Acta Universitatis Apulensis No. 73/2023
ISSN: 1582-5329 pp. 101-138
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2023.73.08

NATURAL CO-ORDER AND SOME CO-CONGRUENCES ON INVERSE SEMIGROUP WITH APARTNESS

Daniel A. Romano

Abstract. In classical semigroup theory, due attention is paid to inverse semigroups. Research into this class of semigroups with apartness within the Bishop's constructive framework began in 2019 with paper by A. Cherubini and A. Frigeri (*Inverse semigroups with apartness.* Semigroup Forum, 98(3)(2019), 571-–588). In this paper, relying on the mentioned article, in addition to designing the concept of natural co-order relation on such semigroups, special attention is paid to to concepts of co-congruences and some examples of co-congruences in such semigroups as semilattice and group co-congruences. It is interesting to note that for both these co-congruences one can construct two different quotient structures.

2010 *Mathematics Subject Classification*: 03F65, 20M18.

Keywords: Set with apartness, Semigroup with apartness, Inverse semigroup with apartness, Natural co-order on an inverse semigroup with apartness, Co-congruence on an inverse semigroup with apartness, Group/semilattice co-congruence on an inverse semigroup with apartness.

1. Introduction

According to Lawson [18], inverse semigroups were introduced in the 1950s by Ehresmann in France ("Much of Charles Ehresmann's work in the 1940s and 1950s was also concerned with inverse semigroup theory, although written from a different mathematical perspective." cited by [19], page 1.), Preston [28] in the UK and Wagner [44] in the Soviet Union as algebraic analogues of pseudogroups of transformations. Inverse semigroups and their properties are in the focus of the interest of many researchers (for example, see articles [13], [18], [23], [25], [40], [41], [43] and books [8], [16], [19], [21]).

Semigroups with apartness are the subject of researching of some mathematicians in the last twenty years (for example [10]-[12], [34], [37]-[39]). The concept of inverse semigroups with apartness within Bishop's constructive framework was introduced and analyzed in 2019 by A. Cherubini and A. Frigeri in article [7]. Previously there were several papers studying special types of inverse semigroups with apartness (see for instance the wide literature on groups with apartness and the papers on semilattice-ordered semigroups [24], [35], [36]).

One of the goals in this paper is to show that in an inverse semigroup with apartness $(S, =, \neq, \cdot)$ one can naturally determine the co-order relation that co-compatible with the multiplication in S. In addition, we analyze the conditions for one co-congruence q on an inverse semigroup with apartness S to be a semilattice co-congruence on S (group co-congruence on S, res.). The specificity of these analyzes within the Bishop's constructive framework is that for one semilattice co-congruence (group co-congruence, res.) q two semilattices with apartness (two group with apartness, res.) can be designed.

This paper is organized as the follows: The Preliminaries section, which comes after this the Introduction, consists of four subsections. The Subsection 2.1 contains the necessary concepts for a reader to feel comfortable in Bishop's constructive framework, which is is the logical environment. The Subsection 2.2 contains terms, mostly taken from the article [7], that a refer to inverse semigroups with apartness. One example (Example 1) is designed to show that a semigroup with apartness does not have to be an inverse semigroup with apartness. In the subsection 2.3 the concept of homomorphisms between inverse semigroups with apartness is presented. In addition to statements taken from the literature [7] or from sources related to the classical theory of inverse semigroups, the following statements Proposition 1, Proposition 2 and Theorem 7 appear for the first time. The Section 3 and the Section 4 are the central part of this paper. In the Section 3, the concept of co-orders on an inverse semigroup with apartness is designed (Definition 4) and shows that the relation designed in this way is compatible with the operations in that semigroup (Theorem 12). Finally, in this section, the properties of the semilattice $E(S)$ of all idempotents of an inverse semigroup with apartness S and its constructive dual $E(S)^{\triangleleft}$ are discussed. Section 4 is devoted to co-congruences on inverse semigroups with apartness. Several examples of co-congruences, which should illustrate the difficulties in designing co-congruences in an inverse semigroup with apartness, were analyzed. In addition, the conditions for one co-congruence on an inverse semigroup with apartness to be a group co-congruence (Subsection 4.4, Theorem 24), i.e. - respectively, to be a semilattice co-congruence (Subsection 4.5), are discussed.

2. Preliminaries

This report is within the framework of Bishop's constructive mathematics [1], [4], [5], [6], [20], [22]. In it, as a continuation of previous research (for example, [7],

[10], [11], [34], [37]), we show some of the fundamental features of semigroups and semigroups with apartness. The notions and notations used in this paper are mostly taken from [7], [16], [18], [21], [23], [25], [28], [38].

2.1. Bishop's constructive framework

Let S be a set given through an algorithm for constructing their members, and assume that S is an inhabited set, i.e., a set in which at least an element can be constructed. Let '=' be any equivalence relation on S, we call it an equality on S. For a given equality, we will require that all predicate and functional symbols on S are extensional in the following sense:

- for any functional symbol f

$$(\forall x, y \in S)(x = y \implies f(x) = f(y))$$

holds, and

- for any predicative symbol P

$$(\forall x, y \in S)((P(x) \wedge x = y) \implies P(y))$$

is valid.

An apartness relation on S is a binary relation '$\neq$' on S satisfying the following properties:

$(\forall a \in S)\neg(a \neq a)$ (consistency)

$(\forall a, b \in S)(a \neq b \implies b \neq a)$ (symmetry) and

$(\forall a, b, c \in S)(a \neq c \implies (a \neq b \vee b \neq c))$ (co-transitivity).

By $\lhd$ we denote the relation between elements and subsets of S defined by

$$a \lhd T \iff (\forall u \in T)(a \neq u).$$

The set $(S, =, \neq)$ is a discrete set if the following holds

$$(\forall\, a, b \in S)(a = b \vee a \neq b).$$

The subset T of the set S is said to be strongly extensional if the following holds:

$$(\forall a, b \in S)(a \in T \implies (a \neq b \vee b \in T)).$$

The complement of T in S is the subset $T^{\lhd} := \{a \in S : a \lhd T\}$.

If A and B are subsets of the set S, an equality and a difference are:

$A = B \iff (A \subseteq B \wedge B \subseteq A)$,

$A \neq B \iff ((\exists a \in A)(a \lhd B) \vee (\exists b \in B)(b \lhd A))$.

The relation $\neq$ here described is not co-transitive so it is not an apartness relation. We call it a difference-relation between sets.

Let S and T be two sets with apartness. The Cartesian product $S \times T$ is a set with apartness with respect to the equality and the apartness defined respectively by

$$(\forall a, b \in S)(\forall u, v \in T)((a, u) = (b, v) \iff (a = b \wedge u = v))$$

and

$$(\forall a, b \in S)(\forall u, v \in T)((a, u) \neq (b, v) \iff (a \neq b \vee u \neq v)).$$

Let $f : S \longrightarrow T$ be a function between sets with apartness. In addition to the standard terminology related to functions, the following terms are used in the Bishop's constructive framework:

- f is strongly extensional (se-function, in short) if holds

$$(\forall a, b \in S)(f(a) \neq f(b) \implies a \neq b),$$

and

- f is an embedding if holds

$$(\forall a, b \in S)(a \neq b \implies f(a) \neq f(b)).$$

We will now deal with one important relationship between the equality and the apartness in a set $(S, =, \neq)$ which is a specificity of this aspect. The apartness '$\neq$' on $(S, =, \neq)$ is said to be tight if the following applies

$$(\forall x, y \in S)(\neg(x \neq y) \implies x = y).$$

In the general case, the apartness does not have to be tight with the equality in the set $(S, =, \neq)$. However ([38], Proposition 1.1]), the strong compliment $\neq^{\lhd}$ of the relation $\neq$ is an equivalence in S and the following hold

$$= \subseteq \neq^{\lhd} \subseteq \neg \neq, \quad \neq \circ \neq^{\lhd} \subseteq \neq, \quad \neq^{\lhd} \circ \neq \subseteq \neq .$$

Then we can define the factor set

$$(S, \neq^{\lhd}, \neq) := (S/(\neq^{\lhd}, \neq))$$

of the set $(S, =, \neq)$. If $\neq$ is tight, then $(S, \neq^{\lhd}, \neq) = (S, =, \neq)$.

A set $(S, =, \neq)$ with apartness is a semigroup with apartness if it is defined a total se-function $\omega : S \times S$ which is associative, i.e.

$$(\forall x, y, z \in S)((x, (y, z)) = ((x, y)z)).$$

The function ω is called the (inner binary) operation on S. In this paper we will write $x \cdot y$ (or shortly xy) instead of $\omega(x, y)$ and the semigroup is denoted by $(S, =, \neq, \cdot)$. Notice that in the semigroup $(S, =, \neq, \cdot)$ the following holds

$$(\forall x, y, u, v \in S)(xu \neq yv \implies (x \neq y \vee u \neq v)).$$

So, in further on, in addition to the semigroup $(S, =, \neq, \cdot)$, we also observe the semigroup $(S, \neq^{\triangleleft}, \neq, \cdot)$ generated by the previous one in case the apartness is not tight.

2.2. Inverse semigroup with apartness

We call ([7], Definition 4) I-semigroup with apartness a semigroup with apartness $(S, =, \neq, \cdot)$ equipped with a strongly extensional unary operation on S denoted by '${}^{-1}$' such that

$$(\forall x \in S)(x \cdot x^{-1} \cdot x = x \wedge (x^{-1})^{-1} = x).$$

In other words, an I-semigroup with apartness is a tuple $(S, =, \neq, \cdot, ^{-1})$ where

(I1) $(S, =, \neq)$ is an inhabited set with apartness;
(I2) '$\cdot$' is a binary operation on S such that:
(a) for all $x, y, z \in S$, it holds $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,
(b) for all $x, y, u, v \in S$, $x \cdot u \neq y \cdot v$ implies $x \neq y$ or $u \neq v$;
(I3) '${}^{-1}$' is a unary operation such that:
(c) for all $x \in S$ it holds $(x^{-1})^{-1} = x$,
(d) for all $x, y \in S$, $x^{-1} \neq y^{-1}$ implies $x \neq y$;
(I4) for all $x \in S$ it holds $x \cdot x^{-1} \cdot x = x$.

As usual, we will write xy instead of $x \cdot y$. Since we assumed that all properties we are dealing with are extensional, we immediately derive that $\cdot$ and ${}^{-1}$ are well defined, i.e., for all $x, y, u, v \in S$, $x = u \wedge y = v$ implies $xy = uv$ and $x = y$ implies $x^{-1} = y^{-1}$. Moreover, by extensionality and (I3)(d), we also derive that for all $x, y \in S$, $x \neq y$ implies $x^{-1} \neq y^{-1}$. Then, in the definition of I-semigroup with apartness, condition (I3)(b) can be written as

(I3)(d') for all $x, y \in S$ it holds $x^{-1} \neq y^{-1} \iff x \neq y$.

Moreover, condition (I3)(c) implies that $x^{-1}xx^{-1} = x^{-1}$ for all $x \in S$. Lastly, (I3)(c) and extensionality of '${}^{-1}$' give that for all $x, y \in S$ it holds

$$x^{-1} = y^{-1} \iff x = y \tag{1}$$

An inverse semigroup with apartness ([7], Definition 5) is an I-semigroup with apartness $(S, =, \neq, \cdot, ^{-1})$ such that

(I5) for all $x, y \in S$ it holds

$$xx^{-1}yy^{-1} = yy^{-1}xx^{-1}.$$

Observe that property (I5) implies, as usual, that

$$(xy)^{-1} = y^{-1}x^{-1}.$$

We call idempotent of S each element $e \in S$ such that $ee = e$ and we denote the set of all idempotents of S by $E(S)$.

Lemma 1 ([7], Proposition 1)**.** *Let S be an inverse semigroup with apartness. Then $e = ee^{-1}$ and $ef = fe$ for each $e, f \in E(S)$.*

Lemma 2. $(\forall e, s \in S)(e \in E(S) \implies (ses^{-1} \in E(S) \ \wedge \ s^{-1}es \in E(S)))$.

Lemma 3. *Let S be an inverse semigroup with apartness. Then:*
- For each idempotent $e \in E(S)$ and element $a \in S$ there is an idempotent $f \in E(S)$ such that $ea = af$.
- For each idempotent $e \in E(S)$ and element $a \in S$ there is an idempotent $f \in E(S)$ such that $ae = fa$.
- $(\forall a \in S)(a \in E(S) \implies a^{-1} = a)$.

Proof. The proof is the same of the classical case and can be found in [16].

Let us give three interesting examples. In the first of them, the design of the relation apaerness in so-called free semigroups is demonstrated.

Example 1. *Let $(X, =, \neq)$ be a set with apartness. We form the following class X^+ of all strictly finite sequences of elements of X*

$$x^+ \in X^+ \Longleftrightarrow (\exists n_x \in \boldsymbol{N})(\exists f_x)(f_x : \{1, 2, ..., n_x\} \longrightarrow x^+)$$

with

$$(\forall i \in \{1, 2, ..., n_x\})(f_x(i) \in X).$$

As usual the concatenation of x^+ and y^+ is denoted by $x^+ \circ y^+$. If

$$x^+ = (f_x(1), ..., f_x(n_x)) \text{ and } y^+ = (f_y(1), ..., f_y(n_y)),$$

then

$$n_{xy} = n_x + n_y \text{ and}$$
$$i \in \{1, 2, ..., n_x\} \Longrightarrow f_{xy}(i) = f_x(i),$$
$$i = n_x + j \, (j \in \{1, 2, ..., n_y\}) \Longrightarrow f_{xy}(i) = f_y(j),$$

i.e.

$$x^+ \circ y^+ = (f_x(1), ..., f_x(n_x), f_y(1), ..., f_y(n_y)).$$

On the class X^+ we define

$$x^+ =_1 y^+ \Longleftrightarrow (n_x = n_y \wedge f_x = f_y) \text{ and}$$
$$x^+ \neq_1 y^+ \Longleftrightarrow (\neg(n_x = n_y) \vee f_x \neq f_y).$$

The mapping $\circ : X^+ \times X^+ \ni (x^+, z^+) \longmapsto x^+ \circ z^+ \in X^+ \times X^+$ is an internal binary strongly extensional operation on the set $(X^+, =_1, \neq_1)$. Finally, the structure $(X^+, =_1, \neq_1, \circ)$ is a semigroup with apartness ([35]) but it is not an inverse semigroup with apartness.

Example 2. *Groups with apartness are precisely the inverse semigroups with apartness with exactly one idempotent (see, for example, [30]).*

Example 3. *A semilattice with apartness is precisely an inverse semigroups with apartness in which every element is an idempotent (see, for example, [24], [35], [36]).*

2.3. Homomorphism of inverse semigroups with apartness

In this subsection, we observe homomorphisms of inverse semigroups with apartness. Homomorphisms of semigroups with apartness are observed, for example, in the article [38]. In the article [7], homomorphisms of inverse semigroups with apartness are introduced. In the classical theory of inverse semigroups, the basic properties of the homomorphisms between them have been described by Wagner [45] and Preston [28]. More information on homomorphisms in the classical theory of inverse semigroups can be find in [8], [16], [19], [21], [42].

In what follows, we first need a notion whose definition we take from [7]:

Definition 1 ([7], Definition 12)**.** *A strongly extensional subset M of an inverse semigroup with apartness S is an inverse co-subsemigroup if*
(M1) $(\forall x, y \in S)(xy \in M \Longrightarrow (x \in M \vee y \in M))$, *and*
(M2) $(\forall x \in S)(x^{-1} \in M \Longrightarrow x \in M)$.

Let S and T be semigroups with apartness. A mapping $\alpha : S \longrightarrow T$ is a homomorphism if $\alpha(xy) = \alpha(x)\alpha(y)$ for any $x, y \in S$. The homomorphism α is a strongly extensional homomorphism (se-homomorphism for short) if $\alpha(x) \neq \alpha(y)$ implies $x \neq y$ for any $x, y \in S$. The condition $(\forall x \in S)(\alpha(x^{-1}) = \alpha(x)^{-1})$ holds for any homomorphism of inverse semigroups. Moreover, for homomorphisms of inverse semigroups with apartness also this additional condition

$$(\forall a, b \in S)(\alpha(a)^{-1} \neq \alpha(b)^{-1} \implies \alpha(a) \neq \alpha(b))$$

automatically follows from (I3)(d) and also the reverse implication is valid.

Since the proof of the following lemma coincides with the proof of this lemma in the classical version ([18], Lemma 2.3, page 4; [21], Lemma 3.2, page 367) we can omit its proof.

Lemma 4. *Let $\alpha : S \longrightarrow T$ be a se-homomorphism from an inverse semigroup with apartness S into a semigroup with apartness T.*

If e is an idempotent in S, then $\alpha(e)$ is an idempotent in $\alpha(S)$.

If $\alpha(a)$ is an idempotent in $\alpha(S) \subseteq T$ for some $a \in S$, then there is an idempotent $e \in S$, such that $\alpha(e) = \alpha(a)$.

The following Proposition can be of some interest.

Proposition 1. *Let $\alpha : S \longrightarrow T$ be a se-homomorphism from an inverse semigroup with apartness S into a semigroup with apartness T. Then holds*

$$(\forall a \in S)(\alpha(a)^2 \neq \alpha(a) \implies a \lhd E(S)).$$

Proof. Let $e \in S$ be an arbitrary idempotent of semigroup S. Then

$$\alpha(a)^2 = \alpha(a^2) \neq \alpha(e^2) \vee \alpha(e^2) = \alpha(e) \neq \alpha(a)$$

by co-transitivity of the apartness in T. Thus $a^2 \neq e^2$ or $e \neq a$ by strongly extensionality of α. Hence $a \neq e \in E(S)$. This means $a \lhd E(S)$.

From this proposition, it immediately follows $a^2 \neq a \implies a \lhd E(S)$ for $\alpha = Id_S$.

As in the classical case ([21], Theorem 3.3, page 368; [16], Theorem 5.1.4, page 147), in our case also we have the closedness of the class of inverse semigroups with apartness in relation to the action of se-homomorphisms:

Theorem 5. *If $\alpha : S \longrightarrow T$ is a se-homomorphism from an inverse semigroup with apartness S into a semigroup with apartness T, then $\alpha(S)$ is an inverse semigroup with apartness also.*

However, the following proposition has no counterpart in the classical case:

Proposition 2. *Let S be an inverse semigroup with apartness, T be a semigroup with apartness, $\alpha : S \longrightarrow T$ be a se-epimorphism and $a \in S$ be such that $\lambda = \alpha(a) \in T$ is an idempotent in $\alpha(S)$. Then the set $B(\lambda) := \{x \in S : \alpha(x) \neq \lambda\}$ is an inverse co-subsemigroup of S.*

Proof. Let $x, y \in S$ be such that $\alpha(xy) \neq \lambda$. Then $\alpha(x)\alpha(y) \neq \lambda\lambda$. Thus $\alpha(x) \neq \lambda$ or $\alpha(y) \neq \lambda$. So, $x \in B(\lambda) \vee y \in B(\lambda)$. Now, if $x^{-1} \in B(\lambda)$, then $\alpha(x)^{-1} = \alpha(x^{-1}) \neq \lambda = \lambda^{-1}$. Hence $\alpha(x) \neq \lambda$ and $x \in B(\lambda)$.

Corollary 6. *For each pair of elements $a, b \in S$ such that $\alpha(a)$ and $\alpha(a)$ are idempotent in $\alpha(S)$, if we have $\alpha(a) \neq \alpha(b)$, the following holds*

$$S = B(\alpha(a)) \cup B(\alpha(b)).$$

Proof. Since from $\alpha(a) \neq \alpha(b)$, it follows $\alpha(a) \neq x \vee x \neq \alpha(b)$ for any $x \in S$, we have $x \in B(\alpha(a))$ or $x \in B(\alpha(b))$.

Theorem 7. *Let S be an inverse semigroup with apartness. There is a family $\{B_e\}_{e\in E(S)}$ of co-subsemigroups of S such that for every $e, f \in E(S)$ is valid $e \lhd B_e$ and $e \neq f \implies B_e \cup B_f = S$ and $\bigcap_{e\in E(S)} B_e = E(S)^{\lhd}$. For a given idempotent $e \in E(S)$, the co-subsemigroup B_e contains all idempotents of the semigroup S separated from e.*

Proof. For $e \in E(S)$, we put $B_e := \{x \in S : x \neq e\}$ and apply the previous proposition for $\alpha = Id_S$.

2.4. Principal-philosophical-logical orientation

According to the Stanford Encyclopedia of Phylosophy, two main trends can be distinguished within the literature on set theory necessary for practical work in Bishop's constructive algebra. According to the first one, all of what is available in classical ZF set theory is taken and only modify those principles which have a clear incompatibility with intuitionistic logic (see [3], Chapters 8 and 9). The rationale behind this theory appears to be that of granting the mathematician the most powerful tools possible, as long as compatibility with intuitionistic logic is preserved. According to the second approach, in addition to the adherence to intuitionistic logic, restrictions are introduced on the set-theoretical principles admitted, as far as the resulting system complies with the constructive mathematical practice. Theories of this second kind can thus be seen as the outcome of a double process of restriction with respect to classical ZF. First there is a restriction to intuitionistic logic, then a restriction is imposed on the set-theoretical constructions allowed. An example of the latter kind of systems is Constructive Zermelo-Fraenkel set theory CZF by Aczel and Rathjen ([1]).

Accepting Howie's motivation for the study of semigroups [15], we continue to deal with semigroups with apartness within the Bishop's constructive framework (for example, [34], [37], [10], [11], [38], [39]). In this paper, we show that in an inverse semigroup with apartness S (introduced in [7] by A. Cherubini and A. Frigeri) a co-order relation compatible with the multiplication in S can be designed. Also, the concepts of co-ideals and co-filters in inverse semigroups with apartness were designed. Several examples of co-congruences on such semigroups have also been constructed. In addition, we analyzed the conditions for one co-congruence q on an inverse semigroup with apartness S to be a semilattice co-congruence on S (group co-congruence on S, res.). The specificity of these analyzes within the Bishop's constructive framework is that for one semilattice co-congruence (group co-congruence, res.) q, two semilattices with apartness (two group with apartness, res.) can be designed.

By choosing Intuitionistic logic instead of classical logic for the work environment, the possibility of perceiving and analyzing the algebraic world parallel to the classical algebraic world is opened. This author deeply believes that such a world exists and that it should be of interest to both the academic community of mathematicians and the academic community of researchers in the philosophy of mathematics. By accepting the existence of an independent entity 'apartness relation $\neq$ on set $(S,=)$', which has a strong connection with the equation $=$, and the construction of algebraic structures on the relational system $(S,=,\neq)$ allows to mathematicians to accept the existence of two intertwined algebraic worlds. Such a commitment would enable them not only to see the newly discovered algebraic world, but also to better understand the classical world by recognizing the properties of the intertwining of these two algebraic worlds. For example, in inverse semigroups, designed on the relational system $(S,=,\neq)$, the existence of an interconnected pair of natural order relations on them can be shown: the natural order $\leqslant$ and the natural co-order $\nleqslant$. The observed environment enables the recognition of pairs of interconnected substructures such as, for example, ideals and co-ideals, and filters and co-filters on such semigroups. Also, connections between elements in such semigroups often occur in pairs: for example, one such pair is the concepts of congruences and co-congruences on semigroups with apartness.

It is now quite natural to try to answer the question:

Are inverse semigroups and inverse semigroups with apartness one and the same class of semigroups?

If we look at these algebraic structures through the eyes of a traditional mathematician, then it is obvious that they are two classes of algebraic structures built on different supports. If we look at these algebraic structures through the eyes of an open-minded mathematician through non-traditional glasses, then, of course, it is only one class of algebraic structures. The essence is that looking through the first glasses, one cannot notice the complexities of algebraic structures, but perceiving these complexities in constructed algebraic structures allow looking through other spectacles.

A reader, less versed in the specifics of this orientation, can look at Bauer's text [2] in which some of the specifics of Bishop's constructive mathematics are presented in a pictorial way. Our intention, in this subsection, is not to emphasize the differences between classical and constructive mathematics by giving examples that illustrate such differences. By building a world of semigroups with apartness, our intention is to point out to the academic community of mathematicians the richness and complexity of an additional world closely connected with the classical world by choosing to observe algebraic structures of semigroups not only through the eyes of a classical mathematician but also within Bishop's framework.

More about the principle-philosophical aspects of this mathematical orientation a reader can be found in the article [9].

3. The natural co-order

In addition to the fact that in inverse semigroups can be designed so-called natural order, as is done, for example, in the books [16], [19], [21], in inverse semigroups with apartness another order relation can be designed. In order to do this, we will repeat some terms specific to constructive algebra.

Recall that a relation q on a set $(S, =, \neq)$ is a co-equality relation on S (or a co-equivalence on S) if the following holds

$(\forall x, y \in S)((x, y) \in q \implies x \neq y)$, (consistency)
$(\forall x, y \in S)((x, y) \in q \implies (y, x) \in q)$, (symmetry)
$(\forall x, y, z \in S)((x, z) \in q \implies ((x, y) \in q \vee (y, z) \in q))$ (co-transitivity).

An interested reader can find about this relation in the paper [32].

If in the definition of the concept of co-equality relations we omit the requirement of symmetry, then a new notion appears which meets the axiom of consistency and the axiom of co-transitivity occurs. Thus, if $\nleqslant$ is one such relation in a set $(S, =, \neq)$, then $q = \nleqslant \cup \nleqslant^{-1}$ is a co-equality relation S. This analysis justifies introducing a new concept in set with apartness:

Definition 2. *Let $(S, =, \neq)$ be a set with apartness. A relation $\nleqslant$ on S is a co-quasiorder on S if:*

$(\forall\, x, y \in S)(x \nleqslant y \implies x \neq y)$, *and*
$(\forall\, x, y, z \in S)(x \nleqslant z \implies (x \nleqslant y \,\vee\, y \nleqslant z))$.

The notion of co-quasiorder the first time is defined in article [32]. If in the previous determination we add a new request, the linearity request, an another notion appears. In the following we precisely describe this concept.

Definition 3. *A co-quasiorder relation* $\nleqslant\, \subseteq S \times S$ *is a co-order on* S *if the following is valid* $\neq\, \subseteq\, \nleqslant \,\cup\, \nleqslant^{-1}$ *(linearity).*

If $\nleqslant$ is a co-quasiorder (co-order, res.) relation on a set S, it is said that S is a ordered set under co-quasi-order (res. co-order) $\nleqslant$, or that S is co-quasiordered set (res. co-ordered set). The notion of co-order relation the first time is defined in the articles [33]. The term co-quasiorder (co-order) is a constructive dual of the classic term quasi-order relation (relation of partial order, res.). The co-quasiorder relation (co-order relation, res.) $\nleqslant$, determined on the semigroup with apartness $(S, =, \neq, \cdot)$, it is said to compatible with the operation on the semigroup S if the following holds

$$(\forall x, y, u \in S)((xu \nleqslant yv \,\vee\, ux \nleqslant uy) \implies x \nleqslant y).$$

Without major difficulties it can be shown ([37], Lemma 2.1) that if $\nleqslant$ is a co-order relation on a semigroup with apartness S, then $\nleqslant^{\lhd}$ is a partial order on $((S, \neq^{\lhd}, \neq), \cdot)$ associated with $\nleqslant$ in the following cense

$$(\forall x, y \in S)(\neg(x \nleqslant y \,\wedge\, x \nleqslant^{\lhd} y)),$$

and

$$(\forall x, y, z \in S)((x \nleqslant y \,\wedge\, z \nleqslant^{\lhd} y) \implies x \nleqslant z).$$

Obviously $\leqslant\, \subseteq\, \nleqslant^{\lhd}$ is valid, too. If the apartness $\neq$ is tight, then $\nleqslant^{\lhd}$ $(=\leqslant)$ is a partial order relation on the semigroup S.

This concept has been discussed in the articles [34], [37], [38].

In inverse semigroups with apartness, the so-called natural order '$\leqslant$' can be determined in the same way as in the classical case (see, for example [16]): If a and b are two elements of an inverse semigroup wirh apartness S, we write

$$a \leqslant b \text{ if and only if } aa^{-1} = ab^{-1}$$

or if any one of the following equivalent conditions holds:

$$aa^{-1} = ba^{-1}, \quad a^{-1}a = a^{-1}b, \quad a^{-1}a = b^{-1}a.$$

What we will need in this report in the below is the following lemma:

Lemma 8. *Let S be an inverse semigroup with apartness. Then for any idempotent $a \in E(S)$ and any elements $x, y \in S$ the following holds*

$$ax \leqslant x, \;\; xa \leqslant x, \;\; xay \leqslant xy.$$

Proof. The correctness of these inequalities can be proved by direct verification by referring to the determination of the order relation.

In addition to the previously mentioned order relation, in an inverse semigroup with apartness another order relation can be determined. The following definition introduces the so-called natural co-order relation in an inverse semigroup with apartness:

Definition 4. *On an inverse semigroup with apartness S, the relation '$\nleqslant$' we define as follows*

$$(\forall a, b \in S)(a \nleqslant b \iff a \neq ba^{-1}a). \tag{2}$$

We will first analyze more conditions for determining this relation which are equivalent to the condition in Definition 4.

Lemma 9. *The condition on the right side of the formula* (2) *is equivalent to the condition*

$$ab^{-1} \neq aa^{-1}. \tag{3}$$

Proof. Let $a, b \in S$ be elements such that $a \neq b(a^{-1}a)$. Then $aa^{-1}a \neq ba^{-1}a$. Thus $aa^{-1} \neq ba^{-1}$ by (I2)(b).

Let $a, b \in S$ be elements such that $aa^{-1} \neq ba^{-1}$. Then $aa^{-1} \neq ba^{-1}aa^{-1}$. Thus $a \neq ba^{-1}a$ by (I2)(b).

Lemma 10. *Let $a, b \in S$ be arbitrary elements. The condition $a \neq aa^{-1}b$ is equivalent to the condition*

$$a^{-1}b \neq a^{-1}a. \tag{4}$$

Proof. Let $a, b \in S$ be elements such that $aa^{-1}b \neq a$. Then $aa^{-1}b \neq aa^{-1}a$. Thus $a^{-1}b \neq a^{-1}a$ by (I2)(b).

Conversely, let $a, b \in S$ be elements such that $a^{-1}b \neq a^{-1}a$. Then

$$a^{-1}aa^{-1}b \neq a^{-1}a.$$

Hence $aa^{-1}b \neq a$ by (I2)(b).

Lemma 11. *The formula* (2) *is equivalent to the following formula*

$$(\forall a, b \in S)(a \nleqslant b \iff a \neq aa^{-1}b). \tag{5}$$

Proof. Let $a, b \in S$ be such that $a \nleqslant b$. This means $a \neq ba^{-1}a$. Then $a^{-1} \neq a^{-1}ab^{-1}$ by (I3)(d'). Thus $a^{-1}aa^{-1} \neq a^{-1}ab^{-1}$. Hence $aa^{-1} \neq ab^{-1}$ by (I2)(b). Hence $a \neq aa^{-1}b$, according to Lemma 10.

Conversely, let $a \neq aa^{-1}b$ be valid for the elements $a, b \in S$. This condition is equivalent to the condition $aa^{-1} \neq ab^{-1}$, according to Lemma 10. Then $aa^{-1} \neq ba^{-1}$ by (I2)(b). Thus $aa^{-1} \neq ba^{-1}aa^{=1}$. Hence $a \neq ba^{-1}a$ by (I2)(b). This means $a \nleqslant b$.

Although it would be more natural to collect the results of Lemma 9- Lemma 11 in a single proposition, we present them separately because we got them that way with a lot of difficulty. We may now establish the main properties of the relation '$\nleqslant$'.

Theorem 12. *Let S ne an inverse semigroup with apartness and $\nleqslant$ be a relation defined on S as above. Then:*

(3.1) *The relation $\nleqslant$ is a co-order relation on the set S.*
(3.2) $(\forall a, v \in S)(a \nleqslant b \implies a^{-1} \nleqslant b^{-1})$.
(3.3) $(\forall a, b, u \in S)(au \nleqslant bu \implies a \nleqslant b)$ *and*
(3.4) $(\forall a, b, u \in S)(ua \nleqslant ub \implies a \nleqslant b)$.

Proof. (3.1) The relation $\nleqslant$ is a co-order relation on the set $(S, =, \neq)$:

(1.1) The relation $\nleqslant$ is consistent. Let $a, b \in S$ be such that $x \nleqslant y$. This means $a \neq ba^{-1}a$. On the other hand, since $a = aa^{-1}a$, we have $aa^{-1}a \neq ba^{-1}a$. Then $a \neq b$ by (I2)(b) thus showing that the relation $\nleqslant$ is consistent.

(1.2) The relation $\nleqslant$ is co-transitive. Let $a, b, c \in S$ be arbitrary elements such that $a \nleqslant c$. This means $a \neq ca^{-1}a$. Then

$$a \neq ba^{-1}a \vee ba^{-1}a \neq ca^{-1}a$$

by co-transitivity od the apartness.

(i) The first option gives $a \nleqslant b$.

(ii) Suppose the second option is valid Since $a = aa^{-1}a$, we have

$$ba^{-1}(aa^{-1}a) \neq ca^{-1}(aa^{-1}a)$$

which we can write in form

$$b(a^{-1}a)(a^{-1}a) \neq c(a^{-1}a)(a^{-1}a).$$

Hence follows

$$b(a^{-1}a)(a^{-1}a) \neq (cb^{-1}b)(a^{-1}a)(a^{-1}a)$$

or

$$(cb^{-1}b)(a^{-1}a)(a^{-1}a) \neq c(a^{-1}a)(a^{-1}a)$$

due to the co-transitivity of the apartness relation.

(ii-a) From the first option

$$b(a^{-1}a)(a^{-1}a) \neq (cb^{-1}b)(a^{-1}a)(a^{-1}a),$$

we get $b \neq cb^{-1}b$ by (I2)(b). So, $b \nleqslant c$.

(ii-b) Assume that the second option

$$(cb^{-1}b)(a^{-1}a)(a^{-1}a) \neq c(a^{-1}a)(a^{-1}a)$$

is valid. We can write it in form

$$c(b^{-1}b)(a^{-1}a)(a^{-1}a) \neq c(a^{-1}a)(a^{-1}a).$$

From here, we have

$$(b^{-1}b)(a^{-1}a)(a^{-1}a) \neq (a^{-1}a)(a^{-1}a)$$

by (I2)(b). The next three transformations of this formula are

$$(a^{-1}a)(b^{-1}b)(a^{-1}a) \neq (a^{-1}a)(a^{-1}a),$$

$$(a^{-1}ab^{-1})(ba^{-1}a) \neq (a^{-1}aa^{-1})a$$

and

$$(ba^{-1}a)^{-1}(ba^{-1}a) \neq a^{-1}a,$$

respectively. Hence

$$(ba^{-1}a)^{-1} \neq a^{-1} \vee ba^{-1}a \neq a.$$

Both previous cases give $a \nleqslant b$.

This completes the proof of the co-transitivity of the relation $\nleqslant$.

(1.3) The relation $\nleqslant$ is linear. let $a, b \in S$ be elements such that $a \neq b$. Then

$$a \neq b(a^{-1}a) \vee b(a^{-1}a) \neq b$$

by co-transitivity of the apartness. If the first option $a \neq b(a^{-1}a)$ is valid, then we have $a \nleqslant b$. Assume that the second option

$$b(a^{-1}a) \neq b$$

is valid. The previous formula can be written as follows $(bb^{-1}b)(a^{-1}a) \neq b$ and, further on, in the following way $b(b^{-1}b)(a^{-1}a) \neq b$. The next allowed transformation of the previous valid formula is $b(a^{-1}a)(b^{-1}b) \neq b$ according to (I5). From here it follows

$$(b(a^{-1}a))(b^{-1}b) \neq a(b^{-1}b) \vee a(b^{-1}b) \neq b$$

according to the co-transitivity of the apartness. The second option gives immediately $b \nleqslant a$. Suppose the $(b(a^{-1}a))(b^{-1}b) \neq a(b^{-1}b)$ option is valid. From here, according to (I2)(b), it follows $b(a^{-1}a) \neq a$. This means $a \nleqslant b$.

(3.2) Let $a, b \in S$ be elements such that $a \nleqslant b$. This means $a \neq b(a^{-1}a)$ and $a \neq aa^{-1}b$ according to Lemma 11. From here, according to (I3)(d'), it follows $a^{-1} \neq b^{-1}(aa^{-1}) = b^{-1}((a^{-1})^{-1}a^{-1})$. This means $a^{-1} \nleqslant b^{-1}$.

(3.3) Let $a, b, u \in S$ be arbitrary elements such that $au \nleqslant bu$. Then

$$(au)^{-1}(bu) \neq (au)^{-1}(au)$$

by Lemma 10. Thus $u^{-1}a^{-1}bu \neq u^{-1}a^{-1}au$. Hence $a^{-1}b \neq a^{-1}a$ by (I2)(b). This means $a \nleqslant b$ by Lemma 10.

(3.4) Let $a, b, u \in S$ be arbitrary elements such that $ua \nleqslant ub$. Then

$$(ua)(ub)^{-1} \neq (ua)(ua)^{-1}$$

by Lemma 9. Thus $uab^{-1}u^{-1} \neq uaa^{-1}u^{-1}$. Hence $ab^{-1} \neq aa^{-1}$ by (I2)(b). This means $a \nleqslant b$ by Lemma 9.

It is clear that $\nleqslant$ and $\leqslant$ are associated relations.

Although the following statement is known, we will restate and prove it because of the consistency of the material presented in this report.

Corollary 13. *Conditions* (3.3) *and* (3.4) *are equivalent to the condition*

(3.5) $(\forall a, b, u, v \in S)(ua \nleqslant vb \Longrightarrow (u \nleqslant v \vee a \nleqslant b))$.

Proof. $(3.3) \wedge (3.4) \Longrightarrow (3.5)$. Let $a, b, u, v \in S$ be such that $ua \nleqslant vb$. Then $ua \nleqslant ua \nleqslant ub \vee ub \nleqslant vb$ by co-transitivity of the natural co-order. Then $a \nleqslant b \vee u \nleqslant v$.

$(3.5) \Longrightarrow (3.3) \wedge (3.4)$. Obviously by the respect of consistency of the natural co-order.

The following statements can be proved without major difficulties:

Proposition 3. *If $a \in S$ is an idempotent in an inverse semigroup with apartness S, then the set*

$$\langle a]_{\nleqslant} := \{t \in S : t \nleqslant a\}$$

is a co-subsemigroup in S and it has the following property

$$y \in \langle a]_{\nleqslant} \Longrightarrow (y \nleqslant x \vee x \in \langle a]_{\nleqslant}).$$

Proof. Let $x, y \in S$ be such that $xy \in \langle a]_{\nleqslant}$. This means $xy \nleqslant a = aa$. Then $x \nleqslant a \vee y \nleqslant a$ by (3.5). Thus $x \in \langle a]_{\nleqslant}$ or $y \in \langle a]_{\nleqslant}$. Let $x^{-1} \in \langle a]_{\nleqslant}$, i.e. let $x^{-1} \nleqslant a = a^{-1}$. Then $x \nleqslant a$ by (3.2). So, $x \in \langle a]_{\nleqslant}$. Finally, let $x, y \in S$ br elements such that $y \in \langle a]_{\nleqslant}$. This means $y \nleqslant a$. Hence $y \nleqslant x \vee x \nleqslant a$. Therefore, $y \nleqslant x \vee x \in \langle a]_{\nleqslant}$. From the last proven property it is clear that this set is a strongly extensional subset in S.

Proposition 4. *If $a \in S$ is an idempotent in an inverse semigroup with apartness S, then the set*

$$[a\rangle_{\nleqslant} := \{t \in S : a \nleqslant t\}$$

is a co-cosubsemigroup in $S$and it has the following property

$$y \in [a\rangle_{\nleqslant} \implies (x \nleqslant y \vee x \in [a\rangle_{\nleqslant}).$$

Proof. Let $x, y \in S$ be such that $xy \in [a\rangle_{\nleqslant}$. This means $a^2 = a \nleqslant xy$. Then $a \nleqslant x \vee a \nleqslant y$ by (3.5). Hence $x \in [a\rangle_{\nleqslant} \vee y \in [a\rangle_{\nleqslant}$. If $x^{-1} \in [a\rangle_{\nleqslant}$, then $a^{-1} = a \nleqslant x^{-1}$. Thus $a \nleqslant x$ by (3.2). So, $x \in [a\rangle_{\nleqslant}$. Finally, if $x, y \in S$ be elements such that $y \in [a\rangle_{\nleqslant}$, then $a \nleqslant y$. Thus $a \nleqslant x \vee x \nleqslant y$ by co-transitivity of the apartness. So, $x \in [a\rangle_{\nleqslant}$ or $x \nleqslant y$. As in the previous proposition, it follows from the last proven property that this set is a strongly extensional subset in S.

The previous Proposition 3 suggests that the concept of co-ideals in an inverse semigroup with apartness could be determined as follows: The subset K of an inverse semigroup with apartness S is a co-ideal in S if

(K1) $(\forall x, y \in S)(xy \in K \implies (x \in K \wedge y \in K))$,

(K2) $(\forall x, y \in S)(x \in K \implies (y \in K \vee x \nleqslant y))$ and

(K3) $(\forall x \in S)(x^{-1} \in K \implies x \in K)$.

Also, the following proposition can be proved:

Proposition 5. *If K is a co-ideal in an inverse semigroup with apartness S, then the set $K^{\lhd}$ is an ordered ideal in S.*

Proof. Let $x, y, u \in S$ be such that $x \lhd K$ and $u \in K$. Then $u \in K \implies (u \neq xy \vee xy \in K)$. The second option gives $x \in K$ and $y \in K$ by (K1) which is contrary to the hypothesis $x \lhd K$. Therefore, it must be $xy \neq u \in K$. This means $xy \in K^{\lhd}$. The implication $y \in K^{\lhd} \implies xy \in K$ can be proved in an analogous way.

Let $x, u \in S$ be such that $x \lhd K$ and $u \in K$. Then $u \neq x^{-1} \vee x^{-1} \in K$. The second option gives xK by (K3) which is contrary to the hypothesis $x \lhd K$. So, it have to be $x^{-1} \neq u \in K$. Thus $x^{-1} \in K^{\lhd}$.

Let $x, y, u \in S$ be such that $y \in K^{\lhd}$, $x \leqslant y$ and $u \in K$. Then $u \in K$ implies $u \neq x \vee x \in K$. The second option gives $y \in K \vee x \nleqslant y$ by (K2). As this contradicts the hypotheses, it must be $x \neq u \in K$. So $x \in K^{\lhd}$.

It is not difficult to show that:

Proposition 6. *The set $M := \bigcup_{a\in A}[a\rangle_{\nleqslant}$ is a co-subsemigroup of S that satisfies the additional condition*

$$(\forall x, y \in S)(y \in M \implies (x \nleqslant y \vee x \in M)),$$

where A is a (finite) discrete subset of idempotents in S.

Proof. Let $x, y \in S$ be such that $xy \in \bigcup_{a\in A}[a\rangle_{\nleqslant}$. Then there exists an index $e \in A$ such that $xy \in [e\rangle_{\nleqslant}$. Thus

$$x \in [e\rangle_{\nleqslant} \subseteq \bigcup_{a\in A}[a\rangle_{\nleqslant} \text{ or } y \in [e\rangle_{\nleqslant} \subseteq \bigcup_{a\in A}[a\rangle_{\nleqslant}$$

by Proposition 3. Suppose tat $x^{-1} \in \bigcup_{a\in A}[a\rangle_{\nleqslant}$. Then there exists an index $e \in A$ such that $x \in [e\rangle_{\nleqslant} \subseteq \bigcup_{a\in A}[a\rangle_{\nleqslant}$. Finally, let $x, y \in S$ be elements such that $y \in M$. Then there exists an index $e \in A$ such that $y \in [e\rangle_{\nleqslant}$. Thus $x \in [e\rangle_{\nleqslant} \subseteq \bigcup_{a\in A}[a\rangle_{\nleqslant}$ or $x \nleqslant y$.

Since the following proposition can be proved analogously to the previous one, we will omit its proof.

Proposition 7. *The set $M := \bigcup_{a\in A}\langle a]_{\nleqslant}$ is a co-subsemigroup of S that satisfies the additional condition*

$$(\forall x, y \in S)(y \in M \implies (y \nleqslant x \vee x \in M)),$$

where A is a (finite) discrete subset of idempotents in S.

In the classical theory of inverse semigroups, one of the central places is the commutative semilattice $E(S)$ of all idempotents in an inverse semigroup S. Thus, if e and f are elements of the set $E(S)$, then ef is also an element in $E(S)$. Also, the following holds ([18], Proposition 2.11 (5)): if $s \leqslant e$ and $e \in E(S)$, then $s \in E(S)$.

So and in this case, the case of inverse semigroups with apartness, the properties of this set have a significant place in this class of semigroups. Although it is to be expected that the dual $E(S)^{\lhd}$ of the set $E(S)$ has corresponding dual properties, this is not the case. First, it is obvious that:

Lemma 14. *Let S be an inverse semigroup with apartness. Then*

$$(\forall x \in S)(a^{-1} \lhd E(S) \implies a \lhd E(S)).$$

Proof. Indeed, let $a^{-1} \lhd E(S)$. This means $(\forall e \in E(S))(a^{-1} \neq e = e^{-1})$. Then $(\forall e \in E(S))(a \neq e)$ by (I3)(d'). Hence $a \lhd E(S)$.

On the other hand, the following may prove to be valid:

Lemma 15. *Let S be an inverse semigroup with apartness. Then*
(3.6) $(\forall a, b \in S)(a \lhd E(S) \implies (b \lhd E(S) \vee a \nleq b))$.

Proof. Let $a, b \in S$ be such $a \lhd E(S)$. This means $a^2 \neq a$. Then
$a^2 = aa \neq (aa^{-1})bb(a^{-1}a) \vee$
$(aa^{-1})bb(a^{-1}a) \neq (aa^{-1})b(a^{-1}a) \vee$
$(aa^{-1})b(a^{-1}a) \neq a(a^{-1}a) = a$
by co-transitivity of the apartness in S. Thus

$$a \neq (aa^{-1})b \vee a \neq b(a^{-1}a) \vee b^2 \neq b \vee (aa^{-1})b \neq a.$$

This means $b \lhd E(S) \vee a \nleq b$. Indeed, for any $e \in E(S)$ holds $b^2 \neq e^2 = e \vee e \neq b$ by co-transitivity of the apartness. Then $b \neq e \in E(S)$ by (I2)(b). This means $b \lhd E(S)$.

Although it is to be expected that $E(S)^{\lhd}$ has property 1. in Definition 1, we are not able to see this at this level of development of our understanding of inverse semigroups with apartness. Let $a, b \in S$ be arbitrary elements such that $ab \lhd E(S)$ and let $u \in E(S)$. This means $uu = u$ and $ab \neq u = uu$. Thus $a \neq u \vee b \neq u$. Hence, the conclusion $a \lhd E(S) \vee b \lhd E(S)$ does not follow, since the formula

$$(\forall u \in E(S))(ab \neq uu) \implies (\forall u \in E(S))(a \neq u) \vee (\forall u \in E(S))(b \neq u)$$

is not a valid formula. But if we add one precondition

$$(\forall x, y \in S)(xy \nleq^{\lhd} x \wedge xy \nleq^{\lhd} y)$$

then we can get that the semigroup $E(S)$ has the observed property:

Lemma 16. *If the inverse semigroup with apartness S satisfies the previous condition, then the set $E(S)^{\lhd}$ is a consistent subset in S, that is, the following is valid*

$$(\forall x, y \in S)(xy \lhd E(S) \implies (x \lhd E(S) \wedge y \lhd E(S))).$$

Proof. Let $x, y \in S$ be such that $xy \lhd E(S)$. Then $x \lhd E(S) \vee xy \nleq x$ by (3.6). As the second option is not possible due to the accepted hypothesis, it must be $x \lhd E(S)$. The implication $xy \lhd E(S) \implies y \lhd E(S)$ can be proved analogously to the previous one.

Some of the properties of this semigroup are listed below:

Proposition 8. *Let S be an inverse semigroup with apartness and $a, b \in S$. Then:*
(a) $(\forall a, b \in S)(b^{-1}a \lhd E(S) \implies bb^{-1}a \neq aa^{-1}b)$,
(b) $(\forall a, b \in S)(ab^{-1} \lhd E(S) \iff ba^{-1} \lhd E(S))$,
(c) $(\forall a, b \in S)(a^{-1}b \lhd E(S) \implies b \nleqslant bb^{-1}a)$,
(d) $(\forall a, b \in S)(a^{-1}b \lhd E(S) \implies a \nleqslant aa^{-1}b)$,
(e) $(\forall a, b \in S)(ab^{-1} \lhd E(S) \implies b \nleqslant ab^{-1}b)$,
(f) $(\forall a, b \in S)(a^{-1}b \lhd E(S) \implies (\forall u \in S)(a \nleqslant u \vee b \nleqslant u))$,
(g) $(\forall a, b \in S)(ab^{-1} \lhd E(S) \implies (\forall u \in S)(a \nleqslant u \vee b \nleqslant u))$.

Proof. (a) If $b^{-1}a \lhd E(S)$, then $b^{-1}a \neq (b^{-1}a)(b^{-1}a)^{-1} \in E(S)$. Then $b^{-1}bb^{-1}a \neq (b^{-1}a)(a^{-1}b) = b^{-1}(aa^{-1}b) \in E(S)$. Thus $bb^{-1}a \neq aa^{-1}b$ by (I2)(b).

(b) Let $a, b, e \in S$ be such that $ab^{-1} \lhd E(S)$ and $e \in E(S)$. Then $ab^{-1} \neq e \in E(S)$. Thus $ba^{-1} \neq e^{-1} = e \in E(S)$ by (I3)(d'). Hence $ba^{-1} \lhd E(S)$.

(c) Let $a, b \in S$ be such $a^{-1}b \lhd E(S)$. Then

$$(b^{-1}a)^{-1}(b^{-1}a) \lhd E(S) \vee a^{-1}b \nleqslant (b^{-1}a)^{-1}(b^{-1}a) = a^{-1}bb^{-1}a.$$

As the first option is not possible, it can be $a^{-1}b \nleqslant (b^{-1}a)^{-1}(b^{-1}a) = a^{-1}bb^{-1}a$. Thus $b \nleqslant bb^{-1}a$ by (3.4).

(d) Obtained from (b) and by substituting variables a and b in (c).

(e) Let $a, b \in S$ be such $ab^{-1} \lhd E(S)$. Then $ab^{-1} \nleqslant (ab^{-1})(ab^{-1})^{-1} \vee (ab^{-1})(ab^{-1})^{-1} \lhd E(S)$ by (b). As the second option is not possible, it can be $ab^{-1} \nleqslant (ab^{-1})(ab^{-1})^{-1} = ab^{-1}ba_{-1}$. Thus $b^{-1} \nleqslant b^{-1}ba^{-1}$ by (3.4). Hence $b \nleqslant ab^{-1}b$ by (I3)(d').

(f) Let $a, b \in S$ be such that $a^{-1}b \lhd E(S)$. Then $u^{-1}u \lhd E(S) \vee a^{-1}b \nleqslant u^{-1}u$ by (a). Since the first option is impossible, we have $a^{-1}b \nleqslant u^{-1}u$. Thus $a^{-1}b \nleqslant u^{-1}b \vee u^{-1}b \nleqslant u^{-1}u$ by co-transitivity of the co-order. Hence $a^{-1} \nleqslant u^{-1} \vee b \nleqslant u$ by (3.3) and (3.4). Finally, we have $a \nleqslant u \vee b \nleqslant u$ by (2).

(g) The proof for (h) can be designed analogously to the proof for (f).

Remark 1. *Let us note that: The statement (b) of Proposition 8 is a particular case of Lemma 14. The statements (c) and (d) can be written together in the following way*

$$(\forall a, b \in S)(a^{-1}b \lhd E(S) \implies b \nleqslant bb^{-1}a \wedge a \nleqslant aa^{-1}b).$$

4. Some special co-congruences

This Section is a central part of the paper. It deals with co-congruenges on inverse semigroups with apartness with special focus on some co-congruences with special requirements like group and semilattice co-congruences.

4.1. Two examples as an introduction

In this subsection, we will deal with co-equivalence relations on inverse semigroups with apartness. We will be particularly interested in co-equivalences compatible with the semi-group operations.

Let us analyze one example first:

Example 4. *Let the relation σ on an inverse semigroup with apartness S be determined as follows*

$$(\forall a, b \in S)((a, b) \in \sigma \iff ab^{-1} \in \bigcap_{e \in E(S)} [e\rangle_{\nleqslant}).$$

Then σ is a consistent and symmetric relation on S but, in the general case, it does not have to be co-transitive.

Proof. Let $a, b \in S$ be such that $(a, b) \in \sigma$. This means $e \nleqslant ab^{-1}$ for any $e \in E(S)$. Then $bb^{-1} \nleqslant ab^{-1}$ and hence $b \nleqslant a$. So σ is consistent.

Let $a, b \in S$ be such that $(a, b) \in \sigma$. This means $e \nleqslant ab^{-1}$ for any $e \in E(S)$. Then $e = e^{-1} \nleqslant (ab^{-1})^{-1} = ba^{-1}$. Hence $(b, a) \in \sigma$.‘

Let us now analyze the possibility that σ be co-transitive. Let $a, b, c \in S$ be such that $ac^{-1} \in \bigcap_{e \in E(S)} [e\rangle_{\nleqslant}$. Then $e \nleqslant ac^{-1}$ for any $e \in E(S)$. Thus $e \nleqslant ab^{-1}bc^{-1} \vee ab^{-1}bc^{-1} \nleqslant ac$. Hence $ee = e \nleqslant ab^{-1}bc^{-1}$ because the second option is impossible according Lemma 8. It follows $e \nleqslant ab^{-1}$ or $e \nleqslant bc^{-1}$ from here. However, we cannot prove that $(a, b) \in \sigma \vee (b, c) \in \sigma$ is valid because the formula

$$(\forall e)(A \vee B) \implies (\forall e)A \vee (\forall e)B$$

is not a valid formula.

Regarding the obstacles in the previous example, see Theorem 24. In the general case, this relation does not compatible with the operations in S. However, for relation κ, defined by

$$(a, b) \in \kappa \iff (a, b) \in \sigma \vee (a^{-1}, b^{-1}) \in \sigma,$$

it can be shown that the following

$$(a^{-1}, b^{-1}) \in \kappa \implies (a, b) \in \kappa$$

is valid.

The relation '$\sim$', defined by

$$a \sim b \iff a^{-1}b \in E(S) \wedge ab^{-1} \in E(S),$$

is under the special attention of researchers in the classical theory of inverse semigroups (see, for example [18]). This is called the compatibility relation. It is reflexive and symmetric but not generally transitive. To the question of when this relation will be transitive, one answer is offered ([18], Proposition 2.17) by involving the notion of E-unitarity of a given inverse semigroup. Its constructive dual κ, is determined as follows:

$$(a,b) \in \kappa \iff (a^{-1}b \lhd E(S) \vee ab^{-1} \lhd E(S)).$$

It is consistent and symmetric but not co-transitive in general case. With the current level of our understanding of the properties of inverse semigroups with apartness, we cannot establish sufficient conditions for this relation to be co-transitive and compatible with the semigroup operations. However:

Proposition 9. *Suppose that for a co-subsemigroup M of an inverse semigroup with apartness S holds*

(4.7) $M \subseteq E(S)^{\lhd}$ *and*

(4.8) $(\forall a, b \in S)(a \in M \implies (b \nleq a \vee b \in M))$.

Then the relation κ_M on S, determined by

$$(\forall a, b \in S)((a,b) \in \kappa_M \iff (a^{-1}b \in M \vee ab^{-1} \in M)),$$

is co-equality relation on S which satisfies the following condition

$$(\forall a, b \in S)((a^{-1}, b^{-1}) \in \kappa_M \implies (a,b) \in \kappa_M).$$

Proof. Let $a, b \in S$ be such that $(a,b) \in \kappa_M$. This means $a^{-1}b \in M \vee ab^{-1} \in M$. If $a^{-1}b \in M \subseteq E(S)^{\lhd}$, then $a^{-1}b \neq a^{-1}a \in E(S)$. So, $b \neq a$. The implication $ab^{-1} \in M \implies a \neq b$ can be proved analogously to the previous one. Hence $\kappa_M \subseteq \neq$.

Additionally, we have $a^{-1}b \in M \iff ((a^{-1}b)^{-1})^{-1} \in M$. Then $b^{-1}a = (a^{-1}b)^{-1} \in M$ by (M2). That $ab^{-1} \in M \implies ba^{-1} \in M$ also holds can be proved analogously to the previous one. Thus, κ_M is a symmetric relation.

Let $a, b, c \in S$ be arbitrary elements such that $(a,c) \in \kappa_M$. This means $a^{-1}c \in M$ or $ac^{-1} \in M$. Assume that $a^{-1}c \in M$. Then $a^{-1}bb^{-1}c \in M$ or $a^{-1}bb^{-1}c \nleq a^{-1}c$ by (4.8). The second option is not possible due to Lemma 8. So it has to be $a^{-1}bb^{-1}c \in M$. Thus $a^{-1}b \in M \vee b^{-1}c \in M$ by (M1). That the implication $ac^{-1} \in M \implies ab^{-1} \in M \vee bc^{-1} \in M$ is valid can be proved analogously to the previous proof. Thus, the relation κ_M is co-transitive and, therefore, it is a co-equality relation on S.

Let $a, b \in S$ be such that $(a^{-1}, b^{-1}) \in \kappa_M$. This means $(a^{-1})^{-1}b^{-1} \in M$ or $a^{-1}(b^{-1})^{-1} \in M$. Then $(ba^{-1})^{-1} \in M \vee (b^{-1}a)^{-1} \in M$. Thus $ba^{-1} \in M \vee b^{-1}a \in M$ by (M2). Hence $(b,a) \in \kappa_M$ and $(a,b) \in \kappa_M$ because κ_M a is a symmetric relation.

4.2. Co-congruences

Congruences on inverse semigroups have been investigated by several authors (see, for example [17], [26], [29], [42] and Section 5 in the book [16]). In [29], Reilly and Scheiblich considered the restriction to idempotents of inverse semigroup congruences. They classified congruences in terms of restrictions to idempotents and characterized minimum and maximum congruences having any given restriction. In [42], Scheiblich has extended this approach by showing that each congruence on an inverse semigroup can be uniquely characterized in terms of its restriction to idempotents together with its kernel. In this paper, special attention is paid to semilattice congruences and group congruences on inverse semigroups.

The constructive dual of the concept of congruences in semigroups with apartness is the notion of co-congruences. A co-equality q on s semigroup S is a co-congruence on S if the following holds

$$(\forall a, b, u, v \in S)((au, bv) \in q \implies ((a, b) \in q \vee (u, v) \in q)). \tag{6}$$

The previous implication is equivalent to the next two implications

$$(\forall a, b, u \in S)((au, bu) \in q \implies (a, b) \in q)$$

and

$$(\forall a, b, u \in S)((ua, ub) \in q \implies (a, b) \in q.)$$

If the first implication (the second implication) is valid, then we say that q is compatible on the right side (on left side, res.) with the semigroup operation in S. Some more about the concept of co-equality relations on sets with apartness and its compliance with semigroup operations can be found, for example, in the papers [32] and [38].

An important connection between a co-equivalence q on a semigroup S and a co-congruence

$$q^* := \{(a, b) \in S \times S : (\exists x, y \in S^1)((xay, xby) \in q)\},$$

generated by q, is given by the following lemma:

Lemma 17 ([38], Theorem 2.6)**.** *Let q be a co-equality relation on a semigroup with apartness S. Then the relation q^* is a co-congruence on S such that $q \subseteq q^*$. If κ is a co-congruence on S such that $q \subseteq \kappa$, then $q^* \subseteq \kappa$.*

It is known ([38], Theorem 2.3) that if q is a co-congruence on a semigroup with apartness S, then $q^{\triangleleft} := \{(a, b) \in S \times S : (a, b) \triangleleft q\}$ is a congruence on S associated with q in the following seance

$$q \circ q^{\triangleleft} \subseteq q \text{ and } q^{\triangleleft} \circ q \subseteq q.$$

The concept of co-congruences on an algebraic structure first appears in the paper [31]. For a co-congruence q on a semigroup S, the following sets

$$S/(q^{\triangleleft}, q) := (\{aq^{\triangleleft} : a \in S\}, =_1, \neq_1, \cdot_1)$$

and

$$[S : q] := (\{aq : a \in S\}, =_2, \neq_2, \cdot_2)$$

can be designed, where are

$$(\forall a, b \in S)(aq^{\triangleleft} =_1 bq^{\triangleleft} \iff (a, b) \triangleleft q \ \wedge \ aq^{\triangleleft} \neq_1 bq^{\triangleleft} \iff (a, b) \in q)$$

and

$$(\forall a, b \in S)(aq =_2 bq \iff (a, b) \triangleleft q \ \wedge \ aq \neq_2 bq \iff (a, b) \in q)$$

where $aq =: \{x \in S : (a, x) \in q\}$. Set $S/(q^{\triangleleft}, q)$ is a semigroup under the multiplication '$\cdot_1$ defined by

$$(\forall a, b \in S)(aq^{\triangleleft} \cdot_1 bq^{\triangleleft} := (ab)q^{\triangleleft})$$

(see, for example, [38], Theorem 2.4). Set $[S : q]$ is a semigroup under the multiplication '$\cdot_2$', define by

$$(\forall a, b \in S)(aq \cdot_2 bq := (ab)q)$$

(see, for example, [38], Theorem 2.5). It is clear that the semigroup $[S : q]$ is a specific phenomenon in the theory of semigroups with apartness within Bishop's constructive framework and it has no counterpart in the classical semigroup theory.

In article [7], the authors dealt with co-congruences on inverse semigroups with apartness. The authors, in that paper, added a request

$$(\forall a, b \in S)((a^{-1}, b^{-1}) \in q \implies (a, b) \in q)$$

for co-congruence q on an inverse semigroup with apartness S ([7], Definition 10). In addition, they have shown that the following applies:

Proposition 10 ([7], Proposition 4)**.** *Let q be a co-congruence on an inverse semigroup with apartness S. Then:*

$$(\forall a, b \in S)((a^{-1}, b^{-1}) \in q \iff (a, b) \in q),$$

and

$$(\forall a, b \in S)((a^{-1}, b^{-1}) \triangleleft q \iff (a, b) \triangleleft q).$$

In relation to the previous one, the following theorem can be proved:

Theorem 18 ([7], Theorem 7)**.** *Let S be an inverse semigroup with apartness, and let q be a co-congruence on S. If we define $(\forall a \in S)((aq^{\triangleleft})^{-1} := a^{-1}q^{\triangleleft})$, then $(S/(q^{\triangleleft}, q), =_1, \neq_1, \cdot_1, ^{-1})$ is well-defined and it is an inverse semigroup with apartness. Moreover, the natural projection $\pi : S \longrightarrow S/(q^{\triangleleft}, q)$, defined by $\pi(a) := aq^{\triangleleft}$, is an onto se-homomorphism.*

However, the following theorem on inverse semigroups with apartness has no counterpart in the classical theory of inverse semigroups:

Theorem 19. *Let S be an inverse semigroup with apartness, and let q be a co-congruence on S. If we define $(\forall a \in S)((aq)^{-1} := a^{-1}q)$, then*

$$([S : q], =_2, \neq_2, \cdot_2, ^{-1})$$

is well-defined and it is an inverse semigroup with apartness. Moreover, the natural projection $\vartheta : S \longrightarrow [S : q]$, defined by $\vartheta(a) := aq$, is an onto se-homomorphism.

Proof. Evidence can be obtained by direct verification:

Let $x, y, u, v, a, b \in S$ be elements such that $xq =_2 uq$, $yq =_2 vq$ and $(a, b) \in q$. Then $(x, u) \triangleleft q$, $(y, v) \triangleleft q$ and $(a, xv) \in q \vee (xv, uy) \in q \vee (uy, b) \in q$ by co-transitivity of q. If the second option were valid, it would be $(x, u) \in q$ or $(y, v) \in q$ which is in co-contradiction with the hypotheses $(x, u) \triangleleft q$ and $(y, v) \triangleleft q$. Thus, it must be $(a, xu) \in q \vee (yv, b) \in q$. Hence, $a \neq xu \vee yv \neq b$. So, $(xu, yv) \neq (a, b) \in q$ This means $xq \cdot_2 uq =_2 yq \cdot_2 vq$. This shows that the operation '$\cdot_2$' is well-defined.

Let $x, y, u, v \in S$ be such that $xq \cdot_2 yq \neq_2 uq \cdot_2 vq$. This means $(xy, uv) \in q$. Then $(x, u) \in q \vee (y, v) \in q$. Thus $xq \neq_2 uq \vee yq \neq_2 vq$. This proves that the operation '$\cdot_2$' is a total strongly extensional function from $S \times S$ into S.

Let $x, y, z \in S$ be arbitrary elements. Then

$$xq \cdot_2 (yq \cdot_2 zq) =_2 (x(yz))q =_2 ((xy)z)q =_2 (xq \cdot_2 yq) \cdot_2 zq.$$

On the other hand, taking into account the definition of multiplication in semigroups S and $[S : q]$, we have $xq =_2 (xx^{-1}x)q =_2 xq \cdot_2 x^{-1}q \cdot_2 xq$ and $x^{-1}q =_2 (x^{-1}xx^{-1})q =_2 x^{-1}q \cdot_2 xq \cdot_2 x^{-1}q$. In addition to the previous one, for $x, y \in S$ we have

$x^{-1}q =_2 yq \iff (x^{-1}, y^{-1}) \triangleleft q \iff (x, y) \triangleleft q \iff xq =_2 rq$ and
$x^{-1}q \neq_2 yq \iff (x^{-1}, y^{-1}) \in q \iff (x, y) \in q \iff xq \neq_2 rq$.

In this way, it has been shown that semigroup with apartness $[S : q]$ is an inverse semigroup with apoartness.

Finally, it is obvious that the correspondence ϑ is well-defined homomorphism. Let $x, y \in S$ be such that $\vartheta(x) \neq_2 \vartheta(y)$. This means $xq \neq_2 yq$, i.e. $(x, y) \in q$. Thus $x \neq y$. Thus, this shows that the homomorphism ϑ is a strongly extensional mapping.

There is a strong connection between these two semigroups:

Theorem 20. *The correspondence*

$$\theta : S/(q^{\triangleleft}, q) \ni aq^{\triangleleft} \longmapsto aq \in [S : q],$$

defined by $\theta(aq^{\triangleleft}) =: aq$, *is an embedding se-isomorphism.*

Proof. It is obvious that θ is a total and surjective correspondence.

Let $x, y \in S$ be arbitrary elements. Then:

$xq^{\triangleleft} =_1 yq^{\triangleleft} \iff (x, y) \triangleleft q \iff xq =_2 yq \iff \theta(xq^{\triangleleft}) =_2 \theta(yq^{\triangleleft})$. This shows that θ is an injective function.

$xq =_2 \theta(xq^{\triangleleft}) \neq_2 \theta(yq^{\triangleleft}) =_2 yq \iff (x, y) \in q \iff xq^{\triangleleft} \neq_2 yq^{\triangleleft}$. This shows that θ is an embedding and strongly extensional function.

The following two valid formulas show that θ is a homomorphism of inverse semigroups:

$\theta(xq^{\triangleleft} \cdot_1 yq^{\triangleleft}) =_2 \theta((xy)q^{\triangleleft}) =_2 (xy)q =_2 xq \cdot_2 yq =_2 \theta(xq^{\triangleleft}) \cdot_2 \theta(yq^{\triangleleft})$;

$\theta((xq^{\triangleleft})^{-1}) =_2 \theta(x^{-1}q^{\triangleleft}) =_2 x^{-1}q =_2 (xq)^{-1} =_2 (\theta(xq^{\triangleleft}))^{-1}$.

Although these two inverse semigroups with apartness $S/(q^{\triangleleft}, q)$ and $[S; q]$ are embedding se-isomorphic to each other, they are not one and the same semigroup. First, they are made up of different elements: The first is constructed from the classes $aq^{\triangleleft}$ $(a \in S)$ that satisfy the following properties: $(a, b) \in q \implies aq^{\triangleleft} \cap bq^{\triangleleft} = \emptyset$ and $S = \bigcup_{a \in S} aq^{\triangleleft}$. The second semigroup consists of the classes aq $(a \in S)$ that satisfy the following properties: $(a, b) \in q \implies aq \cup bq = S$ and $\bigcap_{a \in S} aq = \emptyset$.

Let us check whether the relation κ_M, determined in Proposition 9, is a co-congruence relation. Let $a, b, u \in S$ be such that $(au, bu) \in \kappa_M$. This means $(au)^{-1}(bu) \in M \vee (au)(bu)^{-1} \in M$. Written differently $u^{-1}a^{-1}bu \in M$ or $auu^{-1}b^{-1} \in M$. Let $auu^{-1}b^{-1} \in M$. From here we have

$$(auu^{-1}a^{-1})(ab^{-1}) = aa^{-1}auu^{-1}b^{-1} = auu^{-1}b^{-1} \in M$$

since idempotents commute. Thus $auu^{-1}a^{-1} \in M$ or $ab^{-1} \in M$ by (M2). The first option is impossible due to Lemma 2 and (4.7). Suppose now the possibility that $u^{-1}a^{-1}bu \in M$ is valid. From this we should get the following $a^{-1}b \in M$ for the co-equivalence κ_M to be compatible with the operation in S. It is not possible to prove that the relation κ_M, described in Proposition 9, for given a co-subsemigroup M of an inverse semigroup with apartness S is a co-congruence on S without additional requirements for M.

Theorem 21. *Let M be a co-subsemigrouup in an inverse semigroup with apartness S which satisfies the additional conditions*

(4.7) $M \subseteq E(S)^{\lhd}$ (co-fullness)
(4.8) $(\forall a, b \in S)(a \in M \implies (b \nleqslant a \vee b \in M))$.
(4.9) $(\forall a, u \in S)(u^{-1}au \in M \implies a \in M)$ (self-conjugation).

Then the relation κ_M, determined in Proposition 9, is a co-congruence on S.

Proof. κ_M is a co-equivalence on S, according to Proposition 9. We only need to prove the compatibility of the relation κ_M with the multiplication in S.

Let $a, b, u \in S$ be elements such that $(au, bu) \in \kappa_M$. This means

$$(au)^{-1}(bu) \in M \vee (au)(bu)^{-1} \in M.$$

Written differently $u^{-1}a^{-1}bu \in M$ or $auu^{-1}b^{-1} \in M$. Let $auu^{-1}b^{-1} \in M$. From here we have

$$(auu^{-1}a^{-1})(ab^{-1}) = aa^{-1}auu^{-1}b^{-1} = auu^{-1}b^{-1} \in M$$

since idempotents commute. Thus $auu^{-1}a^{-1} \in M$ or $ab^{-1} \in M$ by (M2). The first option is impossible due to Lemma 2 and (4.7). If we take that $u^{-1}a^{-1}bu \in M$ is valid, then we have $a^{-1}b \in M$ by (4.9). So, $(a, b) \in \kappa_M$.

Implications $(ua, ub) \in \kappa_M \implies (a, b) \in \kappa_M$ can be proved analogously to the previous one.

At the end of this subsection, let us point out that: If for the elements $a, b \in S$ holds $(a, b) \lhd q$, then, also, it is

$$(aa^{-1}, bb^{-1}) \lhd q, \ (a^{-1}a, b^{-1}b) \lhd q, \ (ab, ba) \lhd q$$

for any co-congruence q on an inverse semigroup with apartness S. Indeed. To illustrate, we will demonstrate the proof of one of them. The others can be proved analogously. First, recall that if $(a, b) \lhd q$, then $(b, a) \lhd q$ and $(a^{-1}, b^{-1}) \lhd q$. Let $u, v \in S$ be such that $(u, v) \in q$. Then

$$(u, ab) \in q \vee (ab, ba) \in q \vee (ba, v) \in q$$

by co-transitivity of the co-congruence q. In the second case, it would be $(a, b) \in q \vee (b, a) \in q$, which is in contrast to the hypothesis. Thus, it must be $(u, ab) \in q \vee (ba, v) \in q$. It follows $u \neq ab$ or $ba \neq v$ from here by consistency of the co-congruence q. This means $(ab, ba) \neq (u, v) \in q$. So $(ab, ba) \lhd q$.

4.3. Some examples

In [45], Wagner proved that every congruence on an inverse semigroup is completely determined by its idempotent classes. The advances made in [17], [27], [42] lead Lawson to state a general result describing the congruences on an inverse semigroup. Indeed, he showed [19], Theorem 2, p. 135) that for an inverse semigroup S, there exists a bijection between the set of congruences on S and the set of congruence pairs (namely, a couple consisting of a normal subsemigroup and a normal congruence satisfying some special conditions) on S. Following this line of research, the author of paper [7] gave a characterization of I-cocongruences on an inverse semigroup with apartness in term of "co-congruence pairs" ([7], Theorem 4 and Theorem 5). To this end, they introduced the concepts of 'coker' and 'trace' of a co-congruence as followed

$$cokerq := \bigcap_{e\in E(S)} \{x \in E(S) : (e,x) \in q\} = \bigcap_{e\in E(S)} qe$$

and

$$tr(q) := q|_{E(S)\times E(S)}.$$

The authors of [7] solved this problem in the following way:

In addition to the previous one, that paper introduced ([7], Definition 11) the term co-normal for co-congruence q on an inverse semigroup with apartness S if they meet the following condition

$$(\forall a \in S)(\forall e, f \in E(S))((a^{-1}ea, a^{-1}fa) \in q \implies (e,f) \in q).$$

This term is not necessary, since the corresponding condition is always satisfied by any co-congruence q on an inverse semigroup with apartness, determined by the condition (6).

In addition to the above, in [7] has shown that the following concepts play an important role in describing co-congruences on inverse semigroups with apartness:

- A subset M of an inverse semigroup with apartness S is an 'inverse anti-subsemigroup' if

1. $(\forall x, y \in S)(xy \in M \implies (x \in M \vee y \in M))$, and
2. $(\forall x \in S)(x^{-1} \in M \implies x \in M)$.

- An inverse anti-subsemigroup M is

3. co-full if $M \subseteq E(S)^{\lhd}$;
4. co-self-conjugate if $(\forall a, x \in S)(x^{-1}ax \in M \implies a \in M)$;
5. conormal if it is cofull and coself-conjugate.

- A co-congruence pair $(M;\kappa)$ on an inverse semigroup with apartness S consists of a co-normal inverse anti-subsemigroup M and a co-normal co-congruence κ on $E(S)$ such that

6. $(\forall a \in S)(\forall e \in E(S))(a \in M \implies (ae \in M \vee (e, a^{-1}a) \in \kappa))$, and
7. $(\forall a \in S)(a \in M \implies (aa^{-1}, a^{-1}a) \in \kappa)$.

Theorem 22 ([7], Theorem 4)**.** *Let S be an inverse semigroup with apartness and let κ be a I-cocongruence on S. Let $M = coker(\kappa)$ and $\nu = tr(\kappa)$. Then (M, ν) is a co-congruence pair.*

Here are some interesting examples of co-congruences on inverse semigroups with apartness.

Example 5. *If we define q on an inverse semigroup with apartness S as follows*

$$(\forall a, b \in S)((a, b) \in q \iff aa^{-1} \neq_S bb^{-1}),$$

then q is a co-equality on S left compatible with the semigroup operation.

Proof. Let $a, b \in S$ be such that $(a, b) \in q$. This means $aa^{-1} \neq_S bb^{-1}$. Then

$$aa^{-1} \neq_S ab^{-1} \vee ab^{-1} \neq_S bb^{-1}$$

by co-transitivity of the apartness. Thus $a^{-1} \neq_S b^{-1} \vee a \neq_S b$ by (I2)(b). This shows that the relation q is consistent. It is clear that q is a symmetric relation. Let $a, b, c \in S$ be elements such that $(a, c) \in q$. This means $aa^{-1} \neq_S cc^{-1}$. Then $aa^{-1} \neq_S bb^{-1} \vee bb^{-1} \neq_S cc^{-1}$ by co-transitivity of the apartness. Thus $(a, b) \in q \vee (b, c) \in q$ so we have shown that q is co-transitivity relation on S.

Let $a, b, u \in S$ be arbitrary elements such that $(ua, ub) \in q$. This means $(ua)(ua)^{-1} \neq_S (ub)(ub)^{-1}$. Then $uaa^{-1}u^{-1} \neq_S ubb^{-1}u^{-1}$. Hence $aa^{-1} \neq_S bb^{-1}$ by (I2)(b). So, $(a, b) \in q$.

Let $e \in S$ be an idempotent in S. Then

$$\begin{aligned} qe &= \{x \in S : (e, x) \in q\} = \{x \in S : ee^{-1} \neq_S xx^{-1}\} \\ &= \{x \in S : e \neq_S xx^{-1}\} \subseteq \{e\}^{\lhd} \end{aligned}$$

so we have

$$cokerq = \bigcap_{e \in E(S)} qe \subseteq \bigcap_{e \in E(S)} \{e\}^{\lhd} = (\bigcup_{e \in E(S)} \{e\})^{\lhd} = E(S)^{\lhd}.$$

On the other hand, we have

$$\begin{aligned} trq &= \{(e, f) \in E(S) \times E(S) : (e, f) \in q\} \\ &= \{(e, f) \in E(S) \times E(S) : ee^{-1} \neq_S ff^{-1}\} \\ &= \{(e, f) \in E(S) \times E(S) : e \neq_S f\} \\ &= \neq_S |_{E(S) \times E(S)} = \neq_{E(S)}. \end{aligned}$$

It can be shown without major difficulties that the set $E(S)^{\lhd}$ satisfies the conditions 1., 2. 3. and 6., while, in the general case, it does not have to satisfy the conditions 4. and 7.

Remark 2. *It can be shown that the relation κ on S, determined by*

$$(\forall a, b \in S)((a, b) \in \kappa \iff a^{-1}a \neq_S b^{-1}b),$$

is co-equivalence on S right compatible with the semigroup operation in an analogous way as in the previous example. In this case, we also have

$$coker\kappa \subseteq E(S)^{\triangleleft} \quad \text{and} \quad tr\kappa = \neq_{E(S)}.$$

Example 6. *If we define q on an inverse semigroup with apartness S as follows*

$$(\forall a, b \in S)((a, b) \in q \iff (\exists x \in S)(a^{-1}xa \neq b^{-1}xb)),$$

then q is a co-equality on S compatible with the semigroup operation.

Proof. Let $a, b \in S$ be such that $(a, b) \in q$. This means that there exists an element $x \in S$ such that $a^{-1}xa \neq b^{-1}xb$. Then $a^{-1}xa \neq b^{-1}xa \vee b^{-1}xa \neq b^{-1}xb$ by co-transitivity of the apartness. Thus $a^{-1} \neq b^{-1} \vee a \neq b$ by (I2)(b). Hence $a \neq b$. This shows that the relation q is consistent. It is clear that q is a symmetric relation. Let $a, b, c \in S$ be elements such that $(a, c) \in q$. Then there exists an element $x \in S$ such that $a^{-1}xa \neq c^{-1}xc$. Then $a^{-1}xa \neq b^{-1}xb \vee b^{-1}xb \neq c^{-1}xc$ by co-transitivity of the apartness. Thus $(a, b) \in q \vee (b, c) \in q$ so we have shown that q is co-transitivity relation on S.

Let $a, b, u \in S$ be arbitrary elements such that $(au, bu) \in q$. Then there exists an element $x \in S$ such that $(au)^{-1}x(au) \neq (bu)^{-1}x(bu)$. Thus

$$u^{-1}a^{-1}xau \neq u^{-1}b^{-1}xbu.$$

Hence $a^{-1}xa \neq b^{-1}xb$ by (I2)(b). So, $(a, b) \in q$. So, the co-equality q is right compatible with the semigroup operation. Let $a, b, u \in S$ be arbitrary elements such that $(ua, ub) \in q$. Then there exists an element $x \in S$ such that $(ua)^{-1}x(ua) \neq (ub)^{-1}x(ub)$. Thus $a^{-1}(u^{-1}xu)a \neq b^{-1}(u^{-1}xu)b$. So, there exists the element $y := u^{-1}xu \in S$ such that $a^{-1}ya \neq b^{-1}yb$. Hence, $(a, b) \in q$. So, the co-equality q is left compatible with the semigroup operation.

It is proved that the relation q is a co-congruence on the inverse semigroup with apartness S.

It can be shown without much difficulty that, in this case, the following is valid

$$cokerq \subseteq E(S)^{\triangleleft} \quad trq \subseteq \neq_{E(S)} .$$

Example 7. *If in the formula used to describe the relation q in Example 6, we limit the quantifier of existence to $E(S)$, we get the next relation*

$$(\forall a, b \in S)((a, b) \in q \iff (\exists e \in E(S))(a^{-1}ea \neq b^{-1}eb)),$$

The complete proof can be repeated including the implication $(ua, ub) \in q \implies (a, b) \in q$ with one additional explanation. The element $u^{-1}eu$ that appears in the evaluation of the validity of this implication lies in $E(S)$, according to Lemma 2.

Example 8. *The relation q on an inverse semigroup with apartness S, defined by*

$$(\forall a, b \in S)((a, b) \in q \iff (\exists x \in S)(xa \neq xb)),$$

is a co-congruence on S.

Proof. Let $a, b \in S$ be such that $(a, b) \in q$. This means that there exists an element $x \in S$ such that $xa \neq xb$. Thus $a \neq b$ by (I2)(b). So, the relation q is consistent. It is clear that q is a symmetric relation. Let $a, b, c \in S$ be such that $(a, c) \in q$. This means that there exists an element $x \in S$ such that $xa \neq xc$. Then $xa \neq xb \vee xb \neq cv$ by co-transitivity of the apartness. Hence, $(a, b) \in q \vee (b, c) \in q$ which proves the co-transitivity of the relation q.

Let $a, b, u \in S$ be such that $(au, bu) \in q$. Then there exists an element $x \in S$ such that $cau \neq xbu$. Thus $xa \neq xb$ by (I2)(b). Hence, $(a, b) \in q$. At the other hand, if $a, b, u \in S$ be element such that $(ua, ub) \in q$, then there exists an element $x \in S$ such that $xua \neq xub$. So, there exists an element $xu \in S$ such that $(xu)a \neq (xu)b$. this means that $(a, b) \in q$.

Example 9. *Let S be an inverse semigroup with apartness and let a relation q be defined on S by*

$$(\forall a, b \in S)((a, b) \in q \iff (\exists\, e \in E(S))(ea \neq eb)).$$

Then q is a co-congruence on S.

Proof. This relation is a special case of the relation described in Example 8. Indeed: if $xa \neq xb$ for some $x \in S$, then $xx^{-1}xa \neq xx^{-1}xb$, which implies $x^{-1}xa \neq x^{-1}xb$ and $x^{-1}x \in E(S)$. Almost all the evidence can be repeated except for part

$$(\forall a, b, u \in S)((ua, ub) \in q \implies (a, b) \in q).$$

Let $a, b, u \in S$ be such that $(ua, ub) \in q$. This means that there exists an idempotent $e \in E(S)$ such that $e(ua) \neq e(ub)$. Then

$$\begin{aligned} u(u^{-1}eu)a &= (uu^{-1})(eu)a = e(uu^{-1}u)a \\ &\neq e(uu^{-1}u)b = (uu^{-1})(eu)b = u(u^{-1}eu)b. \end{aligned}$$

Thus $(u^{-1}eu)a \neq (u^{-1}eu)b$ by (I2)(b). Since $u^{-1}eu \in E(S)$ by Lemma 2 it holds $(a, b) \in q$.

This relation is the same as the apartness relation. If $a \neq b$ then either $a \nleqslant b$ or $b \nleqslant a$, by Theorem 3. In the first case, we have $aa^{-1}a = a \neq aa^{-1}b$, and hence $(a, b) \in q$ because $aa^{-1} \in E(S)$. Similarly $b \nleqslant a$ implies $(a, b) \in q$.

Example 10. *Let S be an inverse semigroup with apartness and let a relation q be defined on S by*

$$(\forall a, b \in S)((a, b) \in q \iff (\exists\, e, f \in E(S))(eaf \neq ebf)).$$

Then q is a co-congruence on S.

Proof. It is obvious that q is symmetric. Let $a, b \in S$ br such that $(a, b) \in q$. This means that there exist idempotents $e, f \in E(S)$ such thar $eaf \neq ebf$. Thus $a \neq b$ by (I2)(b). Let $a, bc \in S$ be arbitrary elements such that $(a, c) \in q$. This means that there exist idempotent $e, f \in E(S)$ such that $eaf \neq ecf$. Thus $eaf \neq ebf \vee ebf \neq ecf$ by co-transitivity of the apartness. Hence $(a, b) \in q \vee (b, c) \in q$ thus proving that q is co-transitive.

Let $a, b, u \in S$ be such $(au, bv) \in q$. This means that there exist idempotent $e, f \in E(S)$ such that $e(au)f \neq e(bu)f$. Then there exists an idempotent $f' \in E(S)$ such that $uf = f'u$ by Lemma 3. Now, we have $eaf'u \neq ebf'u$. Thus $eaf' \neq ebf'$. The implication $(ua, ub) \in q \implies (a, b) \in q$ can be proved analogously. Suppose now that $(a^{-1}, b^{-1}) \in q$. This means that there exist idempotent $e, f \in E(S)$ such that $ea^{-1}f \neq eb^{-1}f$. Then $fae \neq fbe$. So,$(a, b) \in q$ This shows that q is compatible with the semigroup operations.

So q is a co-congrience on S.

The following theorem can be proved by direct verification, without major difficulties:

Theorem 23. *The family $\mathfrak{Q}(S)$ of all co-congruences on an inverse semigroup with apartness S forms a complete lattice.*

Proof. Let $\{\alpha_i\}_i$ be a family of co-congriences on an inverse semigroup with apartness S. Then $\bigcup_i \alpha_i \subseteq \neq$ because $\alpha_i \subseteq \neq$ for every i. Since $\alpha_i \subseteq \alpha_i^{-1}$ is valid for every i, we conclude that $\bigcup_i \alpha_i \subseteq \bigcup_i \alpha_i^{-1} = (\bigcup_i \alpha_i)^{-1}$ holds. which proves that $\bigcup_i \alpha_i$ is symmetric. Let $x, y, z \in S$ be such that $(x, z) \in \bigcup_i \alpha_i$. Then there exists an index j such that $(x, z) \in \alpha_j$. This $(x, y) \in \alpha_j \subseteq \bigcup_i \alpha_i$ or $(y, z) \in \alpha_j \subseteq \bigcup_i \alpha_i$. Therefore, $\bigcup_i \alpha_i$ is a co-equality relation on S.

Let us prove the compatibility of this relation with the operations in S: Let $a, b, u, v \in S$ be such that $(au, bv) \in \bigcup_i \alpha_i$. Then there exists an index j such that $(au, bv) \in \alpha_j$. Thus $(a, b) \in \alpha_j \subseteq \bigcup_i \alpha_i$ or $(u, v) \in \alpha_j \subseteq \bigcup_i \alpha_i$. Assume that $(a^{-1}, b^{-1}) \in \bigcup_i \alpha_i$. Then there exists an index j such that $(a^{-1}, b^{-1}) \in \alpha_j$. Thus $(a, b) \in \alpha_j \subseteq \bigcup_i \alpha_i$.

Let $\mathcal{X}$ be the family of all co-congruences contained in $\bigcap_i \alpha_i$. Then $\cup\mathcal{X}$ is the maximum co-congruence contained in $\bigcap_i \alpha_i$ according to the first part of this proof.

If we put $\sqcup_i \alpha_i = \bigcup_i \alpha_i$ and $\sqcap_i \alpha_i = \cup\mathcal{X}$, then $(\mathfrak{Q}(S), \sqcup, \sqcap)$ is a complete lattice.

4.4. Group co-congruences

Let us recall the following descriptions in the classical semigroup theory (see, for example, [17]):

- A congruence ρ on an inverse semigroup S is called a group congruence just in case S/ρ is a group.

One can look at, for example, the article [25] about group congruences on inverse semigroups.

We will first design a co-congruence q on an inverse semigroup with apartness S such that the semigroups $S/(q^{\lhd}, q)$ and $[S : q]$ are groups: In order for $S/(q^{\lhd}, q)$ to be a group, it is necessary that:

- There is a unique element $eq^{\lhd} \in S/(q^{\lhd}, q)$ such that for every $xq^{\lhd} \in S/(q^{\lhd}, q)$ the following holds

$$eq^{\lhd} \cdot_1 xq^{\lhd} =_1 xq^{\lhd} \cdot_1 eq^{\lhd} =_1 xq^{\lhd} \text{ and } eq^{\lhd} \cdot_1 eq^{\lhd} =_1 eq^{\lhd}.$$

This means that for the element $e \in S$ such that $eq^{\lhd}$ is the unique idempotent in $S/(q^{\lhd}, q)$ must hold

$$(e \cdot x, x) \lhd q, \ (x, x \cdot e) \lhd q \ \ (x \cdot e, e \cdot x) \lhd q. \tag{7}$$

- For each element $xq^{\lhd} \in S/(q^{\lhd}, q)$ there is a unique element $x'q^{\lhd} \in S/(q^{\lhd}, q)$ such that

$$xq^{\lhd} \cdot_1 x'q^{\lhd} =_1 eq^{\lhd} =_1 x'q^{\lhd} \cdot_1 xq^{\lhd}$$

that is, it must be valid

$$(x \cdot x', e) \lhd q, \ (x' \cdot x, e) \lhd q \ \ (x \cdot x', x' \cdot x) \lhd q. \tag{8}$$

In what follows we adopt that the following formula

$$(\forall t)(A \vee F(t) \implies A \vee (\forall t)F(t)$$

where t is not free in A is valid formula.

Theorem 24. *Let S be an inverse semigroup with apartness and let a relation σ be defined on S by the rule that $(x, y) \in \sigma$ if and only if for for any idempotent $e \in E(S)$ holds $ex \neq ey$. Then σ is a co-congruence and the sets $S/(\sigma^{\lhd}, \sigma)$ and $[S : \sigma]$ are groups.*

Proof. It is clear that σ is consistent and symmetric. Suppose that x, y and z are elements of S such that $(x,z)\in\sigma$. Then for any idempotent $e\in E(S)$ holds $ex\neq ez$. Thus $efx\neq efz$ for any idempotent $f\in E(S)$ because $E(S)E(S)\subseteq E(S)$ holds. Hence $efx\neq efy\vee efy\neq efz$ by co-transitivity of the apartness. It follows $ex\neq ey$ or $fy\neq fz$ from here. We now refer to the validity of the previously mentioned formula. Therefore, the relation σ is co-transitive.

Now let $x,y,u\in S$ be elements such that $(xu,yu)\in\sigma$ and $(ux,uy)\in\sigma$. By hypothesis, for any idempotent $e\in E(S)$ the following holds $exu\neq eyu$ and $e(ux)\neq e(uy)$. From $exu\neq eyu$ we get $ex\neq ey$ immediately. This means $(xu,yu)\in\sigma\implies(x,y)\in\sigma$. Since $eux\neq euy$ holds for every $e\in E(S)$, the following also holds $ueu^{-1}(ux)\neq ueu^{-1}(uy)$ because ueu^{-1} is also an idempotent. So, we have $ue(u^{-1}u)x\neq ue(uu^{-1})y$. Thus $uu^{-1}u(ex)\neq uu^{-1}u(ey)$ because idempotents commute. Hence $ex\neq ey$ for any idempotent $e\in E(S)$. We have proved that the implication $(ux,uy)\in\sigma\implies(x,y)\in\sigma$ is valid. From $(x^{-1},y^{-1})\in\sigma$ immediately follows $(x,y)\in\sigma$. This proves that σ is a co-congruence on S.

Sets $S/(\sigma^{\lhd},\sigma)$ and $[S:\sigma]$ are inverse semigroups with apartness according to Theorem 18 and Theorem 19. We next prove that the semigroups $S/(\sigma^{\lhd},\sigma)$ and $[S:\sigma]$ are groups. Let u,v,e,f be arbitrary elements in S such that $(u,v)\in\sigma$ and $e,f\in E(S)$. Then

$$(u,e)\in\sigma\ \vee\ (e,ef)\in\sigma\ \vee\ (ef,f)\in\sigma\ \vee\ (f,v)\in\sigma.$$

Since the second and third options are impossible because the following holds $(ef)e=e(ef)=ef=(ef)f$, we have $u\neq e\ \vee\ f\neq v$. This means $(e,f)\neq(u,v)\in\sigma$. Therefore, all the idempotents belong to the same class of the relation $\sigma^{\lhd}$, which we denote by $e\sigma^{\lhd}$. So the following equations

$$e\sigma^{\lhd}=_1 f\sigma^{\lhd}\quad e\sigma=_2 f\sigma$$

are valid for any idempotents e and f. Then

$$\begin{aligned}e\sigma^{\lhd}\cdot_1 x\sigma^{\lhd}&=_1(xx^{-1})\sigma^{\lhd}\cdot_1 x\sigma^{\lhd}=_1(xx^{-1}x)\sigma^{\lhd}\\&=_1 x\sigma^{\lhd}\cdot_1(x^{-1}x)\sigma^{\lhd}=_1 x\sigma^{\lhd}\cdot_1 e\sigma^{\lhd}.\end{aligned}$$

Also

$$x^{-1}\sigma^{\lhd}\cdot_1 x\sigma^{\lhd}=_1(x^{-1}x)\sigma^{\lhd}=_1 e\sigma^{\lhd}=_1(xx^{-1})\sigma^{\lhd}=_1 x\sigma^{\lhd}\cdot_1 x^{-1}\sigma^{\lhd}.$$

Therefore, $S/(\sigma^{\lhd},\sigma)$ is a group.

On the other hand, we have

$$e\sigma\cdot_2 x\sigma=_2(xx^{-1})\sigma\cdot_2 x\sigma=_2(xx^{-1}x)\sigma=_2 x\sigma\cdot_2(x^{-1}x)\sigma=_2 x\sigma\cdot_2 e\sigma$$

and

$$x^{-1}\sigma\cdot_2 x\sigma=_2(x^{-1}x)\sigma=_2 e\sigma=_2(xx^{-1})\sigma=_2 x\sigma\cdot_2 x^{-1}\sigma.$$

Therefore, $[S:\sigma]$ is a group.

4.5. Semilattice co-congruences

Let us recall the following descriptions in the classical semigroup theory (see, for example, [17]):

- A congruence ρ on semigroup S will be called a semilattice congruence if and only if S/ρ is a semilattice.

This requirement in the case of an inverse semigroup with apartness S and a co-congruence q on S, the requirement that the co-conruence q be a semilattice co-congruence, has the following form:

A co-congruence q na a semigroup with apartness $(S, =, \neq, \cdot)$ is a 'semilattice co-congruence on S' if the semigroup $S/(q^{\triangleleft}, q)$ is a semilattice. This means that the following conditions must be valid:

$$(\forall a, b \in S)(aq^{\triangleleft} \cdot_1 bq^{\triangleleft} =_1 bq^{\triangleleft} \cdot aq^{\triangleleft}) \text{ and } (\forall a \in S)(aq^{\triangleleft} \cdot_1 aq^{\triangleleft} = aq^{\triangleleft}).$$

Thus, for a co-congruence q on an inverse semigroup with apartness S to be a semilattice co-congruence on S, the following two formulas should be valid formulas:

$$(\forall a, b \in S)((ab, ba) \lhd q) \quad (\forall a \in S)((a^2, a) \lhd q). \tag{9}$$

If these conditions are valid in an inverse semigroup with apartness S, then both semigroups $S/(q^{\triangleleft}, q)$ and $[S : q]$ are semilattices.

The first, if ρ is a semilattice congruence on an inverse semigroup S, then $a\rho$ is an idempotent in S/ρ if and only if $a\rho =_1 aa^{-1}\rho$ ([8], Lemma 7.34). So for a semilattice co-congruence q on an inverse semigroup with apartness S and for an element $a \in S$ such that $aq^{\triangleleft} \in E(S/(q^{\triangleleft}, q))$ the following holds $aq^{\triangleleft} =_1 aa^{-1}q^{\triangleleft}$ and $aq =_2 aa^{-1}q$, i.e., it is valid $(a, aa^{-1}) \lhd q$. Since a congruence ρ on an inverse semigroup S is a semilattice congruence just in case it contains the Green's relation $\mathcal{R}$ ([17], pp. 142), that is, it must be $\mathcal{R} \subseteq q^{\triangleleft}$ if we want a co-congruence q to be a semilattice co-congruence on an inverse semigroup with apartness S. If we put that q the union of all co-equalities that are included in the relation $\mathcal{R}^{\triangleleft}$, it is to be expected that the co-congruence q^* (Lemma 17) will be a semilattice co-congruence on an inverse semigroup with apartness S. In this case, both semigroups $S/(q^{\triangleleft}, q)$ and $[S : q]$ are semilattices.

5. Final comments

Semigroup theory is one of the more important mathematical theories. Within that theory, the theory of inverse semigroups plays an important role as it is interesting to a large number of mathematicians. Although semigroup's theory with apartness within the Bishop's constructive framework has been the subject of study for more than twenty years, the first work on inverse semigroups with apartness appeared only in 2019. This paper is a continuation of research on such a class of semigroups. In fact, the dilemma, whether semigroups with apartness are a new class of semigroups or is it a different aspect of observing these algebraic structures, is still open. By choosing Intuitionist logic as the principle-logic-working milieu instead of classical logic, algebraic structures can be viewed under a different light than is the case in the classical-logic environment. The first choice enables the perception of two parallel worlds of substructures and their mutual relations in the observed algebraic structures, where, not infrequently, some structures appear that do not have their counterparts in the classical case.

References

[1] P. Aczel and M. Rathjen, *CST Book draft.*
Available at https://www1.maths.leeds.ac.uk/ rathjen/book.pdf

[2] A. Bauer, *Five stages of accepting Constructive mathematics.* Bull. Amer. Math. Soc., 54(3)(2017), 481-498.

[3] M. Beeson, *Foundations of Constructive Mathematics*, Berlin: Springer 1985.

[4] E. A. Bishop, *Foundations of Constructive Analysis.* McGraw-Hill, New York 1967.

[5] D. S. Bridges and F. Richman, *Varieties of Constructive Mathematics.* vol. 97 of London Mathematical Society Lecture Notes. Cambridge University Press, Cambridge 1987.

[6] D. Bridges, H. Ishihara, M. Rathjen and H. Schwichtenberg (eds.): *Handbook of Bishop's Mathematics. Cambridge*, University Press (in press).

[7] A. Cherubini and A. Frigeri, *Inverse semigroups with apartness.* Semigroup Forum, 98(3)(2019), 571-–588.

[8] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups, Volumes I and II.* Amer. Math. Soc, Providence, Rhode Island 1961 and 1967.

[9] L. Crosilla, *Bishop's mathematics: A philosophical perspective.* In: D. Bridges, H. Ishihara, M. Rathjen and H. Schwichtenberg (eds.). Handbook of Bishop's Mathematics. Cambridge University Press (in press).

[10] S. Crvenković, M. Mitrović and D. A. Romano, *Semigroups with apartness.* Math. Logic Q., 59(6)(20130, 407-–414.

[11] S. Crvenković, M. Mitrović and D. A. Romano, *Basic notions of (constructive) semigroups with apartness.* Semigroup Forum, 92(3)(2016), 659-–674.

[12] E. Darpö and M. Mitrović, *Some results on constructive semigroup theory.* arXiv:2103.07105 [math.GR]

[13] C. Hollings, *Three approaches to inverse semigroups.* Eur. J. Pure Appl. Math., 8(3)(2015), 294–323.

[14] J. M. Howie, *The maximum idempotent-separating congruence on an inverse semiroup.* Proc. Edinb. Math. Soc., 14(1)(1964), 71–79.

[15] J. M. Howie, *Why study semigroups?* Mathematical Chronicle, 16(1987), 1-–14.

[16] J. M. Howie, *Foundamentals of semigroup theory.* Oxford University Press Inc., New York 1995.

[17] D. G. Green, *The lattice of congruences on an inverse semigroup.* Pacific J. Math., 57(1)(1975), 141–152.

[18] M. V. Lawson, *Inverse semigroups.* Available at: arXiv:2006.01628v1

[19] N. V. Lawson, *Inverse Semigroups: The Theory of Partial Symmetries.* World Scientific, Singapore 1999.

[20] H. Lombardi and C. Quitt'e, *Commutative algebra: Constructive methods.* (English translation by Tania K. Roblot) Arxiv 1605.04832v3 [Math. AC]

[21] E. S. Ljapin, *Semigroups.* State publishing house of physical and mathematical literature. Moskva 1960. (In Russian)

[22] R. Mines, F. Richman and W. Ruitenburg, *A course of constructive algebra.* New York: Springer-Verlag 1988.

[23] H. Mitsch, *Inverse semigroups and their natural order.* Bull. Austral. Math. Soc., 19(1978), 59–65.

[24] M. Mitrović, D. A. Romano and M. Vinčić, *A theorem on semilattice-ordered semigroup.* Int. Math. Forum, 4(5)(2009), 227–232.

[25] W. D. Munn, *A class of irreducible matrix representations od an arbitrary inverse semigroup.* Proc. Glasg. Math. Assoc., 5(1)(1961), 41–48.

[26] M. Petrich, *Congruences on inverse semigroups.* J. Algebra, 55(2)(1978), 231–256.

[27] M. Petrich and N. R. Reilly, *A network of congruences on an inverse semigroup.* Trans. Am. Math. Soc., 270(1982), 309–325.

[28] G. B. Preston, *Inverse semi-groups.* J. London Math. Soc., 29(1954), 396–403.

[29] N. R. Reilly and H. E. Scheiblich, *Congruences on regular semigroups.* Pacific J. Math., 23(2)(1967), 349–360.

[30] D. A. Romano, *Aspect of constructive Abelian groups.* In: Z. Stojaković (Ed.). *Proceedings of the 5th conference "Algebra and Logic"*, Cetinje 1986, (pp. 167–174). University of Novi Sad, Institute of Mathematics, Novi Sad 1987.

[31] D. A. Romano, *Rings and fields, a constructive view*. Math. Logic Q., 34(1)(19880, 25–40.

[32] D. A. Romano, *Coequality relations, a survey*. Bull. Soc. Math. Banja Luka, 3(1996), 1–36.

[33] D. A. Romano, *Semivaluation on Heyting field*. Kragujevac J. Math., 20(1998), 24–40.

[34] D. A. Romano, *A note on quasi-antiorder in semigroup*. Novi Sad J. Math., 37(1)(2007), 3–8.

[35] D. A. Romano, *On semilattice-ordered semigroups. A Constructive point of view*. Sci. Stud. Res., Ser. Math. Inform., 21(2)(2011), 117–134.

[36] D. A. Romano, *Semilattice-ordered semigroup with apartness representation problem*. J. Adv. Math. Stud., 5(2)(2012), 13–19.

[37] D. A. Romano, *On quasi-antiorder relation on semigroups*. Mat. vesnik, 64(3)(2012), 190–199.

[38] D. A. Romano, *Some algebraic structures with apartness, A review*. J. Int. Math. Virtual Inst., 9(2)(2019), 361–395.

[39] D. A. Romano, *On co-filters in semigroup with apartness*. Kragujevac J. Math., 45(4)(2021), 607-–613.

[40] T. Saitô, *Proper ordered inverse semigroups*. Pacific J. Math., 15(2)(1965), 649–666.

[41] T. Saitô, *Ordered inverse semigroups*. Trans. Amer. Math. Soc., 153(1)(1971), 99–138.

[42] H. E. Scheiblich, *Kernels of inverse semigroup homomorphisms*. J. Australian Math. Soc., 18(3)(1974), 289–292.

[43] K. V. R. Srinivas and R. Nandakumar, *Algebraic properties and examples of inverse semigroups*. Proyecciones, 28(3)(2009), 227–232.

[44] V. V. Wagner, *Generalized groups*. Dokl. Akad. Nauk SSSR (N.S.), 84(1952), 1119–1122. (In Russian)

[45] V. V. Wagner, *The theory of generalized heaps and generalized groups*. Matematicheskii Sbornik, (NS), 32(3)(1953), 545-–632. (In Russian)

Daniel A. Romano
International Mathematical Virtual Institute,
78000 Banja Luka, Bosnia and Herzegovina
email: *daniel.a.romano@hotmail.com*