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SOME PROPERTIES OF CERTAIN INTEGRAL OPERATORS

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Abstract. In this paper, we consider certain subclasses of analytic functions with bounded radius and bounded boundary rotation and study the mapping properties of these classes under certain integral operators introduced by Breaz et. al recently.

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1. Introduction

Let A be the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. Let $C_b(\rho)$ and $S_b^*(\rho)$ be the classes of convex and starlike functions of complex order b ($b \in \mathbb{C} - \{0\}$) and type ρ ($0 \leq \rho < 1$) respectively studied by Frasin [3].

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in E with $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \rho}{1 - \rho} \right| d\theta \leq k\pi, \quad z = re^{i\theta},$$

where $k \geq 2$ and $0 \leq \rho < 1$. We note that, for $\rho = 0$, we obtain the class P_k defined and studied in [10], and for $\rho = 0$, $k = 2$, we have the well known class P of functions with positive real part. The case $k = 2$ gives the class $P(\rho)$ of functions with positive real part greater than ρ.

A function $f(z) \in A$ is said to belong to the class $V_k(\rho, b)$ if and only if

$$1 + \frac{1}{b} \frac{zf''(z)}{f'(z)} \in P_k(\rho), \tag{1.1}$$

where $k \geq 2$, $0 \leq \rho < 1$ and $b \in \mathbb{C} - \{0\}$. When $\rho = 0$ and $b = 1$, we obtain the class V_k of functions with bounded boundary rotation, first discussed by Paatero [2].

Similarly, an analytic function $f(z) \in R_k(\rho, b)$ if and only if

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \in P_k(\rho) \tag{1.2}$$

where $k \geq 2$, $0 \leq \rho < 1$ and $b \in \mathbb{C} - \{0\}$. When $\rho = 0$ and $b = 1$, we obtain the class R_k of functions with bounded radius rotation [2]. For more details see [6, 7, 8]. Let us consider the integral operators

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{1.3}$$

and

$$F_{\alpha_1 \ldots \alpha_n}(z) = \int_0^z \left[f_1'(t)\right]^{\alpha_1} \ldots \left[f_n'(t)\right]^{\alpha_n} dt, \tag{1.4}$$

where $f_i(z) \in A$ and $\alpha_i > 0$ for all $i \in \{1, 2, \ldots, n\}$. These operators, given by (1.3) and (1.4), are introduced and studied by Breaz and Breaz [2] and Breaz et.al [4], respectively. Recently, Breaz and Güney [3] considered the above integral operators and they obtained their properties on the classes $C_b(\rho)$, $S_b^*(\rho)$ of convex and starlike functions of complex order b and type ρ introduced and studied by Frasin [5].

In this paper, we investigate some propeties of the above integral operators $F_n(z)$ and $F_{\alpha_1 \ldots \alpha_n}(z)$ for the classes $V_k(\rho, b)$ and $R_k(\rho, b)$ respectively. In order to derive our main result, we need the following lemma.

Lemma 1.1 . [9] *Let $f(z) \in V_k(\alpha)$, $0 \leq \alpha < 1$. Then $f(z) \in R_k(\beta)$, where*

$$\beta = \frac{1}{4}\left[(2\alpha - 1) + \sqrt{4\alpha^2 - 4\alpha + 9}\,\right]. \tag{1.5}$$

2. Main Results

Theorem 2.1. *Let $f_i(z) \in R_k(\rho, b)$ for $1 \leq i \leq n$ with $0 \leq \rho < 1$, $b \in \mathbb{C} - \{0\}$. Also let $\alpha_i > 0$, $1 \leq i \leq n$. If*

$$0 \leq (\rho - 1)\sum_{i=1}^{n} \alpha_i + 1 < 1, \tag{2.1}$$

then $F_n(z) \in V_k(\lambda, b)$ *with*

$$\lambda = (\rho - 1) \sum_{i=1}^{n} \alpha_i + 1. \tag{2.2}$$

Proof. From (1.3), we have

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right). \tag{2.3}$$

Then by multiplying (2.3) with $\frac{1}{b}$, we have

$$\begin{aligned} \frac{1}{b} \frac{zF_n''(z)}{F_n'(z)} &= \sum_{i=1}^{n} \alpha_i \frac{1}{b} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) \\ &= \sum_{i=1}^{n} \alpha_i \left[1 + \frac{1}{b} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) \right] - \sum_{i=1}^{n} \alpha_i \end{aligned}$$

or, equivalently

$$1 + \frac{1}{b} \frac{zF_n''(z)}{F_n'(z)} = 1 - \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \alpha_i \left[1 + \frac{1}{b} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) \right]. \tag{2.4}$$

Subtracting and adding ρ on the right hand side of (2.4), we have

$$\left[\left(1 + \frac{1}{b} \frac{zF_n''(z)}{F_n'(z)} \right) - \lambda \right] = \sum_{i=1}^{n} \alpha_i \left[\left(1 + \frac{1}{b} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) \right) - \rho \right], \tag{2.5}$$

where λ is given by (2.2). Taking real part of (2.5) and then simple computation gives

$$\int_0^{2\pi} \left| \operatorname{Re} \left[\left(1 + \frac{1}{b} \frac{zF_n''(z)}{F_n'(z)} \right) - \lambda \right] \right| d\theta \leq \sum_{i=1}^{n} \alpha_i \int_0^{2\pi} \left| \operatorname{Re} \left[\left(1 + \frac{1}{b} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) \right) - \rho \right] \right| d\theta. \tag{2.6}$$

Since $f_i(z) \in R_k(\rho, b)$ for $1 \leq i \leq n$, we have

$$\int_0^{2\pi} \left| \operatorname{Re} \left[\left(1 + \frac{1}{b} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) \right) - \rho \right] \right| d\theta \leq (1 - \rho) k\pi. \tag{2.7}$$

Using (2.7) in (2.6), we obtain

$$\int_0^{2\pi} \left| \mathrm{Re} \left[\left(1 + \frac{1}{b} \frac{zF_n''(z)}{F_n'(z)} \right) - \lambda \right] \right| d\theta \leq (1-\lambda)\, k\pi.$$

Hence $F_n(z) \in V_k(\lambda, b)$ with λ is given by (2.2).By setting $k = 2$ in Theorem 2.1, we obtain the following result proved in [3].

Corollary 2.2. *Let $f_i(z) \in S_b^*(\rho)$ for $1 \leq i \leq n$ with $0 \leq \rho < 1$, $b \in \mathbb{C} - \{0\}$. Also let $\alpha_i > 0$, $1 \leq i \leq n$. If*

$$0 \leq (\rho - 1) \sum_{i=1}^{n} \alpha_i + 1 < 1,$$

then $F_n(z) \in C_b(\lambda)$ with $\lambda = (\rho - 1) \sum_{i=1}^{n} \alpha_i + 1$.

Theorem 2.3. *Let $f_i(z) \in V_k(\rho, 1)$ for $1 \leq i \leq n$ with $0 \leq \rho < 1$. Also let $\alpha_i > 0$, $1 \leq i \leq n$. If*

$$0 \leq (\beta - 1) \sum_{i=1}^{n} \alpha_i + 1 < 1,$$

then $F_n(z) \in V_k(\lambda, 1)$ with $\lambda = (\beta - 1) \sum_{i=1}^{n} \alpha_i + 1$ and β is given by (1.5).

Proof. From (2.6) with $b = 1$, we have

$$\int_0^{2\pi} \left| \mathrm{Re} \left[\left(1 + \frac{zF_n''(z)}{F_n'(z)} \right) - \lambda \right] \right| d\theta \leq \sum_{i=1}^{n} \alpha_i \int_0^{2\pi} \left| \mathrm{Re} \left[\frac{zf_i'(z)}{f_i(z)} - \beta \right] \right| d\theta. \tag{2.8}$$

Since $f_i(z) \in V_k(\rho, 1)$ for $1 \leq i \leq n$, then by using Lemma 1.1, we have

$$\int_0^{2\pi} \left| \mathrm{Re} \left[\frac{zf_i'(z)}{f_i(z)} - \beta \right] \right| d\theta \leq (1-\beta)\, k\pi, \tag{2.9}$$

where β is given by (1.5) with $\alpha = \rho$. Using (2.9) in (2.8), we obtain

$$\int_0^{2\pi} \left| \mathrm{Re} \left[\left(1 + \frac{zF_n''(z)}{F_n'(z)} \right) - \lambda \right] \right| d\theta \leq (1-\lambda)\, k\pi.$$

Hence $F_n(z) \in V_k(\lambda, b)$ with $\lambda = (\beta - 1)\sum_{i=1}^{n} \alpha_i + 1$.

Set $n = 1$, $\alpha_1 = 1$, $\alpha_2 = \cdots = \alpha_n = 0$ in Theorem 2.3, we obtain.

Corollary 2.4. *Let $f(z) \in V_k(\rho)$. Then the Alexandar operator $F_0(z)$, defined in [1], belongs to the class $V_k(\beta)$, where β is given by (1.5).*

For $\rho = 0$ and $k = 2$ in Corollary 2.4, we have the well known result, that is,

$$f(z) \in C(0) \quad \text{implies} \quad F_0(z) \in C\left(\frac{1}{2}\right).$$

Theorem 2.5. *Let $f_i(z) \in V_k(\rho, b)$ for $1 \le i \le n$ with $0 \le \rho < 1$, $b \in \mathbb{C} - \{0\}$. Also let $\alpha_i > 0$, $1 \le i \le n$. If*

$$0 \le (\rho - 1)\sum_{i=1}^{n} \alpha_i + 1 < 1,$$

then $F_{\alpha_1 \ldots \alpha_n}(z) \in V_k(\mu, b)$ with $\mu = (\rho - 1)\sum_{i=1}^{n} \alpha_i + 1$.

Proof. From (1.4), we have

$$\frac{F''_{\alpha_1 \ldots \alpha_n}(z)}{F'_{\alpha_1 \ldots \alpha_n}(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{f''_i(z)}{f'_i(z)}\right).$$

Then by multiplying both sides with $\frac{z}{b}$, we have

$$\frac{1}{b}\frac{zF''_{\alpha_1 \ldots \alpha_n}(z)}{F'_{\alpha_1 \ldots \alpha_n}(z)} = \sum_{i=1}^{n} \alpha_i \frac{1}{b}\left(\frac{zf''_i(z)}{f'_i(z)}\right)$$

This relation is equivalent to

$$\left[\left(1 + \frac{1}{b}\frac{zF''_{\alpha_1 \ldots \alpha_n}(z)}{F'_{\alpha_1 \ldots \alpha_n}(z)}\right) - \mu\right] = \sum_{i=1}^{n} \alpha_i \left[\left(1 + \frac{1}{b}\frac{zf''_i(z)}{f'_i(z)}\right) - \rho\right], \tag{2.10}$$

where $\mu = (\rho - 1)\sum_{i=1}^{n} \alpha_i + 1$. Taking real part of (2.10) and then simple computation gives us

$$\int_0^{2\pi} \left|\mathrm{Re}\left[\left(1 + \frac{1}{b}\frac{zF''_{\alpha_1 \ldots \alpha_n}(z)}{F'_{\alpha_1 \ldots \alpha_n}(z)}\right) - \mu\right]\right| d\theta \le \sum_{i=1}^{n} \alpha_i \int_0^{2\pi} \left|\mathrm{Re}\left[\left(1 + \frac{1}{b}\frac{zf''_i(z)}{f'_i(z)}\right) - \rho\right]\right| d\theta. \tag{2.11}$$

Since $f_i(z) \in V_k(\rho, b)$ for $1 \le i \le n$, we have

$$\int_0^{2\pi} \left| \mathrm{Re}\left[\left(1 + \frac{1}{b} \frac{z f_i''(z)}{f_i'(z)} \right) - \rho \right] \right| d\theta \le (1-\rho)\, k\pi. \tag{2.12}$$

Using (2.12) in (2.11), we obtain

$$\int_0^{2\pi} \left| \mathrm{Re}\left[\left(1 + \frac{1}{b} \frac{z F''_{\alpha_1 \dots \alpha_n}(z)}{F'_{\alpha_1 \dots \alpha_n}(z)} \right) - \mu \right] \right| d\theta \le (1-\mu)\, k\pi.$$

Hence $F_{\alpha_1 \dots \alpha_n}(z) \in V_k(\mu, b)$ with $\mu = (\rho - 1) \sum_{i=1}^{n} \alpha_i + 1$.

By setting $k = 2$ in Theorem 2.5, we obtain the following result proved in [3].

Corollary 2.6. *Let $f_i(z) \in C_b(\rho)$ for $1 \le i \le n$ with $0 \le \rho < 1$, $b \in \mathbb{C} - \{0\}$. Also let $\alpha_i > 0$, $1 \le i \le n$. If*

$$0 \le (\rho - 1) \sum_{i=1}^{n} \alpha_i + 1 < 1,$$

then $F_{\alpha_1 \dots \alpha_n}(z) \in C_b(\mu)$ with $\mu = (\rho - 1) \sum_{i=1}^{n} \alpha_i + 1$.

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