beta}} A_4 + \epsilon^{\frac{t+\beta}{T+\beta}} = \epsilon^{\frac{\beta}{T+\beta}} \left(A_4 + \epsilon^{\frac{t}{T+\beta}} \right).$$

Remark 3. *1. The condition 25 is not verifiable. Hence, we can check it by replacing the conditions of f and g. Thus, we have*

$$\sum_{m=1}^{\infty} e^{2\beta m^2} u_m^2(t) = \sum_{m=1}^{\infty} e^{2\beta m^2} \left(e^{-(t-T)m^2} g_m - \int_t^T e^{-(t-s)m^2} f_m(s) ds \right)^2.$$

Hence, we can replace 25 by the different conditions

$$\sum_{m=1}^{\infty} e^{2(T+\beta)m^2} g_m < \infty, \quad \sum_{m=1}^{\infty} \int_0^T e^{2(s+\beta)m^2} f_m^2(s) ds < \infty.$$

2. For an ill-posed problem, say for the above problem , normally one proves the following fact: suppose that the norm of the solution at $t = 0$ is bounded, then we have a Holder estimate in $(0;T)$ (as in Theorem 2),if we suppose some regularity of the solution at $t = 0$, then we have a stability estimate of logarithmic type at $t = 0$ (as in Theorem 3)and finally, if something similar to 25 is assumed, then we get a stability estimate of Holder type for the whole $[0;T]$.

3. Notice the reader that the error 26 $(\beta > 0)$ is the order of Holder type for all $t \in [0,T]$. It is easy to see that the convergence rate of ϵ^p, $(0 < p)$ is more quickly than the logarithmic order $\left(\ln(\frac{1}{\epsilon})\right)^{-q}$ $(q > 0)$ when $\epsilon \to 0$. Comparing 23,24 with 26, we can see that our method is effective.

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Dang Duc Trong
Department of Mathematics
Ho Chi Minh City National University
227 Nguyen Van Cu, Q. 5, HoChiMinh City, Vietnam
email: *ddtrong@mathdep.hcmuns.edu.vn*

Pham Hoang Quan
Department of Mathematics
SaiGon University
273 An Duong vuong street, HoChiMinh city, VietNam
email:*tquan@pmail.vnn.vnm*

Nguyen Huy Tuan
Department of Mathematics
SaiGon University
273 An Duong vuong street, HoChiMinh city, VietNam
email:*tuanhuy_bs@yahoo.com*