BOUNDEDNESS FOR MULTILINEAR COMMUTATOR OF INTEGRAL OPERATOR ON HARDY AND HERZ-HARDY SPACES

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Abstract. In this paper,the $(H^p_{\vec{b}}, L^p)$ and $(H\dot{K}^{\alpha,p}_{q,\vec{b}}, \dot{K}^{\alpha,p}_q)$ type boundedness for the multilinear commutator associated with some integral operator are obtained.

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1. Introduction

Let $b \in BMO(R^n)$, and T be the Calderón-Zygmund operator. The commutator $[b, T]$ generated by b and T is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

A classical result of Coifman, Rochberg and Weiss (see [3]) proved that the commutator $[b, T]$ is bounded on $L^p(R^n)$ $(1 < p < \infty)$. However, it was observed that the $[b, T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$. But if $H^p(R^n)$ is replaced by a suitable atomic space $H^P_{\vec{b}}(R^n)$ and $H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$, then $[b, T]$ maps continuously $H^P_{\vec{b}}(R^n)$ into $L^p(R^n)$ and $H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$ into $\dot{K}^{\alpha,p}_q$. In addition we easily known that $H^p_{\vec{b}}(R^n) \subset H^p(R^n)$, $\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n) \subset H\dot{K}^{\alpha,p}_q(R^n)$. The main purpose of this paper is to consider the continuity of the multilinear commutators related to the Littlewood-Paley operators and $BMO(R^n)$ functions on certain Hardy and Herz-Hardy spaces. Let us first introduce some definitions (see [1][4-16][18][19]).

Given a positive integer m and $1 \leq j \leq m$, we denote by C^m_j the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C^m_j$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C^m_j$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

Definition 1. *Let b_i $(i = 1, \cdots, m)$ be a locally integrable function and $0 < p \leq 1$. A bounded measurable function a on R^n is said a $(p, \vec{b})$ atom, if*

(1) $suppa \subset B = B(x_0, r)$

(2) $||a||_{L^\infty} \leq |B|^{-1/p}$

(3) $\int_B a(y)dy = \int_B a(y) \prod_{l\in\sigma} b_l(y)dy = 0$ *for any* $\sigma \in C_j^m$, $1 \leq j \leq m$.

A temperate distribution f is said to belong to $H^p_{\vec{b}}(R^n)$, if, in the Schwartz distribution sense, it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x).$$

where $a_j's$ are $(p, \vec{b})$ atoms, $\lambda \in C$ and $\sum_{j=1}^{\infty} |\lambda|^p < \infty$. Moreover, $||f||_{H^p_{\vec{b}}} \approx (\sum_{j=1}^{\infty} |\lambda_j|^p)^{1/p}$.

Definition 2.*Let $0 < p, q < \infty$, $\alpha \in R$. For $k \in Z$, set $B_k = \{x \in R^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and χ_0 the characteristic function of B_0.*

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n) = \left\{ f \in L^q_{loc}(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,p}} < \infty \right\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_{B_k}||^p_{L^q} \right]^{1/p}.$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \left\{ f \in L^q_{loc}(R^n) : ||f||_{K_q^{\alpha,p}} < \infty \right\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||^p_{L^q} + ||f\chi_{B_0}||^p_{L^q} \right]^{1/p}.$$

Definition 3.*Let $\alpha \in R^n$, $1 < q < \infty$, $\alpha \geq n(1 - \frac{1}{q})$, $b_i \in BMO(R^n)$, $1 \leq i \leq m$. A function $a(x)$ is called a central $(\alpha, q, \vec{b})$ -atom (or a central $(\alpha, q, \vec{b})$-atom of restrict type), if*

(1) $suppa \in B = B(x_0, r)$*(or for some $r \geq 1$),*

(2) $||a||_{L^q} \leq |B|^{-\alpha/n}$

(3) $\int_B a(x)x^\beta dx = \int_B a(x)x^\beta \prod_{i\in\sigma} b_i(x)dx = 0$ *for any* $\sigma \in C_j^m$, $1 \leq j \leq m$.

A temperate distribution f is said to belong to $H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$(or $HK^{\alpha,p}_{q,\vec{b}}(R^n)$), if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$), in the $S'(R^n)$ sense, where

a_j is a central $(\alpha, q, \vec{b})$-atom(or a central $(\alpha, q, \vec{b})$-atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{-\infty}^{\infty} |\lambda_j|^p < \infty$(or $\sum_{j=0}^{\infty} |\lambda_j| < \infty$). Moreover,

$$||f||_{H\dot{K}^{\alpha,p}_{q,\vec{b}}}(\text{ or } ||f||_{HK^{\alpha,p}_{q,\vec{b}}}) = \inf(\sum_j |\lambda_j|^p)^{1/p},$$

where the infimum are taken over all the decompositions of f as above.

Definition 4. *Suppose b_j $(j = 1, \cdots, m)$ are the fixed locally integrable functions on R^n. Let $F_t(x, y)$ be the function defined on $R^n \times R^n \times [0, +\infty)$. Set*

$$S_t(f)(x) = \int_{R^n} F_t(x, y) f(y) dy$$

and

$$S_t^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^{m} (b_j(x) - b_j(y)) F_t(x, y) f(y) dy$$

for every bounded and compactly supported function f. Let H be the Banach space $H = \{h : ||h|| < \infty\}$ such that, for each fixed $x \in R^n$, $S_t(f)(x)$ and $S_t^{\vec{b}}(f)(x)$ may be viewed as the mappings from $[0, +\infty)$ to H. The multilinear commutator related to S_t is defined by

$$T_\delta^{\vec{b}}(f)(x) = ||S_t^{\vec{b}}(f)(x)||,$$

where F_t satisfies: for fixed $\varepsilon > 0$ and $0 < \delta < n$,

$$||F_t(x, y)|| \le C|x - y|^{-n+\delta}$$

and

$$||F_t(x, y) - F_t(x, z)|| + ||F_t(y, x) - F_t(z, x)|| \le C|y - z|^{\epsilon}|x - z|^{-n-\varepsilon+\delta}$$

if $2|y - z| \le |x - z|$. We also define $T_\delta(f)(x) = ||S_t(f)(x)||$.

2. Theorems and Proofs

Lemma.(see [18]) *Let $1 < r < \infty$, $b_j \in BMO(R^n)$ for $j = 1, \cdots, k$ and $k \in N$. Then, we have*

$$\frac{1}{|Q|} \int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q| dy \le C \prod_{j=1}^{k} ||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|} \int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q|^r dy \right)^{1/r} \le C \prod_{j=1}^{k} ||b_j||_{BMO}.$$

Theorem 1. *Let $b_i \in BMO(R^n)$, $1 \le i \le m$, $\vec{b} = (b_1, \cdots, b_m)$, $0 < \delta < n$, $n/(n+\varepsilon-\delta) < q \le 1$, $1/q = 1/p - \delta/n$. Suppose that $T_\delta^{\vec{b}}$ is the multilinear commutator as in Definition 4 such that T is bounded from $L^s(R^n)$ to $L^r(R^n)$ for any $1 < s < n/\delta$ and $1/r = 1/s - \delta/n$. Then $T_\delta^{\vec{b}}$ is bounded from $H_{\vec{b}}^p(R^n)$ to $L^q(R^n)$.*

Proof. It suffices to show that there exist a constant $C > 0$, such that for every $(p, \vec{b})$ atom a,

$$||T_\delta^{\vec{b}}(a)||_{L^q} \le C.$$

Let a be a $(p, \vec{b})$ atom supported on a ball $B = B(x_0, l)$. We write

$$\int_{R^n} |T_\delta^{\vec{b}}(a)(x)|^q dx = \int_{|x-x_0|\le 2l} |T_\delta^{\vec{b}}(a)(x)|^q dx + \int_{|x-x_0|>2l} |T_\delta^{\vec{b}}(a)(x)|^q dx = I + II.$$

For I, taking $r, s > 1$ with $q < s < n/\delta$ and $1/r = 1/s - \delta/n$, by Hölder's inequality and the (L^s, L^r)- boundedness of $T_\delta^{\vec{b}}$, we see that

$$\begin{aligned} I &\le \left(\int_{|x-x_0|\le 2l} |T_\delta^{\vec{b}}(a)(x)|^r dx\right)^{q/r} \cdot |B(x_0, 2l)|^{1-q/r} \\ &\le C||T_\delta^{\vec{b}}(a)(x)||_{L^s}^q \cdot |B(x_0, 2l)|^{1-q/r} \\ &\le C||a||_{L^s}^q |B|^{1-q/r} \\ &\le C|B|^{-q/p+q/s+1-q/r} \\ &\le C. \end{aligned}$$

For II, denoting $\lambda = (\lambda_1, \cdots, \lambda_m)$ with $\lambda_i = (b_i)_B$, $1 \le i \le m$, where $(b_i)_B = \frac{1}{|B(x_0,l)|}\int_{B(x_0,l)} b_i(x)dx$, by Hölder's inequality and the vanishing moment of a, we get

$$\begin{aligned} II &= \sum_{k=1}^\infty \int_{2^{k+1}B\setminus 2^k B} |T_\delta^{\vec{b}}(a)(x)|^q dx \\ &\le C\sum_{k=1}^\infty |2^{k+1}B|^{1-q} \left(\int_{2^{k+1}B\setminus 2^k B} |T_\delta^{\vec{b}}(a)(x)| dx\right)^q \\ &\le C\sum_{k=1}^\infty |2^{k+1}B|^{1-q} \\ &\quad \times \left[\int_{2^{k+1}B\setminus 2^k B} || \int_B \prod_{j=1}^m (b_j(x) - b_j(y)) F_t(x,y) a(y) dy || dx\right]^q \\ &\le C\sum_{k=1}^\infty |2^{k+1}B|^{1-q} \end{aligned}$$

$$\times\left[\int_{2^{k+1}B\backslash 2^kB}\int_B ||F_t(x,y)-F_t(x,0)||\prod_{j=1}^m|(b_j(x)-b_j(y))||a(y)|dydx\right]^q;$$

noting that $y\in B,\ x\in 2^{k+1}B\backslash 2^kB$, then

$$\begin{aligned}
&\int_B ||F_t(x,y)-F_t(x,0)||\prod_{j=1}^m|(b_j(x)-b_j(y))||a(y)|dy\\
\le\ & C\int_B\prod_{j=1}^m|(b_j(x)-b_j(y))|||F_t(x,y)-F_t(x,0)|||a(y)|dy\\
\le\ & C\int_B\prod_{j=1}^m|(b_j(x)-b_j(y))|\frac{|y|^\varepsilon}{|x|^{n+\varepsilon-\delta}}|a(y)|dy,
\end{aligned}$$

thus

$$\begin{aligned}
II\ \le\ & C\sum_{k=1}^\infty|2^{k+1}B|^{1-q}\left[\int_{2^{k+1}B\backslash 2^kB}|x|^{-(n+\varepsilon-\delta)}\left(\int_B\prod_{j=1}^m|b_j(x)-b_j(y)||y|^\varepsilon|a(y)|dy\right)dx\right]^q\\
\le\ & C\sum_{k=1}^\infty|2^{k+1}B|^{1-q}\\
&\times\left[\sum_{j=0}^m\sum_{\sigma\in C_j^m}\int_{2^{k+1}B\backslash 2^kB}|x|^{-(n+\varepsilon-\delta)}|(\vec{b}(x)-\lambda)_\sigma|dx\int_B|(\vec{b}(y)-\lambda)_{\sigma^c}||y|^\varepsilon|a(y)|dy\right]^q\\
\le\ & C\sum_{j=0}^m\sum_{\sigma\in C_j^m}\left(\int_B|(\vec{b}(y)-\lambda)_{\sigma^c}||y|^\varepsilon|a(y)|dy\right)^q\\
&\times\sum_{k=1}^\infty|2^{k+1}B|^{1-q}\left[\int_{2^{k+1}B\backslash 2^kB}|x|^{-(n+\varepsilon-\delta)}|(\vec{b}(x)-\lambda)_\sigma|dx\right]^q\\
\le\ & C\sum_{j=0}^m\sum_{\sigma\in C_j^m}||\vec{b}_{\sigma^c}||_{BMO}^q\cdot||\vec{b}_\sigma||_{BMO}^q\sum_{k=1}^\infty|2^{k+1}B|^{1-q(n+\varepsilon-\delta)/n}k^q|B|^{(1+\varepsilon/n-1/p)q}\\
\le\ & C||\vec{b}||_{BMO}^q\sum_{k=1}^\infty k^q\cdot 2^{-knq(1+\varepsilon/n-\delta/n-1/q)}\\
\le\ & C||\vec{b}||_{BMO}^q.
\end{aligned}$$

This finish the proof of Theorem 1.

Theorem 2. *Let* $0<p<\infty,\ 0<\delta<n,\ 1<q_1,q_2<\infty,\ 1/q_1-1/q_2=\delta/n,$ $n(1-1/q_1)+\delta\le\alpha<n(1-1/q_1)+\varepsilon+\delta$ *and* $b_i\in BMO(R^n), 1\le i\le m,\ \vec{b}=$

$(b_1,\cdots,b_m)$. *Suppose that* $T_\delta^{\vec b}$ *is the multilinear commutator as in Definition 4 such that* T *is bounded from* $L^s(R^n)$ *to* $L^r(R^n)$ *for any* $1<s<n/\delta$ *and* $1/r=1/s-\delta/n$. *Then* $T_\delta^{\vec b}$ *is bounded from* $H\dot K^{\alpha,p}_{q_1,\vec b}(R^n)$ *to* $\dot K^{\alpha,p}_{q_2}(R^n)$.

Proof. Let $f\in H\dot K^{\alpha,p}_{q_1,\vec b}(R^n)$ and $f(x)=\sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 3, we write

$$\begin{aligned}
&||T_\delta^{\vec b}(f)(x)||_{\dot K^{\alpha,p}_{q_2}}\\
\le\ & C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}(\sum_{j=-\infty}^{\infty}|\lambda_j|||T_\delta^{\vec b}(a_j)\chi_k||_{L^{q_2}})^p\right]^{1/p}\\
\le\ & C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}(\sum_{j=-\infty}^{k-3}|\lambda_j|||T_\delta^{\vec b}(a_j)\chi_k||_{L^{q_2}})^p\right]^{1/p}\\
&+C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}(\sum_{j=k-2}^{\infty}|\lambda_j|||T_\delta^{\vec b}(a_j)\chi_k||_{L^{q_2}})^p\right]^{1/p}\\
=\ & I+II.
\end{aligned}$$

For II, noting that supp$a_j\subseteq B(0,2^j)$, $||a_j||_{L^{q_1}}\le|B(0,2^j)|^{-\alpha/n}$, by the boundedness of $T_\delta^{\vec b}$ on $(L^{q_1}(R^n),L^{q_2}(R^n))$ and the Hölder's inequality, we get

$$\begin{aligned}
II\ &=\ C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}(\sum_{j=k-2}^{\infty}|\lambda_j|||T_\delta^{\vec b}(a_j)\chi_k||_{L^{q_2}})^p\right]^{1/p}\\
&\le\ C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||a_j||_{L^{q_1}}\right)^p\right]^{1/p}\\
&\le\ C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|\cdot 2^{-j\alpha}\right)^p\right]^{1/p}\\
&\le\ C\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p}\right]^{1/p}, & 0<p\le 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p/2})(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p'/2})^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}\\
&\le\ C\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}\\
&\le\ C||f||_{H\dot K^{\alpha,p}_{q_1,\vec b}}.
\end{aligned}$$

For I, when m=1, let $C_k = B_k \backslash B_{k-1}$, $\chi_k = \chi_{C_k}$, $b_j^i = |B_j|^{-1}\int_{B_j} b_i(x)dx$, $1 \le i \le m, \vec{b'} = (b_j^1, \cdots, b_j^m)$. We have

$$\begin{aligned} T_\delta^{b_1}(a_j)(x) &\le \int_{B_j} ||F_t(x,y) - F_t(x,0)|||b_1(x) - b_1(y)||a_j(y)|dy \\ &\le \int_{B_j} |a_j(y)||b_1(x) - b_1(y)| \frac{|y|^\varepsilon}{|x|^{n+\varepsilon-\delta}} dy \\ &\le C|x|^{-(n+\varepsilon-\delta)} \int_{B_j} |y|^\varepsilon |a_j(y)||b_1(x) - b_j^1|dy \\ &\quad + C|x|^{-(n+\varepsilon-\delta)} \int_{B_j} |y|^\varepsilon |a_j(y)||b_1(y) - b_j^1|dy \\ &\le C|x|^{-(n+\varepsilon-\delta)} \left(|b_1(x) - b_j^1| 2^{j(\varepsilon+n(1-1/q_1)-\alpha)} + 2^{j(\varepsilon+n(1-1/q_1)-\alpha)} ||b_1||_{BMO}\right); \end{aligned}$$

Then

$$\begin{aligned} & ||T_\delta^{b_1}(a_j)\chi_k||_{L_{q_2}} \\ \le\ & C 2^{j(\varepsilon+n(1-1/q_1)-\alpha)} [\left(\int_{B_k} |x|^{-q_2(n+\varepsilon-\delta)} |b_1(x) - b_j^1|^{q_2} dx\right)^{1/q_2} \\ & + \left(\int_{B_k} |x|^{-q_2(n+\varepsilon-\delta)} dx\right)^{1/q_2} ||b_1||_{BMO}] \\ \le\ & C 2^{j(\varepsilon+n(1-1/q_1)-\alpha)} \left[2^{-k(n+\varepsilon-\delta)} \cdot |B_k|^{1/q_2} ||b_1||_{BMO} + 2^{-k(n+\varepsilon-\delta)} \cdot |B_k|^{1/q_2} ||b_1||_{BMO}\right] \\ \le\ & C ||b_1||_{BMO} 2^{j(\varepsilon+n(1-1/q_1)-\alpha)} \cdot 2^{-k(\varepsilon+n(1-1/q_1))}, \end{aligned}$$

thus

$$\begin{aligned} I &= C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| ||T_\delta^{b_1}(a_j)\chi_k||_{L^{q_2}}\right)^p\right]^{1/p} \\ &\le C||b_1||_{BMO} \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| 2^{j(\varepsilon+n(1-1/q_1)-\alpha)-k(\varepsilon+n(1-1/q_1))}\right)^p\right]^{1/p} \\ &\le C||b_1||_{BMO} \begin{cases} \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{j=-\infty}^{k-3} |\lambda_j|^p 2^{p[j(\varepsilon+n(1-1/q_1)-\alpha)-k(\varepsilon+n(1-1/q_1))]}\right]^{1/p}, & 0 < p \le 1 \\ \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j|^p 2^{[j(\varepsilon+n(1-1/q_1)-\alpha)-k(\varepsilon+n(1-1/q_1))]p/2}\right)\right. & \\ \quad \left. \times \left(\sum_{j=-\infty}^{k-3} 2^{[j(\varepsilon+n(1-1/q_1)-\alpha)-k(\varepsilon+n(1-1/q_1))]p'/2}\right)^{p/p'}\right]^{1/p}, & 1 < p < \infty \end{cases} \end{aligned}$$

$$\leq C||b_1||_{BMO}\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{p(j-k)(\varepsilon+n(1-1/q_1)-\alpha)}\right]^{1/p}, & 0<p\leq 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=j+3}^{\infty}2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)p/2}\right)\right. & \\ \quad\left.\times\left(\sum_{k=j+3}^{\infty}2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)p'/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}$$

$$\leq C||b_1||_{BMO}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}$$

$$\leq C||f||_{H\dot{K}^{\alpha,p}_{q_1,\vec{b}}}.$$

When $m>1$, similar to the proof of $T_\delta^{b_1}(a_j)(x)$, we have

$$\begin{aligned}T_\delta^{\vec{b}}(a_j)(x) &\leq C\int_{B_j}\prod_{i=1}^m|b_i(x)-b_i(y)|||F_t(x,y)-F_t(x,0)|||a_j(y)|dy\\ &\leq C|x|^{-(n+\varepsilon-\delta)}\int_{B_j}|y|^\varepsilon|a_j(y)|\prod_{i=1}^m|b_i(x)-b_i(y)|dy\\ &\leq C|x|^{-(n+\varepsilon-\delta)}\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}')_\sigma|\int_{B_j}|y|^\varepsilon|a_j(y)||(\vec{b}(x)-\vec{b})_{\sigma^c}|dy\\ &\leq C|x|^{-(n+\varepsilon-\delta)}\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}')_\sigma|2^{j\varepsilon}\cdot 2^{-j\varepsilon}\cdot 2^{jn(1-1/q_1)}||\vec{b}_{\sigma^c}||_{BMO}\\ &\leq C|x|^{-(n+\varepsilon-\delta)}\cdot 2^{j(\varepsilon+n(1-1/q_1)-\alpha)}\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}')_\sigma|||\vec{b}_{\sigma^c}||_{BMO};\end{aligned}$$

So

$$\begin{aligned}&||T_\delta^{\vec{b}}(a_j)\chi_k||_{L^{q_2}}\\ \leq\ & C2^{j(\varepsilon+n(1-1/q)-\alpha)}||\vec{b}_{\sigma^c}||_{BMO}\left[\int_{B_k}\left(|x|^{-(n+\varepsilon-\delta)}\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}')_\sigma|\right)^{q_2}dx\right]^{1/q_2}\\ \leq\ & C||\vec{b}_{\sigma^c}||_{BMO}2^{j(\varepsilon+n(1-1/q_1)-\alpha)}\cdot 2^{-k(n+\varepsilon-\delta)+kn/q_2}||\vec{b}_\sigma||_{BMO}\\ \leq\ & C||\vec{b}||_{BMO}2^{j(\varepsilon+n(1-1/q_1)-\alpha)-k(\varepsilon+n(1-1/q_1))}\end{aligned}$$

and

$$I = C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|||T_\delta^{\vec{b}}(a_j)\chi_k||_{L^{q_2}}\right)^p\right]^{1/p}$$

$$\leq C||\vec{b}||_{BMO}\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|2^{[j(\varepsilon+n(1-1/q_1)-\alpha)-k(\varepsilon+n(1-1/q_1))]}\right)^p\right]^{1/p}$$

$$\leq C||\vec{b}||_{BMO}\begin{cases}\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\sum_{j=-\infty}^{k-3}|\lambda_j|^p2^{p[j(\varepsilon+n(1-1/q_1)-\alpha)-k(\varepsilon+n(1-1/q_1))]}\right]^{1/p}, & 0<p\leq 1\\ \left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p2^{[j(\varepsilon+n(1-1/q_1)-\alpha)-k(\varepsilon+n(1-1/q_1))]p/2}\right)\right. & \\ \quad\left.\times\left(\sum_{j=-\infty}^{k-3}2^{[j(\varepsilon+n(1-1/q_1)-\alpha)-k(\varepsilon+n(1-1/q_1))]p'/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}$$

$$\leq C||\vec{b}||_{BMO}\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)p}\right]^{1/p}, & 0<p\leq 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=j+3}^{\infty}2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)p/2}\right)\right. & \\ \quad\left.\times\left(\sum_{k=j+3}^{\infty}2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)p'/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}$$

$$\leq C||\vec{b}||_{BMO}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}$$

$$\leq C||f||_{H\dot{K}^{\alpha,p}_{q_1,\vec{b}}}.$$

Remark. Theorem 2 also hold for nonhomogeneous Herz-type spaces, we omit the details.

4. Applications

Now we give some applications of Theorems in this paper.

Application 1. Littlewood-Paley operator.

Fixed $0<\delta<n$ and $\varepsilon>0$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n}\psi(x)dx=0$,

(2) $|\psi(x)|\leq C(1+|x|)^{-(n+1-\delta)}$,

(3) $|\psi(x+y)-\psi(x)|\leq C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y|<|x|$.

The Littlewood-Paley multilinear operators are defined by

$$g^{\vec{b}}_{\psi,\delta}(f)(x)=\left(\int_0^\infty|F^{\vec{b}}_t(f)(x)|^2\frac{dt}{t}\right)^{1/2},$$

where

$$F^{\vec{b}}_t(f)(x)=\int_{R^n}\prod_{j=1}^m(b_j(x)-b_j(y))\psi_t(x-y)f(y)dy$$

and $\psi_t(x)=t^{-n+\delta}\psi(x/t)$ for $t>0$. Set $F_t(f)(y)=f*\psi_t(y)$. We also define

$$g_{\psi,\delta}(f)(x)=\left(\int_0^\infty|F_t(f)(x)|^2\frac{dt}{t}\right)^{1/2},$$

which is the Littlewood-Paley operator(see [18]). Let H be the space

$$H = \left\{ h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t \right)^{1/2} < \infty \right\},$$

then, for each fixed $x \in R^n$, $F_t^{\vec{b}}(f)(x)$ and $F_t^{\vec{b}}(f)(x,y)$ may be viewed as the mappings from $[0, +\infty)$ to H, and it is clear that

$$g_{\psi,\delta}^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||, \quad g_{\psi,\delta}(f)(x) = ||F_t(f)(x)||.$$

It is easily to see that $g_{\psi,\delta}$ satisfies the conditions of Theorem 1 and 2 (see [5-9]), thus Theorem 1 and 2 hold for $g_{\psi,\delta}^{\vec{b}}$.

Application 2. Marcinkiewicz operator.

Fixed $0 < \delta < n$ and $0 < \gamma \le 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. The Marcinkiewicz multilinear operators are defined by

$$\mu_{\Omega,\delta}^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{|x-y|\le t} \prod_{j=1}^m (b_j(x) - b_j(y)) \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy.$$

We also define

$$\mu_{\Omega,\delta}(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

which is the Marcinkiewicz operator(see [8][20]). Let H be the space

$$H = \left\{ h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3 \right)^{1/2} < \infty \right\}.$$

Then, it is clear that

$$\mu_{\Omega,\delta}^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||, \quad \mu_{\Omega,\delta}(f)(x) = ||F_t(f)(x)||,$$

It is easily to see that $\mu_{\Omega,\delta}$ satisfies the conditions of Theorem 1 and 2 (see [8][20]), thus Theorem 1 and 2 hold for $\mu_{\Omega,\delta}^{\vec{b}}$.

Application 3. Bochner-Riesz operator .

Let $\eta > (n-1)/2$, $\widehat{B_t^\eta(f)}(\xi) = (1-t^2|\xi|^2)_+^\eta \hat{f}(\xi)$ and $B_t^\eta(z) = t^{-n}B^\eta(z/t)$ for $t > 0$. Set

$$F_{\eta,t}^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y)) B_t^\eta(x-y) f(y) dy.$$

The maximal Bochner-Riesz multilinear commutator are defined by

$$B_{\eta,*}^{\vec{b}}(f)(x) = \sup_{t>0} |B_{\eta,t}^{\vec{b}}(f)(x)|.$$

We also define that

$$B_{\eta,*}(f)(x) = \sup_{t>0} |B_t^\eta(f)(x)|,$$

which is the maximal Bochner-Riesz operator(see [10]). Let H be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B_{\eta,*}^{\vec{b}}(f)(x) = ||B_{\eta,t}^{\vec{b}}(f)(x)||, \quad B_*^\eta(f)(x) = ||B_t^\eta(f)(x)||.$$

It is easily to see that $B_{\eta,*}^{\vec{b}}$ satisfies the conditions of Theorem 1 and 2 with $\delta = 0$(see [9]), thus Theorem 1 and 2 hold for $B_{\eta,*}^{\vec{b}}$.

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