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THE BALANCING-LIKE SEQUENCES IN GROUPS

E. K. Çetinalp, N. Yilmaz, Ö. Deveci

Abstract. In this paper, we extend the balancing sequences to groups. In the first part of paper, we study the balancing sequences in 2-generator groups and then give some useful results concerning the periods of the balancing sequences in finite groups. Also, we calculate the periods of balancing sequences in the polyhedral groups $(2,2,2)$, $(n,n,2)$, $(2,n,2)$ and $(2,2,n)$ with respect to the generating pair (x,y). In the later part of the paper, to improve balancing sequences, we define the k-step balancing sequences and examine the periods of these sequences when read modulo m. Furthermore, we redefine the k-step balancing sequences by means of group elements and then examine them in finite groups. Finally, we compute the periods of the 3-step balancing sequences in the polyhedral groups $(2,2,2)$, $(2,2,n)$, $(2,n,2)$ and $(2,2,n)$ with respect to the generating triple (x,y,z) applications of the results obtained.

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1. Introduction and Preliminaries

Since the beginning of time, mathematicians have been drawn to the study of number sequences. Since then, a lot of them have focused their attention on the intriguing topic of triangular numbers. In 1999, Behera and Panda [1] introduced the notion of balancing numbers $(B_n)_{n\in\mathbb{N}}$ as solutions to a certain Diophantine equation. Then, the recurrence relation of this number is $B_{n+1} = 6B_n - B_{n-1}$ for $n \geqslant 1$, where $B_0 = 0$, $B_1 = 1$. A study on the Lucas-balancing numbers $C_n = \sqrt{8B_n^2 + 1}$ was published in 2006 by Panda [14]. The recurrence relation of this number is $C_{n+1} = 6C_n - C_{n-1}$ for $n \geqslant 1$, where $C_0 = 1$, $C_1 = 3$. Also, the authors examined the periodicity of these numbers in [15, 16].

Additionally, the matrices can be used to represent the balancing numbers and can be extended to related sequences. In [17], the author introduced balancing Q-matrix as follows

$$Q = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$$

and gave a general formula for the nth powers of this matrix

$$Q^n = \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix}.$$

Kalman [11] said that these sequences are particular examples of the sequence, which is defined recursively as a linear combination of the preceding k-step terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + c_2 a_{n+2} + \cdots + c_{k-2} a_{n+k-2} + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, c_2, \ldots, c_{k-1}$ are real constants. In [11], Kalman derived a number of closed-form formulas for the generalized sequence by companion matrix method as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix}_{k \times k} . \tag{1}$$

By inductive argument it is obtained

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{k-2} \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \\ \vdots \\ a_{n+k-2} \\ a_{n+k-1} \end{bmatrix} \tag{2}$$

for $n \geq 0$.

Let G be a finite k-generator group and let X be the subset of $\underbrace{G \times G \times \cdots \times G}_{k}$ such that $x_1, x_2, \ldots, x_k \in X$ if and only if G is generated by $x_1, x_2, \ldots, x_k$. We call $(x_1, x_2, \ldots, x_k)$ a generating k-tuple for G.

For a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \ldots, a_n\}$, the sequence $x_u = a_{u+1}$, $0 \leq u \leq n-1$, $x_{n+u} = \prod_{v=1}^{n} x_{u+v-1}$, $u \geq 0$ is called the Fibonacci orbit of G with respect to the generating set A, denoted $F_A(G)$ (see [2, 3, 4]).

A k-nacci (k-step Fibonacci) sequence in a finite group is a sequence of group elements $x_0, x_1, x_2, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, x_1, x_2, \ldots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \cdots x_{n-1} & \text{for } j \leq n < k, \\ x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n \geq k. \end{cases}$$

The k-nacci sequence is forced to mirror the structure of the group since we also demand that the beginning elements of the sequence $x_0, x_1, x_2, \ldots, x_{j-1}$ generate the group. The k-nacci sequence of a group G generated by $x_0, x_1, x_2, \ldots, x_{j-1}$ is denoted by $F_k(G; x_0, x_1, x_2, \ldots, x_{j-1})$ in [13].

Notice that the orbit of a k-generated group is a k-nacci sequence.

Definition 1. *The polyhedral (triangle) group (l, m, n) for $l, m, n > 1$ is defined by the presentation*

$$\left\langle x, y, z \mid x^l = y^m = z^n = xyz = 1 \right\rangle,$$

or

$$\left\langle x, y \mid x^l = y^m = (xy)^n = 1 \right\rangle.$$

The polyhedral group (l, m, n) is finite if and only if the number

$$\mu = lmn \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1 \right) = mn + nl + lm - lmn$$

is positive, that is, in the cases $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$. Its order is $2lmn/\mu$. Also, A_4, S_4 and A_5 are the groups $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$, respectively. By using Tietze transformations, we may show that $(l, m, n) \cong (m, n, l) \cong (n, l, m)$. For more information on these groups see [5, 6].

A sequence is considered periodic if, after a given point, all it consists of is repetitions of a predetermined subsequence. The period of the sequence is equal to the number of elements in the repeating subsequence. In [18], the investigation of common Fibonacci sequences in cyclic groups served as the foundation for the research of linear recurrence sequences in groups. Many writers have recently examined various unique groupings of linear recurrence sequences; for instance, [7, 8, 9, 10, 12, 13, 19] .

In Section 2, we study the balancing sequences in 2-generator groups and obtain some results concerning the periods of the balancing sequences in finite groups. Then, we calculate the periods of balancing sequences in the polyhedral groups $(2, 2, 2)$, $(n, n, 2)$, $(2, n, 2)$ and $(2, 2, n)$ with respect to the generating pair (x, y). In Section 3, we define the k-step balancing sequences and examine the periods of these sequences when read modulo m. Additionally, we redefine the k-step balancing sequences by means of group elements and then examine them in finite groups. Finally, we compute the periods of the 3-step balancing sequences in the polyhedral groups $(2,2,2)$, $(2,2,n)$, $(2,n,2)$ and $(2,2,n)$ with respect to the generating triple (x,y,z).

2. The Balancing Sequences in Finite Groups

In this section, we define the balancing orbit of a 2-generator group. Then, we obtain some results concerning the periods of the balancing orbits of finite 2-generator groups. Finally, we calculate the lengths of the periods of balancing orbits of the polyhedral groups $(2,2,2)$, $(n,2,2)$, $(2,n,2)$ and $(2,2,n)$ for the generating pair (x,y).

Definition 2. *Let G be a 2-generator group. For the generating pair (x,y), we define the balancing orbit of G is as follows:*

$$x_0 = x,\ x_1 = y,\ x_{i+2} = x_i^{-1} x_{i+1}^6,\ (i \geq 0)\,.$$

For the generating pair (x,y), we denote the balancing orbit of the group G by $B_{x,y}(G)$.

Theorem 1. *Let G be a 2-generator group and let (x,y) be a generating pair of G. If G is finite, then the sequence $B_{x,y}(G)$ is simply periodic.*

Proof. Let n be the order of G. Since there n^2 distinct 2-tuples of elements of G, at least one of the 2-tuples appears twice in a balancing orbit of G. Thus, the subsequence following this 2-tuples. Because of the repeating, the balancing orbit is periodic. Since the balancing orbit periodic, there exist natural numbers i and j, with $i > j$, such that $x_{i+1} = x_{j+1}$ and $x_{i+2} = x_{j+2}$. By the defining relation of a balancing orbit, we know that

$$x_i = (x_{i+1})^6 (x_{i+2})^{-1} \qquad \text{and} \qquad x_j = (x_{j+1})^6 (x_{j+2})^{-1}.$$

Then, from $x_i = x_j$ and it follows that

$$x_{i-j} = x_{j-i} = x_0 \qquad \text{and} \qquad x_{i-j+1} = x_{j-i+1} = x_1.$$

Thus, the balancing orbit is simply periodic.

The length of the period of the balancing orbit $B_{x,y}(G)$ is denoted by $LB(G,x,y)$. It is called the balancing length of G with respect to generating pair (x,y).

Theorem 2. *Let $\mathbb{Z}_n$ and $\mathbb{Z}_m$ be finite cyclic groups generated by x and y, respectively. Then the balancing length of $\mathbb{Z}_n \times \mathbb{Z}_m$ equals the least common multiple of the balancing lengths of the groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$.*

Proof. For the groups $\mathbb{Z}_n = \langle x \mid x^n = 1\rangle$ and $\mathbb{Z}_m = \langle y \mid y^m = 1\rangle$, we get the presentation of direct product $\mathbb{Z}_n \times \mathbb{Z}_m$ as followings

$$\left\langle x, y \mid x^n = y^m = (xy)^2 = 1 \right\rangle.$$

The balancing orbit of $\mathbb{Z}_n \times \mathbb{Z}_m$ is

$$x_0 = x,\ x_1 = y,\ x_2 = y^6x^{-1},\ x_3 = y^{35}x^{-6},\ x_4 = y^{204}x^{-35}, \ldots.$$

Now the proof is finished when we note that the balancing orbit will repeat when $x_\alpha = x$ and $x_{\alpha+1} = y$. By examining this statement in more detail gives

$$\begin{aligned} y^{B(\alpha)}x^{-B(\alpha-1)} &= x, \\ y^{B(\alpha+1)}x^{-B(\alpha)} &= y. \end{aligned}$$

The least non-trivial integer satisfying the above conditions occurs when
$\alpha = \operatorname{lcm}[kB(n), kB(m)]$ ($kB(n)$ is defined as in [15]).

Now we compute the balancing lengths of polyhedral groups $(2, 2, 2)$, $(2, n, 2)$ and $(2, 2, n)$ for the generating pair (x, y).

Theorem 3. *For $n > 2$, we obtain the balancing lengths of polyhedral groups $(2, 2, 2)$, $(n, 2, 2)$, $(2, n, 2)$ and $(2, 2, n)$ as follows:*

$$\begin{aligned} \textbf{i)}\ LB((2,2,2), x, y) &= LB((2,2,n), x, y) = 2, \\ \textbf{ii)}\ LB((2,n,2), x, y) &= LB((n,2,2), x, y) = 4. \end{aligned}$$

Proof. **i)** The polyhedral group $(2, 2, 2)$ defined by the presentation
$\left\langle x, y \mid x^2 = y^2 = (xy)^2 = 1 \right\rangle$ has the balancing orbit as follows:

$$x_0 = x, x_1 = y, x_2 = x^{-1}, x_3 = y^{-1}, \ldots.$$

So, $LB((2,2,2), x, y) = 2$. Similarly, the polyhedral group $(2, 2, n)$ is presented by $\left\langle x, y \mid x^2 = y^2 = (xy)^n = 1 \right\rangle$. The balancing orbit of this group is

$$x_0 = x, x_1 = y, x_2 = x, x_3 = y, \ldots.$$

Then, $LB((2,2,n), x, y) = 2$.
ii) The polyhedral group $(2, n, 2)$ defined by the presentation $\left\langle x, y \mid x^2 = y^n = (xy)^2 = 1 \right\rangle$ has the balancing orbit as follows:

$$x_0 = x, x_1 = y, x_2 = xy^6, x_3 = y^{-1}, x_4 = x, x_5 = y, x_6 = xy^6, \ldots.$$

Then, $LB((2,n,2),x,y)=4$. In a similar way, for the presentation $\langle x,y \mid x^n=y^2=(xy)^2=1\rangle$, we have following the balancing orbit:

$$x_0=x, x_1=y, x_2=x^{-1}, x_3=yx^{-6}, x_4=x, x_5=y, x_6=x^{-1}, \ldots .$$

So, $LB((n,2,2),x,y)=4$.

3. k-step Balancing Sequences

In this section, firstly, we define the k-step balancing sequences and its generating matrix and then give the relationship between the elements of these sequences and the generating matrix. Next, we study the k-step balancing sequences modulo m. After that, we extend the concept to groups and then obtain the lengths of the periods of the k-step balancing sequences in the polyhedral groups $(2,2,2)$, $(2,n,2)$, $(n,2,2)$ and $(2,2,n)$ considering 3-generator case by the aid of the periods of some special sequences according to modulo m.

Definition 3. *The k-step balancing sequences are defined as*

$$B_k(n+k)=6B_k(n+k-1)-B_k(n+k-2)+B_k(n+k-3)+\cdots+B_k(n), \quad (3)$$

where $B_k(u)=0, B_k(k-1)=1$, $0\le u<k-1$ and $n\ge 0$.

- By taking $k=2$ in the equation (3), these sequences reduces to the usual balancing sequence $\{B_2(n)\}$ in OEIS A001109.

In equation (3), we may write the following companion matrix:

$$C_k=[c_{ij}]_{k\times k}=\begin{bmatrix} 6 & -1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

The matrix C_k is said to be the k-step balancing matrix and we have

$$\begin{bmatrix} B_k(n) \\ B_k(n-1) \\ B_k(n-2) \\ \vdots \\ B_k(n-k+2) \\ B_k(n-k+1) \end{bmatrix} = \begin{bmatrix} 6 & -1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} B_k(n-1) \\ B_k(n-2) \\ B_k(n-3) \\ \vdots \\ B_k(n-k+1) \\ B_k(n-k) \end{bmatrix}.$$

By inductive argument, we can write

$$C_k^n = \begin{bmatrix} B_k(n+k-1) & B_k(n+k)-6B_k(n+k-1) & \cdots & B_k(n+k-2) \\ B_k(n+k-2) & B_k(n+k-1)-6B_k(n+k-2) & \cdots & B_k(n+k-3) \\ B_k(n+k-3) & B_k(n+k-2)-6B_k(n+k-3) & \cdots & B_k(n+k-4) \\ \vdots & \vdots & \ddots & \vdots \\ B_k(n+1) & B_k(n+2)-6B_k(n+1) & \cdots & B_k(n) \\ B_k(n) & B_k(n+1)-6B_k(n) & \cdots & B_k(n-1) \end{bmatrix}.$$

Then we get the following matrix relation:

$$C_k^n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} B_k(n+k-2) \\ B_k(n+k-3) \\ B_k(n+k-4) \\ \vdots \\ B_k(n) \\ B_k(n-1) \end{bmatrix}$$

for $n \geq 1$.

3.1. k-step Balancing Sequences Modulo m

In this subsection, we study the k-step balancing sequences modulo m and then show that they are simply periodic sequences for any $k \geq 3$.

Reducing the k-step balancing sequences modulo m, we can get a repeating sequence, denoted by

$$\{B_{k,m}(n)\} = \{B_{k,m}(0), B_{k,m}(1), \ldots, B_{k,m}(k), \ldots\}$$

where $B_{k,m}(i) \equiv B_k(i) \pmod{m}$ and it has the same recurrence relation as in (3).

Theorem 4. *$\{B_{k,m}(n)\}$ is a simply periodic sequence.*

Proof. Let $A_{k+1} = \{(a_0, a_1, \ldots, a_k) \mid 0 \leq a_i \leq m-1,\ 0 \leq i \leq k\}$. Since there m^{k+1} distinct of elements of A_{k+1}, then we have $|A_{k+1}| = m^{k+1}$. For any $i \geq 0$, there exist $j \geq i+k$ such that

$$B_{k,m}(i+1) = B_{k,m}(j+1),\ B_{k,m}(i+2) = B_{k,m}(j+2), \ldots, B_{k,m}(i+p+1) = B_{k,m}(j+p+1),$$

where $0 \leq p \leq k-1$. From definition of the k-step balancing sequences we have $B_k(n) = B_k(n+k) - 6B_k(n+k-1) + B_k(n+k-2) - B_k(n+k-3) - \cdots - B_k(n+1)$ so if $B_{k,m}(i) = B_{k,m}(j),\ B_{k,m}(i-1) = B_{k,m}(j-1), \ldots, B_{k,m}(1) = B_{k,m}(j-i+1),\ B_{k,m}(0) = B_{k,m}(j-i)$, which implies that these sequences are simply periodic.

Example 1. *We have*

$\{B_{4,3}(n)\} = \{0,0,0,1,0,2,1,2,1,1,2,2,0,1,1,1,0,1,2,0,2,0,0,2,2,1,0,0,0,1,0,2,\ldots\}$,

and then repeat. So, we get $kB_{4,3}(n) = 26$.

Given any integer matrix $A = [a_{ij}]_{k\times k}$, $A \,(\bmod m)$ means that all entries of A are modulo m, that is $A\,(\bmod m) = (a_{ij}(\bmod m))$. Let us consider the set $\langle A\rangle_m = \{(A)^n(\bmod m) \mid n \geq 0\}$. In here, If $(\det A, m) = 1$, then the set $\langle A\rangle_m$ is a cyclic group. Since $\det C_k^n = (-1)^{n(k+1)}$, the set $\langle C_k\rangle_m$ is a cyclic group for every positive integer $k > 2$. From the above matrix relation, it is easy to see that the order of the set $\langle C_k\rangle_m$ equals to the period of $\{B_{k,m}(n)\}$.

3.2. k-step Balancing Sequences in Groups

In this subsection, we redefine the k-step balancing sequences by means of group elements and then examine them in finite groups. Further, we define some particular sequences and determine the periods of those according to modulo m with respect to various matrix relations. Finally, we give the lengths of the periods of the k-step balancing sequences in the polyhedral groups $(2,2,2)$, $(2,n,2)$, $(n,2,2)$ and $(2,2,n)$ for the generating triple (x,y,z) by using the periods of the defined those particular sequences.

Let G be a 3-generator group. For the generating triple (x,y,z), we define the balancing orbit of G is as follows:

$$x_0 = x,\, x_1 = y,\, x_2 = z,\, x_{i+3} = x_i x_{i+1}^{-1} x_{i+2}^{6},\, (i \geq 0)\,.$$

For the generating triple (x,y,z), we denote the balancing orbit of the group G by $B_{x,y,z}(G)$.

Definition 4. *A k-step balancing sequence in a finite group is a sequence of group elements $a_0, a_1, \ldots, a_n, \ldots,$ for which, given an initial set $a_0 = x_0$, $a_1 = x_1$, $a_2 = x_2, \ldots, a_{j-1} = x_{j-1}$, $a_j = x_j$, each element is defined by*

$$a_n = \begin{cases} a_0 a_1 a_2 \cdots a_{n-3} a_{n-2}^{-1} a_{n-3}^{6} & j < n < k \\ a_{n-k} a_{n-k+1} \cdots a_{n-3} a_{n-2}^{-1} a_{n-3}^{6} & n \geq k \end{cases}.$$

It is require that the initial elements of the sequence $x_0, x_1, x_2, \ldots, x_{j-1}$ generate the group, thus, forcing the k-step balancing sequences to reflect the structure of the group. The k-step balancing sequences in a group G generated by $x_0, x_1, x_2, \ldots, x_{j-1}$ is denoted by $B_{x_0,x_1,\ldots,x_{j-1}}(G)$.

Theorem 5. *A k-step balancing sequence in a finite group is periodic.*

Proof. Let G be a finite group and $|G|$ be the order of G. Since there are $|G|^{k+1}$ distinct $k+1$-tuples of elements of the group G, at least one of the $k+1$-tuples appears twice in a k-step balancing sequences in the group G. Because of the repeating, the k-step balancing sequences are periodic.

Let $LB_{x_0,x_1,\ldots,x_{j-1}}(G)$ denote the length of the period of the sequence $B_{x_0,x_1,\ldots,x_{j-1}}(G)$. From the definitions, it is clear that the periods of the sequences $B_{x_0,x_1,\ldots,x_{j-1}}(G)$ for $i \geq 2$ depend on the chosen generating set and the order in which the assignments of x_0, x_1, $x_2, \ldots, x_{j-1}$.

Theorem 6. *The group defined by the presentation* $\left\langle x, y, z \mid x^2 = y^2 = z^2 = xyz = 1 \right\rangle$ *has the balancing length* $LB_{x,y,z}((2,2,2)) = 7$.

Proof. It is important to note that $x = zy$, $y = xz$ and $z = xy$. By a simple calculation, we obtain the balancing orbit of the polyhedral $(2,2,2)$ as shown:

$$x_0 = x, x_1 = y, x_2 = z = xy, x_3 = z, x_4 = x, x_5 = 1, x_6 = zx, x_7 = x, x_8 = y, x_9 = z, \ldots.$$

So, $B_{x,y,z}((2,2,2)) = 7$.

In the following Theorem 7, 8 and Theorem 9, a generator matrix will be obtained and the solution with cyclic groups will be provided to calculate the periods of the k-step balancing sequences in the polyhedral groups $(2,n,2)$, $(n,2,2)$ and $(2,2,n)$.

Consider the sequences

$$a_0 = 0,\ a_1 = 1,\ a_2 = 1, a_n = \begin{cases} 6a_{n-1} - a_{n-2} + a_{n-3} & \text{if} \quad n \equiv 0\ (7), \\ -a_{n-2} + a_{n-3} & \text{if} \quad n \equiv 1\ (7), \\ 6a_{n-1} - a_{n-2} + a_{n-3} & \text{if} \quad n \equiv 2\ (7), \\ -a_{n-2} + a_{n-3} & \text{if} \quad n \equiv 3\ (7), \\ a_{n-2} - a_{n-3} & \text{if} \quad n \equiv 4\ (7), \\ a_{n-2} - a_{n-3} & \text{if} \quad n \equiv 5\ (7), \\ 6a_{n-1} + a_{n-2} - a_{n-3} & \text{if} \quad n \equiv 6\ (7), \end{cases}$$

$$b_0 = 1,\ b_1 = 0,\ b_2 = 1, b_n = \begin{cases} -b_{n-2} + b_{n-3} & \text{if} \quad n \equiv 0\ (7), \\ b_{n-2} - b_{n-3} & \text{if} \quad n \equiv 1\ (7), \\ b_{n-2} - b_{n-3} & \text{if} \quad n \equiv 2\ (7), \\ 6b_{n-1} + b_{n-2} - b_{n-3} & \text{if} \quad n \equiv 3\ (7), \\ 6b_{n-1} - b_{n-2} + b_{n-3} & \text{if} \quad n \equiv 4\ (7), \\ -b_{n-2} + b_{n-3} & \text{if} \quad n \equiv 5\ (7), \\ 6b_{n-1} + b_{n-2} - b_{n-3} & \text{if} \quad n \equiv 6\ (7), \end{cases}$$

and

$$c_0=1,\ c_1=-1,\ c_2=0, c_n=\begin{cases} -c_{n-2}+c_{n-3} & \text{if} \quad n\equiv 0\ (7), \\ -6c_{n-1}+c_{n-2}-c_{n-3} & \text{if} \quad n\equiv 1\ (7), \\ c_{n-2}+c_{n-3} & \text{if} \quad n\equiv 2\ (7), \\ c_{n-2}+c_{n-3} & \text{if} \quad n\equiv 3\ (7), \\ -c_{n-2}+c_{n-3} & \text{if} \quad n\equiv 4\ (7), \\ 6c_{n-1}-c_{n-2}+c_{n-3} & \text{if} \quad n\equiv 5\ (7), \\ -6c_{n-1}+c_{n-2}+c_{n-3} & \text{if} \quad n\equiv 6\ (7), \end{cases}$$

where $n\geq 3$. It is easy to prove that the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ for modulo m are periodic. Reducing the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ by a modulus m, then we get the repeating sequences, respectively denoted by

$$\{a_n(m)\}=\{a_0(m), a_1(m), \ldots a_\tau(m), \ldots\},$$

$$\{b_n(m)\}=\{b_0(m), b_1(m), \ldots b_\tau(m), \ldots\}$$

and

$$\{c_n(m)\}=\{c_0(m), c_1(m), \ldots c_\tau(m), \ldots\}.$$

They have the same recurrence relation as in the definitions of the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$. We denote the lengths of the periods of the sequences $\{a_n(m)\}$, $\{b_n(m)\}$ and $\{c_n(m)\}$ by $h_{a_n}(m)$, $h_{b_n}(m)$ and $h_{c_n}(m)$, respectively. Let us consider the generating matrix

$$A_1=A_6=\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A_2=A_3=\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A_4=A_7=\begin{bmatrix} 6 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A_5=\begin{bmatrix} 6 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$A'_1=A'_4=\begin{bmatrix} 6 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A'_2=\begin{bmatrix} 6 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A'_3=A'_5=\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A'_6=A'_7=\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$A''_1=A''_7=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A''_2=A''_5=\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A''_3=\begin{bmatrix} 6 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A''_4=A''_6=\begin{bmatrix} -6 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

By direct calculation it is easy to see that the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ conform to the following patterns:

$$A_1\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}a_3\\a_2\\a_1\end{bmatrix},\ A_2A_1\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}a_4\\a_3\\a_2\end{bmatrix},\ldots,A_7A_6A_5A_4A_3A_2A_1\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}a_9\\a_{10}\\a_{11}\end{bmatrix},$$
$$A_1A_7A_6\cdots A_1\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}a_{10}\\a_{11}\\a_{12}\end{bmatrix},\ldots,A_6A_5\cdots A_1A_7A_6\cdots A_1\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}a_{15}\\a_{16}\\a_{17}\end{bmatrix},$$
$$A_7A_6\cdots A_1A_7A_6\cdots A_1\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}a_{16}\\a_{17}\\a_{18}\end{bmatrix},\ A_1A_7A_6\cdots A_1A_7A_6\cdots A_1\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}a_{17}\\a_{18}\\a_{19}\end{bmatrix},\ldots\ ,$$

$$A_1^{'}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}b_3\\b_2\\b_1\end{bmatrix},\ A_2^{'}A_1^{'}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}b_4\\b_3\\b_2\end{bmatrix},\ldots,\ A_7^{'}A_6^{'}\cdots A_1^{'}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}b_9\\b_{10}\\b_{11}\end{bmatrix},$$
$$A_1^{'}A_7^{'}A_6^{'}\cdots A_1^{'}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}b_{10}\\b_{11}\\b_{12}\end{bmatrix},\ldots,A_6^{'}A_5^{'}\cdots A_1^{'}A_7^{'}A_6^{'}\cdots A_1^{'}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}b_{15}\\b_{16}\\b_{17}\end{bmatrix},$$
$$A_7^{'}A_6^{'}\cdots A_1^{'}A_7^{'}A_6^{'}\cdots A_1^{'}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}b_{16}\\b_{17}\\b_{18}\end{bmatrix},\ A_1^{'}A_7^{'}A_6^{'}\cdots A_1^{'}A_7^{'}A_6^{'}\cdots A_1^{'}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}b_{17}\\b_{18}\\b_{19}\end{bmatrix},\ldots$$

and

$$A_1^{''}\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}c_3\\c_2\\c_1\end{bmatrix},\ A_2^{''}A_1^{''}\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}c_4\\c_3\\c_2\end{bmatrix},\ldots,\ A_7^{''}A_6^{''}\cdots A_1^{''}\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}c_9\\c_{10}\\c_{11}\end{bmatrix},$$
$$A_1^{''}A_7^{''}A_6^{''}\cdots A_1^{''}\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}c_{10}\\c_{11}\\c_{12}\end{bmatrix},\ldots,\ A_6^{''}A_5^{''}\cdots A_1^{''}A_7^{''}A_6^{''}\cdots A_1^{''}\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}c_{15}\\c_{16}\\c_{17}\end{bmatrix},$$
$$A_7^{''}A_6^{''}\cdots A_1^{''}A_7^{''}A_6^{''}\cdots A_1^{''}\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}c_{16}\\c_{17}\\c_{18}\end{bmatrix},\ A_1^{''}A_7^{''}A_6^{''}\cdots A_1^{''}A_7^{''}A_6^{''}\cdots A_1^{''}\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}c_{17}\\c_{18}\\c_{19}\end{bmatrix},\ldots\ .$$

Suppose that $A_7A_6A_5A_4A_3A_2A_1=B$, $A_7^{'}A_6^{'}A_5^{'}A_4^{'}A_3^{'}A_2^{'}A_1^{'}=B^{'}$ and $A_7^{''}A_6^{''}A_5^{''}A_4^{''}A_3^{''}A_2^{''}A_1^{''}=B^{''}$. Using the above, we define the following matrices:

$$M^n=A_uA_{u-1}\cdots A_1B^k,$$

$$\left(M^{'}\right)^n=A_u^{'}A_{u-1}^{'}\cdots A_1^{'}\left(B^{'}\right)^k$$

and

$$\left(M^{''}\right)^n=A_u^{''}A_{u-1}^{''}\cdots A_1^{''}\left(B^{''}\right)^k$$

where $n = 7k + u$ such that $u,\ k, \in \mathbf{N}$. So we get

$$M^n \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{pmatrix},$$

$$\left(M'\right)^n \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b_{n+2} \\ b_{n+1} \\ b_n \end{pmatrix}$$

and

$$\left(M''\right)^n \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_{n+2} \\ c_{n+1} \\ c_n \end{pmatrix}$$

for $n = 7k + u$ such that $u,\ k, \in \mathbf{N}$. From these equations we immediately deduce:

- $h_{a_n}(m)$ is the smallest positive integer α such that $M^\alpha \equiv I(\text{mod } m)$.
- $h_{b_n}(m)$ is the smallest positive integer α such that $\left(M'\right)^\alpha \equiv I(\text{mod } m)$.
- $h_{c_n}(m)$ is the smallest positive integer α such that $\left(M''\right)^\alpha \equiv I(\text{mod } m)$.

Now we give the lengths of the periods of the sequences $B_{x,y,z}((2,n,2))$, $B_{x,y,z}((2,2,n))$ and $B_{x,y,z}((n,2,2))$ by the aid of the above useful results.

Theorem 7. *For $n > 2$,*

$$LB((2,n,2), x, y, z) = h_{a_n}(n).$$

Proof. The polyhedral group $(2,n,2)$ is defined by the presentation $\langle x, y, z \mid x^2 = y^n = z^2 = xyz = 1 \rangle$, then the balancing orbit of $(2,n,2)$ is as follows:

$$\begin{aligned}
&x_0 = x, x_1 = y, x_2 = xy, x_3 = xy^{-1}, x_4 = x, x_5 = y^{-2}, x_6 = y^{-11},\\
&x_7 = xy^{-64}, x_8 = y^9, x_9 = xy, x_{10} = xy^{-73}, x_{11} = xy^{-8}, x_{12} = y^{-74}, x_{13} = y^{-379},\\
&x_{14} = xy^{-2208}, x_{15} = y^{305}, x_{16} = xy, x_{17} = xy^{-2513}, x_{18} = xy^{-304}, x_{19} = y^{-2514}, x_{20} = y^{-12875},\\
&x_{21} = xy^{-75040}, x_{22} = y^{10361}, x_{23} = xy, x_{24} = xy^{-85401}, x_{25} = xy^{-10360}, x_{26} = y^{-85402}, x_{27} = y^{-437371},\\
&\ldots.
\end{aligned}$$

By direct calculation it is easy to see that the sequence $B_{x,y,z}((2,n,2))$ conforms to the following pattern:

$$x_0 = xy^{a_0}, x_1 = y^{a_1}, x_2 = xy^{a_2}, x_3 = xy^{a_3}, x_4 = xy^{a_4}, x_5 = y^{a_5}, x_6 = y^{a_6}, \ldots.$$

Since the sequence $\{a_n\}$ appear as the power of y and the order of y is n, the period of the sequence $\{a_n(n)\}$ with the balancing length of group $(2,n,2)$ are the same.

So we have the conclusion.

Theorem 8. *For $n > 2$, the balancing length of the polyhedral group $(2,2,n)$ is $h_{b_n}(n)$.*

Proof. We prove this by direct calculation. We first note that in the group defined by $\langle x, y, z \mid x^2 = y^2 = z^n = xyz = 1 \rangle$, $x = yz$, $y = zx$ and $z = yx$. We have the sequence

$$x, y, z, z^5, yz^{29}, z^{-4}, yz^{33}, yz^4, z^{33}, z^{169}, yz^{985}, z^{-136}, yz^{680}, yz^{1121}, yz^{816}, z^{441}, z^{2341}, yz^{14421}, z^{-1900}, yz^{680}, yz^{16321}, yz^{2580}, z^{15641}, z^{93846}, z^{80105}, yz^{467569}, z^{-64464}, yz^{680}, \ldots.$$

So we get

$$x_0 = yz^{b_0}, x_1 = yz^{b_1}, x_2 = z^{b_2}, x_3 = z^{b_3}, x_4 = yz^{b_4}, x_5 = z^{b_5}, x_6 = yz^{b_6}, \ldots.$$

Since the sequence $\{b_n\}$ appear as the power of z and the order of z is n, we obtain

$$LB((2, 2, n), x, y, z) = h_{b_n}(n).$$

Theorem 9. *The group defined by the presentation* $\langle x, y, z \mid x^n = y^2 = z^2 = xyz = 1 \rangle$ *has the balancing length* $h_{c_n}(n)$ *for* $n > 2$.

Proof. It is important to note that $x = zy$, $y = x^{-1}z$ and $z = xy$. By a simple calculation, we obtain the balancing orbit of the polyhedral $(2, 2, n)$ as shown:

$$x, y, z, z, x^{-1}, x^{-6}, x^{35}z, x^5, x^{-1}z, x^{40}z, x^4z, x^{-41}, x^{-210}, x^{1223}z, x^{169}, x^{-1}z, x^{1392}z, x^{168}z, x^{-1393}, x^{-7134}, x^{41579}z, x^{5741}, x^{-1}z, x^{47320}z, x^{5740}z, x^{-47321}, x^{-242346}, x^{1412495}z, x^{195025}, x^{-1}z, x^{1607520}z, x^{195024}z, \ldots.$$

By using the balancing orbit given above, we derive

$$x_0 = x^{c_0},\ x_1 = x^{c_1}z,\ x_2 = x^{c_2}z,\ x_3 = x^{c_3}z,\ x_4 = x^{c_4},\ x_5 = x^{c_5},\ x_6 = x^{c_6}z, \ldots.$$

Therefore, considering the order of x, we conclude that

$$LB((n, 2, 2), x, y, z) = h_{c_n}(n).$$

Now we concentrate on finding the lengths of the periods of the balancing orbits of the polyhedral groups $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$ for the 2-generator (x, y) and the 3-generator (x, y, z). The results are summarized in the following table:

G_n	$LB_{(x,y)}(G_n)$	$LB_{(x,y,z)}(G_n)$
$(2, 3, 3)$	4	40
$(2, 3, 4)$	4	294
$(2, 3, 5)$	4	2588

These calculations are obtained by using the program "*Magma Computer Algebra*".

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Esra Kırmızı Çetinalp
Department of Mathematics, Kamil Özdağ Science Faculty,

Karamanoğlu Mehmetbey University,
Karaman, Turkey
email: *esrakirmizi@kmu.edu.tr*

Nazmiye Yılmaz
Department of Mathematics, Kamil Özdağ Science Faculty,
Karamanoğlu Mehmetbey University,
Karaman, Turkey
email: *yilmaznzmy@gmail.com.tr*

Ömür Deveci
Department of Mathematics, Faculty of Science and Letter,
Kafkas University,
Kars, Turkey
email: *odeveci36@hotmail.com.tr*