WEAK INVERSE PROPERTY LOOPS AND SOME ISOTOPY-ISOMORPHY PROPERTIES

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Abstract. Two distinct isotopy-isomorphy conditions, different from those of J. M. Osborn and Wilson's condition, for a weak inverse property loop(WIPL) are shown. Only one of them characterizes isotopy-isomorphy in WIPLs while the other is just a sufficient condition for isotopy-isomorphy. Under the sufficient condition called the $\mathcal{T}$ condition, Artzy's result that isotopic cross inverse property loops(CIPLs) are isomorphic is proved for WIP loops.

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1.Introduction

Michael K. Kinyon [13] gave a talk on Osborn Loops and proposed the open problem : "Is every Osborn Loop universal?" which is obviously true for universal WIP loops and universal CIP loops. A popular isotopy-isomorphy condition in loops is the Wilson's identity([5]) and a loop obeying it is called a Wilson's loop by Goodaire and Robinson [6,5] and they proved that a loop is a Wilson loop if and only if it is a conjugacy closed loop(CC-loop) and a WIPL. Our aim in this work is to prove some isotopy-isomorphy conditions, different from those of J. M. Osborn [16] and Wilson [18](i.e the loop not necessarily a CC-loop), for a WIPL and see if the result of Artzy [2] that isotopic CIP loops are isomorphic is true for WIP loops or some specially related WIPLs(i.e special isotopes). But before these, we shall take few basic definitions and concepts in loop theory which are needed here.

Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations ; $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. Furthermore, if there exists a unique element $e \in L$ called the identity element such that for all $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop. For each $x \in L$, the elements $x^\rho, x^\lambda \in L$ such that $xx^\rho = e = x^\lambda x$ are called the right, left inverses of x respectively. L is called a weak inverse property loop (WIPL) if and only if it obeys the weak inverse property (WIP);

$$xy \cdot z = e \text{ implies } x \cdot yz = e \text{ for all } x, y, z \in L$$

while L is called a cross inverse property loop (CIPL) if and only if it obeys the cross inverse property (CIP);

$$xy \cdot x^\rho = y \text{ for all } x, y, \in L.$$

But if L is a quasigroup, then it is called a weak inverse property quasigroup (WIPQ) w.r.t. a permutation δ on L if and only if it obeys the identity

$$x \cdot (yx)\delta = y\delta \text{ for all } x, y, \in L.$$

According to [3], the WIP is a generalization of the CIP. The latter was introduced and studied by R. Artzy [1] and [2] while the former was introduced by J. M. Osborn [16] who also investigated the isotopy invariance of the WIP. Huthnance Jr. [7] did so as well and proved that the holomorph of a WIPL is a WIPL. A loop property is called universal(or at times a loop is said to be universal relative to a particular property) if the loop has the property and every loop isotope of such a loop possesses such a property. A universal WIPL is called an Osborn loop in Huthnance Jr. [7] but this is different from the Osborn loop of Kinyon [13] and Basarab. The Osborn loops of Kinyon and Basarab were named generalised Moufang loops or M-loops by Huthnance Jr. [7] where he investigated the structure of their holomorphs while Basarab [4] studied Osborn loops that are G-loops. Also, generalised Moufang loops or M-loops of Huthnance Jr. are different from those of Basarab. After Osborn's study of universal WIP loops, Huthnance Jr. still considered them in his thesis and did an elaborate study by comparing the similarities between properties of Osborn loops(universal WIPL) and generalised Moufang loops. He was able to draw conclusions that the latter class of loops is larger than the former class while in a WIPL the two are the same.

But in this present work, two distinct isotopy-isomorphy conditions, different from that of Osborn [16] and Wilson [18], for a weak inverse property loop(WIPL) are shown. Only one of them characterizes isotopy-isomorphy in WIPLs while the other is just a sufficient condition for isotopy-isomorphy. Under the sufficient condition called the $\mathcal{T}$ condition, Artzy [2] result that isotopic cross inverse property loops(CIPLs) are isomorphic is proved for WIP loops.

2.Preliminaries

Definition 1. *Let $(L,\cdot)$ and $(G,\circ)$ be two distinct loops. The triple $\alpha = (U,V,W) : (L,\cdot) \to (G,\circ)$ such that $U,V,W : L \to G$ are bijections is called a loop isotopism $\Leftrightarrow$ $xU \circ yV = (x\cdot y)W\ \forall\ x,y \in L$. Hence, L and G are said to be isotopic whence, G is an isotope of L.*

Definition 2. *Let L be a loop. A mapping $\alpha \in S(L)$(where $S(L)$ is the group of all bijections on L) which obeys the identity $x^\rho = [(x\alpha)^\rho]\alpha$ is called a weak right inverse permutation. Their set is represented by $S_\rho(L)$.*

Similarly, if α obeys the identity $x^\lambda = [(x\alpha)^\lambda]\alpha$ it is called a weak left inverse permutation. Their set is represented by $S_\lambda(L)$.

If α satisfies both, it is called a weak inverse permutation. Their set is represented by $S'(L)$.

It can be shown that $\alpha \in S(L)$ is a weak right inverse if and only if it is a weak left inverse permutation. So, $S'(L) = S_\rho(L) = S_\lambda(L)$.

Remark 1. *Every permutation of order 2 that preserves the right(left) inverse of each element in a loop is a weak right (left) inverse permutation.*

Example 1. *If L is an extra loop, the left and right inner mappings $L(x,y)$ and $R(x,y)\ \forall\ x,y \in L$ are automorphisms of orders 2 ([14]). Hence, they are weak inverse permutations by Remark 1.*

Throughout, we shall employ the use of the bijections; $J_\rho : x \mapsto x^\rho$, $J_\lambda : x \mapsto x^\lambda$, $L_x : y \mapsto xy$ and $R_x : y \mapsto yx$ for a loop and the bijections; $J'_\rho : x \mapsto x^{\rho'}$, $J'_\lambda : x \mapsto x^{\lambda'}$, $L'_x : y \mapsto xy$ and $R'_x : y \mapsto yx$ for its loop isotope. If the identity element of a loop is e then that of the isotope shall be denoted by e'.

Lemma 1. *In a loop, the set of weak inverse permutations that commute form an abelian group.*

Remark 2. *Applying Lemma 1 to extra loops and considering Example 1, it will be observed that in an extra loop L, the Boolean groups* $Inn_\lambda(L), Inn_\rho \leq S'(L)$*.* $Inn_\lambda(L)$ *and* $Inn_\rho(L)$ *are the left and right inner mapping groups respectively. They have been investigated in [15] and [14]. This deductions can't be drawn for CC-loops despite the fact that the left (right) inner mappings commute and are automorphisms. And this is as a result of the fact that the left(right) inner mappings are not of exponent 2.*

Definition 3. *($\mathcal{T}$-condition)*

Let $(G,\cdot)$ *and* $(H,\circ)$ *be two distinct loops that are isotopic under the triple* (A,B,C)*.* $(G,\cdot)$ *obeys the* $\mathcal{T}_1$ *condition if and only if* $A = B$*.* $(G,\cdot)$ *obeys the* $\mathcal{T}_2$ *condition if and only if* $J'_\rho = C^{-1}J_\rho B = A^{-1}J_\rho C$*.* $(G,\cdot)$ *obeys the* $\mathcal{T}_3$ *condition if and only if* $J'_\lambda = C^{-1}J_\lambda A = B^{-1}J_\lambda C$*. So,* $(G,\cdot)$ *obeys the* $\mathcal{T}$ *condition if and only if it obey* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *conditions or* $\mathcal{T}_1$ *and* $\mathcal{T}_3$ *conditions since* $\mathcal{T}_2 \equiv \mathcal{T}_3$*.*

It must here by be noted that the $\mathcal{T}$-conditions refer to a pair of isotopic loops at a time. This statement might be omitted at times. That is whenever we say a loop $(G,\cdot)$ has the $\mathcal{T}$-condition, then this is relative to some isotope $(H,\circ)$ of $(G,\cdot)$

Lemma 2. *([17]) Let L be a loop. The following are equivalent.*

1. *L is a WIPL*
2. $y(xy)^\rho = x^\rho \ \forall\ x,y \in L$.
3. $(xy)^\lambda x = y^\lambda \ \forall\ x,y \in L$.

Lemma 3. *Let L be a loop. The following are equivalent.*

1. *L is a WIPL*
2. $R_yJ_\rho L_y = J_\rho \ \forall\ y \in L$.
3. $L_xJ_\lambda R_x = J_\lambda \ \forall\ x \in L$.

3.Main Results

Theorem 1. *Let $(G,\cdot)$ and $(H,\circ)$ be two distinct loops that are isotopic under the triple (A,B,C).*

1. *If the pair of $(G,\cdot)$ and $(H,\circ)$ obey the $\mathcal{T}$ condition, then $(G,\cdot)$ is a WIPL if and only if $(H,\circ)$ is a WIPL.*
2. *If $(G,\cdot)$ and $(H,\circ)$ are WIPLs, then $J_\lambda R_x J_\rho B = CJ'_\lambda R'_{xA}J'_\rho$ and $J_\rho L_x J_\lambda A = CJ'_\rho L'_{xB}J'_\lambda$ for all $x\in G$.*

Proof

1. $(A,B,C)\ :\ G\to H$ is an isotopism $\Leftrightarrow xA\circ yB=(x\cdot y)C\Leftrightarrow yBL'_{xA}=yL_xC\Leftrightarrow BL'_{xA}=L_xC\Leftrightarrow L'_{xA}=B^{-1}L_xC\Leftrightarrow$

$$L_x=BL'_{xA}C^{-1}. \tag{1}$$

Also, $(A,B,C)\ :\ G\to H$ is an isotopism $\Leftrightarrow xAR'_{yB}=xR_yC\Leftrightarrow AR'_{yB}=R_yC\Leftrightarrow R'_{yB}=A^{-1}R_yC\Leftrightarrow$

$$R_y=AR'_{yB}C^{-1}. \tag{2}$$

Applying (1) and (2) to Lemma 3 separately, we have :

$$R_yJ_\rho L_y=J_\rho,\ L_xJ_\lambda R_x=J_\lambda\Rightarrow (AR'_{xB}C^{-1})J_\rho(BL'_{xA}C^{-1})=J_\rho,$$
$$(BL'_{xA}C^{-1})J_\lambda(AR'_{xB}C^{-1})=J_\lambda\Leftrightarrow AR'_{xB}(C^{-1}J_\rho B)L'_{xA}C^{-1}=J_\rho,$$
$$BL'_{xA}(C^{-1}J_\lambda A)R'_{xB}C^{-1}=J_\lambda\Leftrightarrow$$

$$R'_{xB}(C^{-1}J_\rho B)L'_{xA}=A^{-1}J_\rho C,\ L'_{xA}(C^{-1}J_\lambda A)R'_{xB}=B^{-1}J_\lambda C. \tag{3}$$

Let $J'_\rho=C^{-1}J_\rho B=A^{-1}J_\rho C$, $J'_\lambda=C^{-1}J_\lambda A=B^{-1}J_\lambda C$. Then, from (3) and by Lemma 3, H is a WIPL if $xB=xA$ and $J'_\rho=C^{-1}J_\rho B=A^{-1}J_\rho C$ or $xA=xB$ and $J'_\lambda=C^{-1}J_\lambda A=B^{-1}J_\lambda C\Leftrightarrow B=A$ and $J'_\rho=C^{-1}J_\rho B=A^{-1}J_\rho C$ or $A=B$ and $J'_\lambda=C^{-1}J_\lambda A=B^{-1}J_\lambda C\Leftrightarrow A=B$ and $J'_\rho=C^{-1}J_\rho B=A^{-1}J_\rho C$ or $J'_\lambda=C^{-1}J_\lambda A=B^{-1}J_\lambda C$. This completes the proof of the forward part. To prove the converse, carry out the same procedure, assuming the $\mathcal{T}$ condition and the fact that $(H,\circ)$ is a WIPL.

2. If $(H, \circ)$ is a WIPL, then

$$R'_y J'_\rho L'_y = J'_\rho,\ \forall\ y \in H \tag{4}$$

while since G is a WIPL,

$$R_x J_\rho L_x = J_\rho\ \forall\ x \in G. \tag{5}$$

The fact that G and H are isotopic implies that

$$L_x = BL'_{xA}C^{-1}\ \forall\ x \in G\ and \tag{6}$$

$$R_x = AR'_{xB}C^{-1}\ \forall\ x \in G. \tag{7}$$

From (4),

$$R'_y = J'_\rho L'^{-1}_y J'_\lambda\ \forall\ y \in H\ and \tag{8}$$

$$L'_y = J'_\lambda R'^{-1}_y J'_\rho\ \forall\ y \in H \tag{9}$$

while from (5),

$$R_x = J_\rho L^{-1}_x J_\lambda\ \forall\ x \in G\ and \tag{10}$$

$$L_x = J_\lambda R^{-1}_x J_\rho\ \forall\ x \in G. \tag{11}$$

So, using (9) and (11) in (6) we get

$$J_\lambda R_x J_\rho B = CJ'_\lambda R'_{xA} J'_\rho\ \forall\ x \in G \tag{12}$$

while using (8) and (10) in (7) we get

$$J_\rho L_x J_\lambda A = CJ'_\rho L'_{xB} J'_\lambda\ \forall\ x \in G. \tag{13}$$

Remark 3. *In Theorem 1, a loop is a universal WIPL under the $\mathcal{T}$ condition. But the converse of this is not true. This can be deduced from a counter example.*

Counter Example *Let $G = \{0, 1, 2, 3, 4, \}$. From the Table 1, $(G, \cdot)$ is a WIPL.*

·	0	1	2	3	4
0	0	1	2	3	4
1	1	3	0	4	2
2	2	0	4	1	3
3	3	4	1	2	0
4	4	2	3	0	1

Table 1: A commutative weak inverse property loop

Let

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 3 & 0 & 4 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 0 & 4 & 1 & 3 \end{pmatrix}$$

Then, (A, B, I) is an isotopism from $(G, \cdot)$ to itself. But $A \neq B$ so the $\mathcal{T}$ condition does not hold for $(G, \cdot)$.

We can go ahead to prove a generalization of the first part of Theorem 1 by establishing that:

Theorem 2. *If $(G, \cdot)$ and $(H, \circ)$ are two distinct quasigroups that are isotopic under the triple (A, B, C) and the pair of $(G, \cdot)$ and $(H, \circ)$ obey the $\mathcal{T}$ condition, then $(G, \cdot)$ is a WIPQ if and only if $(H, \circ)$ is a WIPQ.*

Proof

The proof is very similar to the proof of Theorem 1 by taking the advantage of the fact that we did not use the identity elements in the proof and by simply re-defining the $\mathcal{T}$ condition as follows:

Let $(G, \cdot)$ and $(H, \circ)$ be two distinct quasigroups with bijective mapping δ_1 and δ_2 respectively. Let them be isotopic under the triple (A, B, C). $(G, \cdot)$ obeys the $\mathcal{T}_1$ condition if and only if $A = B$. $(G, \cdot)$ obeys the $\mathcal{T}_2$ condition if and only if $\delta_2 = C^{-1}\delta_1 B = A^{-1}\delta_1 C$. So, $(G, \cdot)$ obeys the $\mathcal{T}$ condition if and only if it obey $\mathcal{T}_1$ and $\mathcal{T}_2$ conditions.

Example 2. *In Vasantha Kandasamy [19], the author gave a new method of construction of loops as follows.*

Let $L_n(m) = \{e, 1, 2, \cdots, n\}$ be a set where $n > 3$, n is odd and m is a positive integer such that $(m, n) = 1$ and $(m - 1, n) = 1$ with $m < n$. Define on $L_n(m)$ a binary operation $\bullet$ as follows:

- *$e \bullet i = i \bullet e = i$ for all $i \in L_n(m)$;*

•	e	1	2	3	4	5	6	7
e	e	1	2	3	4	5	6	7
1	1	e	4	7	3	6	2	5
2	2	6	e	5	1	4	7	3
3	3	4	7	e	6	2	5	1
4	4	2	5	1	e	7	3	6
5	5	7	3	6	2	e	1	4
6	6	5	1	4	7	3	e	2
7	7	3	6	2	5	1	4	e

Table 2: A weak inverse property loop

∘	e	1	2	3	4	5	6	7
e	7	e	2	3	6	1	4	5
1	1	7	5	4	2	e	6	3
2	3	6	7	2	1	4	5	e
3	4	1	e	7	5	6	3	2
4	6	2	1	5	7	3	e	4
5	5	4	6	e	3	7	2	1
6	2	3	4	1	e	5	7	6
7	e	5	3	6	4	2	1	7

Table 3: A weak inverse property quasigroup

- $i^2 = i \bullet i = e$ *for all* $i \in L_n(m)$*;*
- $i \bullet j = t$ *where* $t = (mj - (m-1)i)\ (mod\ n)$

for all $i, j \in L_n(m)$*;* $i \neq j$*,* $i \neq e$ *and* $j \neq e$*, then* $(L_n(m), \bullet)$ *is a loop. She further established that* $(L_n(m), \bullet)$ *is a WIPL if and only if* $(m^2 - m + 1) \equiv 0 \bmod n$*. Thus, the loop* $(L_7(3), \bullet)$ *where* $L_7(3) = \{e, 1, 2, 3, 4, 5, 6, 7\}$ *is a WIPL. The multiplication table is given by Table 2.*

Consider the quasigroup $(L_7(3), \circ)$ *whose multiplication table is given by Table 3.*

·	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	0	4	5	2	3	7	6	10	11	8	9
2	2	5	0	4	3	1	8	11	6	10	9	7
3	3	4	5	0	1	2	9	10	11	6	7	8
4	4	3	1	2	5	0	10	9	7	8	11	6
5	5	2	3	1	0	4	11	8	9	7	6	10
6	6	8	7	11	9	10	0	2	1	4	5	3
7	7	10	6	9	11	8	1	4	0	2	3	5
8	8	6	9	10	7	11	2	0	5	3	1	4
9	9	11	8	7	10	6	3	5	4	1	2	0
10	10	7	11	8	6	9	4	1	3	5	0	2
11	11	9	10	6	8	7	5	3	2	0	4	1

Table 4: A weak inverse property loop

$(L_7(3), \circ)$ *is a WIPQ w.r.t. the permutation* $\delta_2 = \begin{pmatrix} e & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 2 & 1 & 5 & 7 & 3 & e & 4 \end{pmatrix}$.

Let

$$A = B = \begin{pmatrix} e & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 7 & e & 2 & 6 & 1 & 3 & 5 \end{pmatrix} \text{ and } C = \begin{pmatrix} e & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 4 & 6 & 1 & e & 2 & 5 & 3 \end{pmatrix}.$$

Then, the triple $\alpha = (A, B, C)$ *is an isotopism from* $(L_7(3), \bullet)$ *onto* $(L_7(3), \circ)$ *under the* $\mathcal{T}$ *condition.*

Example 3. *Consider the WIPL* $(G, \cdot)$, $G = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ *whose multiplication table is given by Table 4.*

Consider the quasigroup $(G, \circ)$ *whose multiplication table is given by Table 5.*

$(G, \circ)$ *is a WIPQ w.r.t. the permutation*

$$\delta_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 0 & 1 & 2 & 3 & 5 & 4 & 6 & 8 & 7 & 11 & 10 & 9 \end{pmatrix}.$$

$\circ$	0	1	2	3	4	5	6	7	8	9	10	11
0	8	10	7	4	11	9	3	0	5	1	2	6
1	11	1	8	3	6	5	4	2	10	9	0	7
2	10	8	2	7	4	6	5	1	0	3	11	9
3	6	3	11	1	7	0	10	9	2	4	5	8
4	7	4	6	10	8	2	1	5	3	0	9	11
5	3	6	5	11	1	8	2	4	9	10	7	0
6	9	5	4	0	2	1	8	6	7	11	3	10
7	0	2	1	9	5	4	6	8	11	7	10	3
8	2	9	3	5	0	10	7	11	6	8	1	4
9	5	0	10	2	9	3	11	7	8	6	4	1
10	1	7	0	6	3	11	9	10	4	2	8	5
11	4	11	9	8	10	7	0	3	1	5	6	2

Table 5: A weak inverse property quasigroup

$$Let\ A = B = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 6 & 5 & 4 & 2 & 1 & 0 & 3 & 11 & 9 & 10 & 8 \end{pmatrix}$$

$$and\ C = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 8 & 6 & 4 & 5 & 1 & 2 & 0 & 9 & 3 & 7 & 10 & 11 \end{pmatrix}.$$

Then, the triple $\alpha = (A, B, C)$ is an isotopism from $(G, \cdot)$ onto $(G, \circ)$ under the $\mathcal{T}$ condition.

Remark 4. *Example 2 and Example 3 are examples that illustrate Theorem 2. In these examples, it will be observed that the co-domains $(L_7(3), \circ)$ and $(G, \cdot)$ are WIPQs and not WIPLs. These are okay by Theorem 2. But, it must be mentioned that the choice of the mappings $A = B$ and δ_2 determines if the co-domain is a quasigroup or loop. The examples were gotten by manual construction. The construction to get a co-domain that is not just a WIPQ but a WIPL(to illustrate the result in the the first part of Theorem 1) lies a lot on a systematic choice of $A = B$ and δ_2 which we think will be easier to do with the use of some non-associative soft ware packages(e.g. LOOPS). Nevertheless, there is no harm in getting a WIPQ as the co-domain since WIPQs are good for cryptography. More WIPQs can thus be constructed by applying Theorem 2 to some non-commutative non-Moufang WIPLs constructed in Johnson and Sharma [8].*

Given a loop (L, θ), a loop (L, θ^*) is called the opposite parastrophe(one of the five parastrophes) of (L, θ) if and only if $x\theta y = y\theta^* x$ for all $x, y \in L$. It is easy to show that (L, θ) is a WIPL(or CIPL or IPL) if and only if (L, θ^*) is a WIPL(or CIPL or IPL). Thus, (L, θ) and (L, θ^*) are anti-isotopic. The anti-isotopism could be said to have the $\mathcal{T}$ condition when (L, θ) is an IPL(inverse property loop) but will only have the $\mathcal{T}$ condition when (L, θ) is a WIPL(or CIPL) provided the left and right inverse permutation mappings in (L, θ) coincide.

Theorem 3. *Let $(G, \cdot)$ be a WIPL with identity element e and $(H, \circ)$ be an arbitrary loop isotope of $(G, \cdot)$ with identity element e' under the triple $\alpha = (A, B, C)$. If $(H, \circ)$ is a WIPL then*

1. *$(G, \cdot) \overset{C}{\cong} (H, \circ) \Leftrightarrow (J_\rho L_b J_\lambda, J_\lambda R_a J_\rho, I) \in AUT(G, \cdot)$ where $a = e'A^{-1}, b = e'B^{-1}$. Hence, $(J_\lambda R_a J_\rho, J_\rho L_b J_\lambda, R_a L_b) \in AUT(G, \cdot)$. Furthermore, if $(G, \cdot)$ is a loop of exponent 2 then, $(R_a, L_b, R_a L_b) \in AUT(G, \cdot)$.*

2. *$(G, \cdot) \overset{C}{\cong} (H, \circ) \Leftrightarrow (J'_\rho L'_{b'} J'_\lambda, J'_\lambda R'_{a'} J'_\rho, I) \in AUT(H, \circ)$ where $a' = eA, b' = eB$. Hence, $(J'_\lambda R'_{a'} J'_\rho, J'_\rho L'_{b'} J'_\lambda, R'_{a'} L'_{b'}) \in AUT(H, \circ)$. Furthermore, if $(H, \circ)$ is a loop of exponent 2 then, $(R'_{a'}, L'_{b'}, R'_{a'} L'_{b'}) \in AUT(H, \cdot)$.*

3. *$(G, \cdot) \overset{C}{\cong} (H, \circ) \Leftrightarrow (L_b, R_a, I) \in AUT(G, \cdot)$, $a = e'A^{-1}, b = e'B^{-1}$ provided $(x \cdot y)^\rho = x^\rho \cdot y^\lambda$ or $(x \cdot y)^\lambda = x^\lambda \cdot y^\rho$ $\forall$ $x, y \in G$. Hence, $(G, \cdot)$ and $(H, \circ)$ are isomorphic CIP loops while $R_a L_b = I$, $ba = e$.*

4. *$(G, \cdot) \overset{C}{\cong} (H, \circ) \Leftrightarrow (L'_{b'}, R'_{a'}, I) \in AUT(H, \circ)$, $a' = eA, b' = eB$ provided $(x \circ y)^{\rho'} = x^{\rho'} \circ y^{\lambda'}$ or $(x \circ y)^{\lambda'} = x^{\lambda'} \circ y^{\rho'}$ $\forall$ $x, y \in H$. Hence, $(G, \cdot)$ and $(H, \circ)$ are isomorphic CIP loops while $R'_{a'} L'_{b'} = I$, $b'a' = e'$.*

Proof
Consider the second part of Theorem 1.

1. Let $y = xA$ in (12) and replace y by e'. Then $J_\lambda R_{e'A^{-1}} J_\rho B = C \Rightarrow C = J_\lambda R_a J_\rho B \Rightarrow B = J_\lambda R_a^{-1} J_\rho C$. Let $y = xB$ in (13) and replace y by e'. Then $J_\rho L_{e'B^{-1}} J_\lambda A = C \Rightarrow C = J_\rho L_b J_\lambda A \Rightarrow A = J_\rho L_b^{-1} J_\lambda C$. So, $\alpha = (A, B, C) = (J_\rho L_b^{-1} J_\lambda C, J_\lambda R_a^{-1} J_\rho C, C) = (J_\rho L_b^{-1} J_\lambda, J_\lambda R_a^{-1} J_\rho, I)(C, C, C)$. Thus, $(J_\rho L_b J_\lambda, J_\lambda R_a J_\rho, I) \in AUT(G, \cdot) \Leftrightarrow (G, \cdot) \overset{C}{\cong} (H, \circ)$.

Using the results on autotopisms of WIP loops in [Lemma 1,[16]],

$$(J_\lambda R_a J_\rho, I, L_b), (I, J_\rho L_b J_\lambda, R_a) \Rightarrow (J_\lambda R_a J_\rho, J_\rho L_b J_\lambda, R_a L_b) \in AUT(G, \cdot).$$

The further conclusion follows by breaking this.

2. This is similar to (1.) above but we only need to replace x by e in (12) and (13).

3. This is achieved by simply breaking the autotopism in (1.) and using the fact that a WIPL with the AIP is a CIPL.

4. Do what was done in (3.) to (2.).

Corollary 1. *Let $(G, \cdot)$ and $(H, \circ)$ be two distinct loops that are isotopic under the triple (A, B, C). If G is a WIPL with the $\mathcal{T}$ condition, then H is a WIPL :*

1. *there exists $\alpha, \beta \in S'(G)$ i.e α and β are weak inverse permutations and*

2. *$J'_\rho = J'_\lambda \Leftrightarrow J_\rho = J_\lambda$.*

Proof
By Theorem 1, $A = B$ and $J'_\rho = C^{-1} J_\rho B = A^{-1} J_\rho C$ or $J'_\lambda = C^{-1} J_\lambda A = B^{-1} J_\lambda C$.

1. $C^{-1} J_\rho B = A^{-1} J_\rho C \Leftrightarrow J_\rho B = CA^{-1} J_\rho C \Leftrightarrow J_\rho = CA^{-1} J_\rho C B^{-1} = CA^{-1} J_\rho C A^{-1} = \alpha J_\rho \alpha$ where $\alpha = CA^{-1} \in S(G, \cdot)$.

2. $C^{-1} J_\lambda A = B^{-1} J_\lambda C \Leftrightarrow J_\lambda A = CB^{-1} J_\lambda C \Leftrightarrow J_\lambda = CB^{-1} J_\lambda C A^{-1} = CB^{-1} J_\lambda C B^{-1} = \beta J_\lambda \beta$ where $\beta = CB^{-1} \in S(G, \cdot)$.

3. $J'_\rho = C^{-1} J_\rho B$, $J'_\lambda = C^{-1} J_\lambda A$. $J'_\rho = J'_\lambda \Leftrightarrow C^{-1} J_\rho B = C^{-1} J_\lambda A = C^{-1} J_\lambda B \Leftrightarrow J_\lambda = J_\rho$.

Lemma 4. *Let $(G, \cdot)$ be a WIPL with the $\mathcal{T}$ condition and isotopic to another loop $(H, \circ)$. $(H, \circ)$ is a WIPL and G has a weak inverse permutation.*

Proof
From the proof of Corollary 1, $\alpha = \beta$, hence the conclusion.

Theorem 4. *With the $\mathcal{T}$ condition, isotopic WIP loops are isomorphic.*

Proof
From Lemma 4, $\alpha = I$ is a weak inverse permutation. In the proof of Corollary 1, $\alpha = CA^{-1} = I \Rightarrow A = C$. Already, $A = B$, hence $(G, \cdot) \cong (H, \circ)$.

Remark 5. *Theorem 3 and Theorem 4 describes isotopic WIP loops that are isomorphic by*

1. *an autotopism in either the domain loop or the co-domain loop and*
2. *the $\mathcal{T}$ condition(for a special case).*

These two conditions are completely different from that shown in [Lemma 2,[16]] and [Theorem 4,[18]]. Furthermore, it can be concluded from Theorem 3 that isotopic CIP loops are not the only isotopic WIP loops that are isomorphic as earlier shown in [Theorem 1,[2]]. In fact, isotopic CIP loops need not satisfy Theorem 4(i.e the $\mathcal{T}$ condition) to be isomorphic.

4.Conclusion and Future Study

Karklinüsh and Karkliň [9] introduced m-inverse loops i.e loops that obey any of the equivalent conditions

$$(xy)J_\rho^m \cdot xJ_\rho^{m+1} = yJ_\rho^m \qquad \text{and} \qquad xJ_\lambda^{m+1} \cdot (yx)J_\lambda^m = yJ_\lambda^m.$$

They are generalizations of WIPLs and CIPLs, which corresponds to $m = -1$ and $m = 0$ respectively. After the study of m-loops by Keedwell and Shcherbacov [10], they have also generalized them to quasigroups called (r, s, t)-inverse quasigroups in [11] and [12]. It will be interesting to study the universality of m-inverse loops and (r, s, t)-inverse quasigroups. These will generalize the works of J. M. Osborn and R. Artzy on universal WIPLs and CIPLs respectively.

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