Acta Universitatis Apulensis No. 75/2023
ISSN: 1582-5329 pp. 89-101
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2023.75.06

ON DOUBLE INTEGRALS OF HENSTOCK-KURZWEIL ON TIME SCALES

D.A. Afariogun, A.A. Mogbademu

Abstract. This paper provides a basic definition for Henstock-Kurzweil double integrals on time scales. Some of the basic properties such as uniqueness and existence of this integral are proved. An example is given to support our results.

2010 *Mathematics Subject Classification*: 26A39; 26E70; 28B15; 46G10.

Keywords: Double integral, Existence, Henstock-Kurzweil integral, Time scales, Uniqueness.

1. Introduction

The Hestock-Kurzweil integral was independently introduced by Kurzweil and Henstock in 1957 and 1958 respectively. The concept of Henstock integral [9] for real valued functions was first defined by Henstock in 1963, and it is seen to be easier and simpler than the McSchane integral [11], Lebesgue integral [10], Denjoy integral [6] and Perron integral [13]. Recently, there have been tremendous research in the study of Henstock integral. Many authors have investigated Henstock integrals dealing with certain valued functions on time scales. For instance, Peterson and Thompson [14] introduced Henstock delta integral on time scales and gave some of its basic properties. In [15], Thompson studied Henstock integrals on time scales. Other studies of integrals on time scales can be found in [2, 3, 4, 5], [12] and [15]. The study of Double Henstock integral on time scales has not received enough attention in the literature. But, Afariogun et al. [1] recently studied Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrals of interval-valued functions on time scales. For further study of such valued-functions integrals and other important properties (see [7] and [8]).

The aim of this paper is to introduce Henstock-Kurzweil double integral on time scales which is a generalization of the Henstock-typed integrals in [7], [8], [9], and [11]. Also, some basic properties of the integral are established with an example.

The following concepts are important in $\mathbb{T}_1 \times \mathbb{T}_2$. Let $a, b \in \mathbb{T}_1, c, d \in \mathbb{T}_2$, where $a < b, c < d$, and $\mathcal{R} = [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} = \{(t, s) : t \in [a, b), s \in [c, d), t \in \mathbb{T}_1, s \in \mathbb{T}_2\}$. Let $F : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be bounded on $\mathcal{R}$. Let P_1 and P_2 be two partitions of $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$ such that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, ..., s_n\} \subset [c, d]_{\mathbb{T}_2}$. Let $\{\xi_1, \xi_2, ..., \xi_n\}$ denote an arbitrary selection of points from $[a, b]_{\mathbb{T}_1}$ with $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}_1}, i = 1, 2, ..., n$. Similarly, let $\{\zeta_1, \zeta_2, ..., \zeta_n\}$ denote an arbitrary selection of points from $[c, d]_{\mathbb{T}_2}$ with $\zeta_j \in [s_{j-1}, s_j)_{\mathbb{T}_2}, j = 1, 2, ..., k$.

Definition 1. *[1] A pair $\delta = (\delta_L, \delta_R)$ of real-valued functions defined on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ is said to be a Δ-gauge for $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ if*

$$\begin{cases} \delta_L(t, s) > 0 \quad on \quad (a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2} \\ \delta_R(t, s) > 0 \ on \ \ [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \\ \delta_L(a) \geq 0, \ \delta_R(b) \geq 0 \ \delta_L(c) \geq 0 \ \delta_R(d) \geq 0 \\ \delta_R(t, s) \geq \sigma(t, s) - (t, s) \quad for\ all\ t \in [a, b)_{\mathbb{T}_1}, s \in [c, d)_{\mathbb{T}_2}. \end{cases}$$

A pair $\gamma = (\gamma_L, \gamma_R)$ of real-valued functions defined on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ is said to be a $\bigtriangledown$-gauge for $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ if

$$\begin{cases} \gamma_L(t, s) > 0 \quad on \quad (a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2} \\ \gamma_R(t, s) > 0 \ on \ \ [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \\ \gamma_L(a) \geq 0, \ \gamma_R(b) \geq 0, \ \gamma_L(c) \geq 0, \ \gamma_R(d) \geq 0 \\ \gamma_R(t, s) \geq (t, s) - \rho(t, s) \quad for\ all\ t \in (a, b]_{\mathbb{T}_1}, s \in (c, d]_{\mathbb{T}_2} \end{cases}$$

where the function $\sigma : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and the function $\rho(t)$ defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ are respectively the forward and backward jump operators.
Given a Δ-gauge δ and a $\bigtriangledown$-gauge γ, the partitions $P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c, d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1, 2, ..., k$, is said to be:

- *δ-fine if $(\xi_i, \zeta_j) - \delta_L \leq (t_{i-1}, s_{j-1}) < (t_i, s_j) \leq ((\xi_i, \zeta_j) + \delta_R(\xi_i, \zeta_j))$ and*
- *γ-fine if $(\xi_i, \zeta_j) - \gamma_L \leq (t_{i-1}, s_{j-1}) < (t_i, s_j) \leq ((\xi_i, \zeta_j) + \gamma_R(\xi_i, \zeta_j))$.*

Definition 2. *[1] Let $F : [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \to \mathbb{R}$ be a real-valued function on $\mathcal{R}$ with partitions $P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, 2, ..., n$ and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c, d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$ for $j =$*

$1, 2, ..., k$. *Then*

$$S(P_1, P_2, F) = \sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i, \zeta_j)[(t_i) - (t_{i-1})][((s_j) - (s_{j-1})]$$

is defined as Henstock-Kurzweil double sum of F.

Let $P = P_1 \times P_2$ and $\Diamond t_i \Diamond t_j = [(t_i) - g_1(t_{i-1})][(s_j) - (s_{j-1})]$ where $\Diamond$ represents either Δ or $\bigtriangledown$, then the Henstock-Kurzweil double sum of F is denoted by $S(P, F)$ is written as

$$S(P, F) = \sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i, \zeta_j)\Diamond t_i \Diamond s_j, \ (i = 1, ..., n; \ j = 1, ..., k).$$

Definition 3. *[2] Let $F : [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \to \mathbb{R}$ be a real-valued function on $\mathcal{R} = [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$. Then F is said to be Henstock-Kurzweil integrable if there exists a number $\alpha \in \mathbb{R}$ such that for every $\varepsilon > 0$, there are $\Diamond$-gauges δ_1 and δ_2 (or γ_1 and γ_2) for $[a, b)_{\mathbb{T}_1}$ and $[c, d)_{\mathbb{T}_2}$ respectively such that*

$$|S(P, F) - \alpha| < \varepsilon$$

provided that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, ..., n$ is a δ_1-fine (or γ_1) and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c, d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1, 2, ..., k$ is a δ_2-fine (or γ_2) are δ-fine (or γ) partitions of $[a, b)_{\mathbb{T}_1}$ and $[c, d)_{\mathbb{T}_2}$ respectively.

α is the Henstock-Kurzweil double integral of F defined on $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$, and write

$$\int\int_{\mathcal{R}} F(t, s)dtds = \alpha.$$

2. Main Results

Now our main results are as follows:

Theorem 1. *If $F : (a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2} \to \mathbb{R}$ is Henstock-Kurzweil double integrable on $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$, then the Henstock-Kurzweil double integral of F is unique.*

Proof. Suppose that α_1 and α_2 are both Henstock-Kurzweil double integrals of F on $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$. With the assumption that α_1 and α_2 are not unique, then F is said to be Henstock-Kurzweil double integrable on $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$ if it satisfies the following point wise integrability criterion: for every $\varepsilon > 0$ there are $\diamondsuit$-gauges δ_1 and δ_2 (or γ_1 and γ_2) defined on $(a,b]_{\mathbb{T}_1}$ and $(c,d]_{\mathbb{T}_2}$ respectively, such that for every $\varepsilon > 0$, there are $\diamondsuit$-gauges δ_1^1 and δ_2^1 (or γ_1^1 and γ_2^1) for $[a,b)_{\mathbb{T}_1}$ and δ_1^2 and δ_2^2 (or γ_1^2 and γ_2^2) for $[c,d)_{\mathbb{T}_2}$ such that

$$|S(P^1,F)-\alpha_1| < \frac{\varepsilon}{2} \text{ and } |S(P^2,F)-\alpha_2| < \frac{\varepsilon}{2} \text{ for all pairs } P^1 = P_1^1 \times P_2^1 \text{ and } P^2 = P_1^2 \times P_2^2 \text{ of } \delta_1\text{-fine (or } \gamma_1)$$

and for every $\varepsilon > 0$ and $i \in \{1,2\}$, there are $\diamondsuit$-gauges δ_1^i and δ_2^i (or γ_1^i and γ_2^i) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that

$$|S(P^i,F)-\alpha_i| < \frac{\varepsilon}{2}$$

provided that $P^i = P_1^i \times P_2^i$ is a pair of δ_1^i-fine (or γ_1^i) and δ_2^i-fine (or γ_2^i) partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively.
Let $\delta_1 = \min\{\delta_1^1, \delta_1^2\}$ i.e. $(\delta_1)_L = \min\{(\delta_1^1)_L, (\delta_1^2)_L\}$ and $(\delta_1)_R = \min\{(\delta_1^1)_R, (\delta_1^2)_R\}$ and
$\delta_2 = \min\{\delta_2^1, \delta_2^2\}$ i.e. $(\delta_2)_L = \min\{(\delta_2^1)_L, (\delta_2^2)_L\}$ and $(\delta_2)_R = \min\{(\delta_2^1)_R, (\delta_2^2)_R\}$, δ_1 and δ_2 are $\diamondsuit$-gauges for $(a,b]_{\mathbb{T}_1}$ and $(c,d]_{\mathbb{T}_2}$ respectively, and given a pair $P = P_1 \times P_2$ of δ_1-fine and δ_2-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$, P_1 is a δ_1^1-fine and δ_1^2-fine partition of $(a,b]_{\mathbb{T}_1}$, P_2 is a δ_2^1-fine and δ_2^2-fine partition of $[c,d)_{\mathbb{T}_2}$, hence

$$\begin{aligned} |\alpha_1-\alpha_2| &\leq |(\alpha_1 - S(P,F) + S(P,F) - \alpha_2)| \\ &\leq |S(P,F)-\alpha_1| + |S(P,F,g)-\alpha_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

since for all $\varepsilon > 0$, there are $\diamondsuit$-gauges δ_1 and δ_2 (or γ_1 and γ_2), then it follows that $\alpha_1 = \alpha_2$.
Hence, the Henstock-Kurzweil double integral of F on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ is unique.

Theorem 2 (Bolzano Cauchy Criterion)**.** .
Let $F : (a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2} \to \mathbb{R}$ be a real-valued function over a rectangle $(a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2}$. Then, F is Henstock-Kurzweil double integrable on $(a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2}$ if and only if for each $\varepsilon > 0$ there exists $\diamondsuit$-gauges δ_1 and δ_2 (or γ_1 and γ_2) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively, such that $|S(P^1,F) - S(P^2,F)| < \varepsilon$ for all pairs $P^1 = P_1^1 \times P_2^1$ and $P^2 = P_1^2 \times P_2^2$ of δ_1 (or γ_1)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and δ_2 (or γ_2)-fine partitions of $[c,d)_{\mathbb{T}_2}$.

Proof. Suppose F is Henstock-Kurzweil double integrable on $(a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2}$, and let

$$\alpha = \int\int_{\mathcal{R}} F(t,s)dtds.$$

Let $\varepsilon > 0$. There are $\Diamond$-gauges δ_1 and δ_2 for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that $|S(P,F,g) - \alpha| < \frac{\varepsilon}{2}$ provided that $P = P_1 \times P_2$ where P_1 is a δ_1 (or γ_1) fine partition of $[a,b)_{\mathbb{T}_1}$ and P_2 is a δ_2 (or γ_2) fine partition of $[c,d)_{\mathbb{T}_2}$. Therefore, if $P = P_1 \times P_2$ and $P = P_1^{'} \times P_2^{'}$ are pairs of δ_1 (or γ_1) fine partition of $[a,b)_{\mathbb{T}_1}$ and P_2 is a δ_2 (or γ_2) fine partition of $[c,d)_{\mathbb{T}_2}$, then

$$\begin{aligned} |S(P,F) - S(P^1,F)| &\leq |S(P,F) - \alpha| + |\alpha - S(P^1,F)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Conversely, suppose that for all $\varepsilon > 0$ there are $\Diamond$-gauges δ_1 and δ_2 (or γ_1 and γ_2) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that $|S(P^1,F) - S(P^2,F)| < \varepsilon$ for all pairs $P^1 = P_1^1 \times P_2^1$ and $P^2 = P_1^2 \times P_2^2$ of δ_1 (or γ_1)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and δ_2 (or γ_2)-fine partitions of $[c,d)_{\mathbb{T}_2}$.

Let $n \in \mathbb{N}$. Taking $\varepsilon = \frac{1}{n}$, there are $\Diamond$-gauges $\delta_{1,n}$ and $\delta_{2,n}$ (or $\gamma_{1,n}$ and $\gamma_{2,n}$) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that $|S(P^{'},F) - S(P^2,F)| < \varepsilon$ for all pairs $P^1 = P_1^1 \times P_2^1$ and $P^2 = P_1^2 \times P_2^2$ of $\delta_{1,n}$ (or $\gamma_{1,n}$)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $\delta_{2,n}$ (or $\gamma_{2,n}$)-fine partitions of $[c,d)_{\mathbb{T}_2}$.

By replacing $\delta_{i,n}$ by $\min\{\delta_{i,1}, \delta_{i,2}, ..., \delta_{i,n}\}$ with $i \in \{1,2\}$, we may assume that $\delta_{i,n+1} \leq \delta_{i,n}$. Thus, for all $j > n$ $\delta_{i,j} \leq \delta_{i,n}$ so any pair $P^n = P_1^n \times P_2^n$ of $\delta_{1,n}$ (or $\gamma_{1,n}$)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $\delta_{2,n}$ (or $\gamma_{2,n}$)-fine partitions of $[c,d)_{\mathbb{T}_2}$ is also a pair of $\delta_{1,j}$ (or $\gamma_{1,j}$)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $\delta_{2,j}$ (or $\gamma_{2,j}$)-fine partitions of $[c,d)_{\mathbb{T}_2}$, hence

$$|S(P^n,F) - S(P^j,F)| < \frac{1}{j}.$$

This shows that $\{S(P^n,F)\}_{n \in \mathbb{N}}$ is a Cauchy sequence.

Let α be the limit of $\{S(P^n,F)\}_{n \in \mathbb{N}}$. For all $\varepsilon > 0$, choosing $N > \frac{2}{\varepsilon}$, for $\Diamond$-gauges $\delta_{1,N}$ and $\delta_{2,N}$ (or $\gamma_{1,N}$ and $\gamma_{2,N}$) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively,

$$\begin{aligned} |S(P,F) - \alpha| &\leq |S(P,F) - S(P^N,F)| + |S(P^N,F) - \alpha| \\ &< \frac{1}{N} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

for pair $P = P_1 \times P_2$ such that P_1 is a $\delta_{1,N}$ (or $\gamma_{1,N}$) fine partition of $[a,b)_{\mathbb{T}_1}$ and P_2 is a $\delta_{2,N}$ (or $\gamma_{2,N}$) fine partition of $[c,d)_{\mathbb{T}_2}$.

The next theorem gives the linearity properties of Henstock-Kurzweil double integral on time scales.

Theorem 3. *Let F and G be Henstock-Kurzweil double integrable on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$, then* *(i)* λF *is Henstock-Kurzweil double integrable on* $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ *for each* $\lambda \in \mathbb{R}$ *with*

$$\int\int_{\mathcal{R}} \lambda F(t,s)dtds = \lambda \int\int_{\mathcal{R}} F(t,s)dtds;$$

(ii) $F+G$ *is Henstock-Kurzweil double integrable on* $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ *with*

$$\int\int_{\mathcal{R}} [F(t,s)+G(t,s)]dtds = \int\int_{\mathcal{R}} F(t,s)dtds + \int\int_{\mathcal{R}} G(t,s)dtds.$$

Proof. (i). Let $\varepsilon > 0$. By assumption F is Henstock-Kurzweil integrable. So for $\frac{\varepsilon}{|\lambda|}$ there are $\Diamond$-gauges δ_1 and δ_2 (or γ_1 and γ_2) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that

$$\left| S(P,F) - \int\int_{\mathcal{R}} F(t,s)dtds \right| \leq \frac{\varepsilon}{|\lambda|}$$

provided that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1,...,n$ is a δ_1-fine (or γ_1) and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1,2,...,k$ is a δ_2-fine (or γ_2) are δ-fine (or γ) partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively.
Now we have that

$$\begin{aligned} \left| S(P,\lambda F) - \lambda \int\int_{\mathcal{R}} F(t,s)dtds \right| &= \left| \sum_{i=1}^{n} \sum_{j=1}^{k} \lambda F(\xi_i, \zeta_j)[(t_i) - (t_{i-1})][((s_j) - (s_{j-1})] - \lambda \int\int_{\mathcal{R}} F(t,s)dtds \right| \\ &= |\lambda| \left| S(P,F) - k \int\int_{\mathcal{R}} F(t,s)dtds \right| \\ &\leq |\lambda| \frac{\varepsilon}{|\lambda|} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, therefore λF is Henstock-Kurzweil double integrable on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ for each $\lambda \in \mathbb{R}$ with

$$\int\int_{\mathcal{R}} \lambda F(t,s)dtds = \lambda \int\int_{\mathcal{R}} F(t,s)dtds.$$

(ii). Let $\varepsilon > 0$. By assumption F and G are Henstock-Kurzweil double integrable. For $\frac{\varepsilon}{2}$ there exist $\Diamond$-gauges δ_1 and δ_2 (or γ_1 and γ_2) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that

$$\left| S(P_1,F) - \int\int_{\mathcal{R}} F(t,s)dtds \right| \leq \frac{\varepsilon}{2}$$

and

$$\left| S(P_2,G) - \int\int_{\mathcal{R}} F(t,s)dtds \right| \leq \frac{\varepsilon}{2}$$

provided that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1,...,n$ is a δ_1-fine (or γ_1) and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in$

$[s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1, 2, ..., k$ is a δ_2-fine (or γ_2) are δ-fine (or γ) partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively.
Now define $\delta_{i,n}$ by $\min\{\delta_{i,1}, \delta_{i,2}, ..., \delta_{i,n}\}$ with $i \in \{1,2\}$, we may assume that $\delta_{i,n+1} \leq \delta_{i,n}$. Thus, for all $j > n$ $\delta_{i,j} \leq \delta_{i,n}$ so any pair $P^n = P_1^n \times P_2^n$ of $\delta_{1,n}$ (or $\gamma_{1,n}$)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $\delta_{2,n}$ (or $\gamma_{2,n}$)-fine partitions of $[c,d)_{\mathbb{T}_2}$ is also a pair of $\delta_{1,j}$ (or $\gamma_{1,j}$)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $\delta_{2,j}$ (or $\gamma_{2,j}$)-fine partitions of $[c,d)_{\mathbb{T}_2}$, hence

$$\begin{aligned} S(F+G, P^n) &= \sum_{i=1}^{n}\sum_{j=1}^{k}(F+G)(\xi_i, \zeta_j)[(t_i)-(t_{i-1})][((s_j)-(s_{j-1})] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{k}[F(\xi_i,\zeta_j)[(t_i)-(t_{i-1})][((s_j)-(s_{j-1})] + G(\xi_i,\zeta_j)[(t_i)-(t_{i-1})][((s_j)-(s_{j-1})] \end{aligned}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{k}F(\xi_i,\zeta_j)[(t_i)-(t_{i-1})][((s_j)-(s_{j-1})] + \sum_{i=1}^{n}\sum_{j=1}^{k}G(\xi_i,\zeta_j)[(t_i)-(t_{i-1})][((s_j)-(s_{j-1})]$$

$$= S(F,P) + S(G,P).$$

For all $\varepsilon > 0$, choosing $N > \dfrac{2}{\varepsilon}$, for $\Diamond$-gauges $\delta_{1,N}$ and $\delta_{2,N}$ (or $\gamma_{1,N}$ and $\gamma_{2,N}$) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively. Then for pair $P = P_1 \times P_2$ such that P_1 is a $\delta_{1,N}$ (or $\gamma_{1,N}$) fine partition of $[a,b)_{\mathbb{T}_1}$ and P_2 is a $\delta_{2,N}$ (or $\gamma_{2,N}$) fine partition of $[c,d)_{\mathbb{T}_2}$. We have that

$$\begin{aligned} & \left|S(F+G,P) - \left(\int\int_{\mathcal{R}} F(t,s)dtds + \int\int_{\mathcal{R}} G(t,s)dtds\right)\right| \\ \leq & \left|S(P,F) - \int\int_{\mathcal{R}} F(t,s)dtds\right| + \left|S(P,G) - \int\int_{\mathcal{R}} G(t,s)dtds\right| \\ \leq & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, therefore $F+G$ is Henstock-Kurzweil double integrable on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ with

$$\int\int_{\mathcal{R}}[F(t,s)+G(t,s)]dtds = \int\int_{\mathcal{R}} F(t,s)dtds + \int\int_{\mathcal{R}} G(t,s)dtds.$$

This ends the proof.

Theorem 4 (Existence Theorem)**.** *Let* $F : [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2} \to \mathbb{R}$ *be a continuous function and* F *is also be of bounded variation on* $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$*, then* F *is Henstock-Kurzweil double integrable on* $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$*.*

Proof. Let $\varepsilon > 0$. Since F is of bounded variation, $\mathbf{Var}_F \in \mathbb{R}^2$. This means that there exists $M > 0$ such that $\mathbf{Var}_F(t,s) \leq M$ for all $t, s \in [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$. Since F is continuous on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$, for all $t_0, s_0 \in [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ there exists a positive $\delta_0(t_0,s_0)$ such that whenever $t, s \in [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ with

$$|(t,s) - (t_0,s_0)| < \delta_0,$$

we have

$$|(F(t,s) - F(t_0,s_0)| < \varepsilon.$$

Let a positive gauge δ be defined on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ by $\delta = \dfrac{\delta_0}{2}$, for all $t, s \in [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$. Let

$$P_1 = \{([a,t_1],\xi_1), ([t_1,t_2],\xi_2), ..., ([t_{n-1},b],\xi_n)\} \subset [a,b]_{\mathbb{T}_1}$$

and

$$P_2 = \{([c,s_1],\zeta_1), ([s_1,s_2],\zeta_2), ..., ([s_{k-1},d],\zeta_k)\} \subset [c,d]_{\mathbb{T}_2}$$

be δ-fine tagged divisions of $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$. Then, there exists a tagged division P_0 such that $P_1 < P_0$ and $P_2 < P_0$. Now, for every $([t_{i-1},t_i],\xi_i) \in P_1$ and $([s_{j-1},s_j],\zeta_j) \in P_2$; $\;i = 1,2,...,n$; $\;j = 1,2,...,k$. Now we have the difference

$$|(t_{i-1},s_{j-1}) - (t_i,s_j)| = F(\xi_i,\zeta_j)[(t_i) - (t_{i-1}))(s_j) - (s_{j-1}))] - S(F,P_{i,j})$$

where

$$P_{i,j} = \left\{\left(\left[X_{q-1}^{(i,j)}, X_q^{(i,j)}\right], s_q^{(i,j)}\right) \quad, \; X_0^{(i,j)} = (t_{i-1},s_{j-1}),\; X_{m_i}^{(i,j)} = (t_i,s_j),\; q-1 < m_{i,j}\right.$$

is a refinement of $([(t_{i-1},s_{j-1}),(t_i,s_j)],(\xi_i,\zeta_j))$ in P_0. Then

$$|(t_{i-1},s_{j-1}) - (t_i,s_j)| = \sum_{i=1}^{n}\sum_{j=1}^{k}\left(\sum_{q=1}^{m_{i,j}} F(\xi_i,\zeta_j) - F(s_q^{(i,j)})\right)\left(X_q^{(i,j)} - X_{q-1}^{(i,j)}\right).$$

Now, $s_q^{i,j}, (\xi_i,\zeta_j) \in ((t_{i-1},s_{j-1}),(t_i,s_j)) \subseteq (\xi_i,\zeta_j) - \delta(\xi_i,\zeta_j), (\xi_i,\zeta_j) + \delta(\xi_i,\zeta_j)$ which implies that

$$\left|(\xi_i,\zeta_j) - s_q^{i,j}\right| \leq |(t_{i-1},s_{j-1}) - (t_i,s_j| < \delta(\xi_i,\zeta_j).$$

By continuity of F at (ξ_i,ζ_j),

$$|\; s_q^{i,j} - (\xi_i,\zeta_j) \;| < \delta(\xi_i,\zeta_j) = \frac{\delta_0(\xi_i,\zeta_j)}{2} < \delta_0(\xi_i,\zeta_j)$$

it implies that

$$|F(s_q^{i,j}) - F(\xi_i,\zeta_j)| < \varepsilon.$$

So,

$$|(t_{i-1}, s_{j-1}) - (t_i, s_j)| = \sum_{i=1}^{n} \sum_{j=1}^{k} \left(\sum_{q=1}^{m_{i,j}} F(\xi_i, \zeta_j) - F(s_q^{(i,j)}) \right) \left(X_q^{(i,j)} - X_{q-1}^{(i,j)} \right).$$

Hence, by bounded variation of F, we have

$$\begin{aligned}
& |S(P,F) - S(P_0,F)| \\
= & \left| \sum_{i=1}^{n} \sum_{j=1}^{k} F(\xi_i, \zeta_j)[(t_i) - (t_{i-1})((s_j) - (s_{j-1})] - \sum_{i=1}^{n} \sum_{j=1}^{k} S(P_{i,j}, F) \right| \\
= & \left| \sum_{i=1}^{n} \sum_{j=1}^{k} \{F(\xi_i, \zeta_j)[(t_i) - (t_{i-1})((s_j) - (s_{j-1})] - S(P_{i,j}, F)\} \right| \\
= & \left| \sum_{i=1}^{n} \sum_{j=1}^{k} |(t_{i-1}, s_{j-1}) - (t_i, s_j)| \right| \\
\leq & \sum_{i=1}^{n} \sum_{j=1}^{k} ||(t_{i-1}, s_{j-1}) - (t_i, s_j)|| \\
= & \sum_{i=1}^{n} \sum_{j=1}^{k} \left| \sum_{q=1}^{m_{i,j}} F(\xi_i, \zeta_j) - F(s_q^{(i,j)})(F(X_q^{(i,j)} - F(X_{q-1}^{(i,j)})) \right| \\
\leq & \sum_{i=1}^{n} \sum_{j=1}^{k} \left(\sum_{q=1}^{m_{i,j}} \left| F(\xi_i, \zeta_j) - F(s_q^{(i,j)})(F(X_q^{(i,j)} - F(X_{q-1}^{(i,j)})) \right| \right) \\
\leq & \sum_{i=1}^{n} \sum_{j=1}^{k} \left(\sum_{q=1}^{m_{i,j}} \frac{\varepsilon}{K} \left| (F(X_q^{(i,j)} - F(X_{q-1}^{(i,j)})) \right| \right) \\
\leq & \frac{\varepsilon}{K} . \sum_{i=1}^{n} \sum_{j=1}^{k} \left(\sum_{q=1}^{m_{i,j}} \left| (F(X_q^{(i,j)} - F(X_{q-1}^{(i,j)})) \right| \right) \\
\leq & \frac{\varepsilon}{K} . \sum_{i=1}^{n} \sum_{j=1}^{k} Var[F, (t_{i-1}, s_{j-1}), (t_i, s_j)] \\
= & \frac{\varepsilon}{K} Var_F < \frac{\varepsilon}{K} K = \varepsilon.
\end{aligned}$$

Similarly,

$$|S(Q,F) - S(P_0,F)| < \varepsilon.$$

Thus,

$$\begin{aligned} |S(P,F) - S(Q,F)| &\le |S(P,F) - S(P_0,F)| + |S(P_0,F) - S(Q,F)| \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon. \end{aligned}$$

By Cauchy criterion Theorem 2, F is Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$.

Example 1. *Let $F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to I_{\mathbb{R}}$ be a Dirichlet function defined by:*

$$F(t,s) = \begin{cases} [0,1] & if \quad (t,s) \in \mathbb{Q}^2 \\ [-1,0] & if \quad (t,s) \notin \mathbb{Q}^2. \end{cases}$$

Then $F(t,s)$ is Henstock-Kurzweil double integrable with

$$\int_a^b \int_c^d F(t,s)dtds = 0.$$

To show this, let $\varepsilon > 0$. Enumerate the rational numbers in $[a,b]_{\mathbb{T}_1}$ as $\{r_1, r_2, ...,\}$ and enumerate the rational numbers in $[a,b]_{\mathbb{T}_2}$ as $\{q_1, q_2, ...,\}$. Now, define a Δ-gauge $\delta_{1\varepsilon}$-fine for $[a,b]_{\mathbb{T}_1}$, the partition $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, ..., n$ is a $\delta_{1\varepsilon}$-fine (or $\gamma_{1\varepsilon}$) and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1, 2, ..., k$ is a $\delta_{2\varepsilon}$-fine (or $\gamma_{2\varepsilon}$) are δ-fine (or γ) partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively.
It follows from Definition 1, that $\delta_1 = (\delta_{1L}, \delta_{1R})$ and $\delta_2 = (\delta_{2L}, \delta_{2R})$, then

$$\delta_{1R}(t) = \delta_{1L}(t) = \begin{cases} \dfrac{\sqrt{\varepsilon}}{2^i} & if \quad t = r_i \\ 1 & if \quad t \notin \{r_1, r_2, ...\}. \end{cases}$$

Similarly,

$$\delta_{2R}(s) = \delta_{2L}(s) = \begin{cases} \dfrac{\sqrt{\varepsilon}}{2^j} & if \quad s = q_j \\ 1 & if \quad s \notin \{q_1, q_2, ...\}. \end{cases}$$

Let $g_1(t) = \frac{t}{2}$, $\quad t \in [a,b]_{\mathbb{T}_1}$ and $g_2(s) = 2s$, $\quad s \in [a,b]_{\mathbb{T}_2}$. Then,

$$\Diamond_\varepsilon(t,s) = \begin{cases} \dfrac{\sqrt{\varepsilon}}{2^i}, \dfrac{\sqrt{\varepsilon}}{2^j} & if \quad (t,s) = r_i, q_j \\ 1 & if \quad t \notin \{t_1, t_2, ...\}, s \notin \{q_1, q_2, ...\}. \end{cases}$$

If $\xi_i \in \{r_1, r_2, ...\}$ is a tag on $[t_{i-1}, t_i]$ and $\zeta_i \in \{q_1, q_2, ...\}$ is a tag on $[s_{j-1}, s_j]$, then $F(\xi_i, \zeta_j) = [0,1]$ and

$$((t_i) - (t_{i-1}))((s_j) - (s_{j-1})) \leq \Diamond_\varepsilon(\xi_i, \zeta_j) = \frac{\sqrt{\varepsilon}}{2^i}.\frac{\sqrt{\varepsilon}}{2^j}.$$

Thus we have:

$$F(\xi_i, \zeta_j)\left((\frac{1}{2}t_i) - (\frac{1}{2}t_{i-1})\right)((2s_j) - (2s_{j-1})) \leq \Diamond_\varepsilon(\xi_i, \zeta_j) = \frac{\sqrt{\varepsilon}}{2^i}.\frac{\sqrt{\varepsilon}}{2^j} \leq \frac{\varepsilon}{2^{i+j}}.$$

If $\xi_i \notin \{r_1, r_2, ...\}$ and $\zeta_j \notin \{q_1, q_2, ...\}$ are tags on $[t_{i-1}, t_i]$ and $[s_{j-1}, s_j]$ respectively, then $F(\xi_i, \zeta_j) = [-1, 0]$ and

$$((t_i) - (t_{i-1}))((s_j) - (s_{j-1})) \leq \Diamond_\varepsilon(\xi_i, \zeta_j) = [0,1].$$

Therefore,

$$F(\xi_i, \zeta_j)\left((\frac{1}{2}t_i) - (\frac{1}{2}t_{i-1})\right)[(2s_j) - (2s_{j-1})] \leq \Diamond_\varepsilon(\xi_i, \zeta_j) = \{0\} = \alpha.$$

So subintervals with tags $\xi_i \in \{r_1, r_2, ...\}$ and $\zeta_j \in \{q_1, q_2, ...\}$ do not contribute to the Henstock-Kurzweil sum $S(F, P)$. Let μ be the set of indices i, j such that $\xi_i \in \{r_1, r_2, ...\}$ and $\zeta_j \in \{q_1, q_2, ...\}$ and ν be the set of indices i, j such that $\xi_i \in \{r_1, r_2, ...\}$ and $\zeta_j \in \{q_1, q_2, ...\}$. We conclude that:

$$\begin{aligned}
|S(F,P) - \alpha| &= |S(F,P) - \{0\}| \\
&= \left|\sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i, \zeta_j)((t_i) - (t_{i-1}))((s_j) - (s_{j-1}))\right| \\
&\leq \left|\sum_{i\in\mu}\sum_{j\in\mu} F(\xi_i, \zeta_j)((t_i) - (t_{i-1}))((s_j) - (s_{j-1}))\right| \\
&+ \left|\sum_{i\in\nu}\sum_{j\in\nu} F(\xi_i, \zeta_j)((t_i) - (t_{i-1}))((s_j) - (s_{j-1}))\right| \\
&\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\sqrt{\varepsilon}}{2^i} \cdot \frac{\sqrt{\varepsilon}}{2^j} \\
&\leq \sum_{i=1}^{\infty}\frac{\sqrt{\varepsilon}}{2^i}\sum_{j=1}^{\infty}\frac{\sqrt{\varepsilon}}{2^j} \\
&\leq \varepsilon\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{2^{i+j}} < \varepsilon.
\end{aligned}$$

Since ε was arbitrary, which holds for all $\varepsilon > 0$, therefore the Dirichlet function $F(t,s)$ is Henstock-Kurzweil double integrable on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ with

$$\int_a^b \int_c^d F(t,s)dtds = 0.$$

3. Conclusion

We obtained uniqueness and existence results for Hentock-Kurzweil double integrals on time scales. These results generalized the existing Hentock-typed integrals in classical sense. An example is provided to support our result.

Acknowledgements. The reviewers and editors of this manuscript deserve the heartfelt appreciation of the authors for their unrelenting efforts that helped this research.

Conflict of Interests The authors declare that there is no competing interests between them during the time of writing this paper.

References

[1] D. A. Afariogun, A. A. Mogbademu and H. O. Olaoluwa, *On Henstock-Kurzweil-Stieltjes-◇-double integrals of interval-valued functions on time scales*, Annals of Maths. and Comp. Sci. Vol. 2 (2021), 29-40.

[2] D. A. Afariogun, A. A. Mogbademu, *Henstock-Kurzweil-Stieltjes-◇-Double Integral for Gronwall-Bellman's Type Lemma on Time Scales*. Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 833-841.

[3] S. Avsec, B. Bannish, B. Johnson and S. Meckler , *The Henstock-Kurzweil delta integral on unbounded time scales*. Panamer. Math. J. No. 3, 16(2006), 77-98.

[4] Z. Bartosiewicz and E. Piotrowska, *The Lyapunov converse theorem of asymptotic stability on time scales*, presented at WCNA 2008, Orlando, Florida, (2008), July 2-9.

[5] M. Bohner and A. Peterson, *Advances in dynamic equations on time scales*, Birkhauser Boston, MA, (2003).

[6] A. Denjoy, *Une extension de l'integragrale de M. Lebesgue*, comptes Rendus De l'Academie des Sciences, (1912), 154.

[7] R A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Vol. 4 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, USA, (1994).

[8] R. Henstock, *The theory of Integration*, Butterworth, London, (1963).

[9] J. Kurzweil, *"Generalized ordinary differential equations and continuous dependence on a parameter"*, Czechoslovak Mathematical Journal, Vol. 7. no. 82, pp. 418-449, (1957).

[10] H. Lebesgue, *Lessons on Integration and Analysis of Primitive functions.* University of Paris, Milan, (1904).

[11] E. McSchane, *A Riemann-type integral that includes Lebesgue-Stieltjes, Bochner and stochastic integrals,* vol. 88. Amer. Mathematical Society, (1969).

[12] D. Mozyrska, E. Pawluszewicz and D. F. M. Torres, *The Riemann-Stieltjes integral on time scales*, Austr. J. Math. Anal. Appl., (2009), 1-14.

[13] O. Perron, *Uber den integralbegriff* Sitzungsber. Heidelberg. Akad. Wiss., VA, (1914), pp. 1-16.

[14] A. Peterson and B. Thompson, *Henstock-Kurzweil delta and nabla integrals*, J. Math. Anal. Appl., No. 1, 323 (2006), 162-178.

[15] B. Thompson, *Henstock-Kurzweil integrals on time scales*, Panamer. Math. J. No. 1, 18 (2008), 1-19.

David Adebisi Afariogun
Department of Mathematical Sciences,
Faculty of Natural Sciences,
Ajayi Crowther University,
Oyo, Nigeria.
email: *da.afariogun@acu.edu.ng*

Adesanmi Alao Mogbademu
Department of Mathematics,
Faculty of Science,
University of Lagos,
Lagos, Nigeria.
email: *amogbademu@unilag.edu.ng*