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INSPECTION ON NULL HYPERSURFACES OF TRANS-PARA SASAKIAN MANIFOLDS

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Abstract. In the present discourse, we explore various kinds of null hypersurfaces of trans-para-Sasakian manifolds, including
(i) Re-current,
(ii) Lie re-current,
(iii) Hopf null hypersurfaces.
We also discuss a few axioms of screen semi-invariant null hypersurfaces of trans-para Sasakian manifolds. In addition, we obtain a few results on conformal hypersurfaces and screen totally geodesic null hypersurfaces. Lastly, we examine the integrability conditions for the distributions engaged with the screen semi-invariant null hypersurface of a trans-para Sasakian manifold.

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1. Introduction

One of the most specific and fascinating areas in the theory of null submanifolds is the differential geometry of null hypersurfaces. Some of its significant applications include mathematical physics [4], electromagnetism [5], black hole theory [3], string theory, and general theory of relativity (GTR) [9]. A submanifold of a semi-Riemannian manifold is called a null submanifold if the induced metric is degenerate then a submanifold of semi-Riemannian is null submanifold which completely different from the non-degenerate submanifolds. Null submanifolds of almost contact metric manifolds were first introduced by Duggal and Bejancu in 1996 (reference source [5]). This idea has been extended further by Duggal and Sahin, who have also studied numerous new classes of null submanifolds (see references [6, 7, 8]). On null submanifolds with various spaces, several geometers, including Jin, have been explored (see cites [1, 2, 13, 14, 15, 16] for examples).

The research of para-complex structure and nearly para-contact structure on a semi-Riemannian manifold, on the other hand, was first started in 1985 by Kaneyuki and Konzai [17]. Para-contact metric manifolds have been thoroughly researched by Zamkovoy [21]. The useful contribution of semi-Riemannian manifolds' para-contact geometry has since been demonstrated in numerous articles that have been studied ([21, 22, 23]).

In 2018, Zamkovoy also introduced the geometry of trans-para-Sasakian manifolds [22]. An almost contact structure on a manifold M is called a trans-Sasakian structure [11] if the product manifolds $M \times \mathbb{R}$ belongs to the class W_4 [10]. In [12], Marrero and Chinea completely characterized trans-Sasakian structures of types (α, β). We note that the trans-Sasakian structures of type $(\alpha, 0)$, $(0, \beta)$ and $(0, 0)$ are α-Sasakian [11], β-Kenmotsu [12], and cosympletic [10], respectively. In [11], S. Zamkovoy consider the trans-para-Sasakian manifolds as an analogue of the trans-Sasakian manifolds. A trans-para-Sasakian manifolds is a trans-para-Sasakian structure of type (α, β), where and β are smooth functions. The trans-para-Sasakian manifolds of type (α, β), and are respectively the para-Sasakian manifolds,in case $\alpha = 1$, para-Kenmostu manifolds in case $\beta = 1$ [12] and para-cosympletic manifolds $(\alpha = \beta = 0)$. Siddiqi also explores some properties of trans-para Sasakian manifolds which are closely correlated with this research note (cf. [18, 19, 20]).
by the above research articles, in the present paper we consider the three kinds of null hypersurfaces of a trans-para-Sasakian manifold
(i) Re-current hypersurface,
(ii) Lie re-current hypersurface,
(iii) Hopf null hypersurfaces hypersurface.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M has an almost paracontact structure (φ, ξ, η) if it admits a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the following compatibility conditions

$$\varphi^2 X = X - \eta(X)\xi, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1, \tag{1}$$

The distribution $\mathbb{D} : p \in M \longrightarrow \mathbb{D}_p \subset T_pM$: $\mathbb{D}_p = Ker\eta = \{X \in T_pM : \eta(X) = 0\}$ is called paracontact distribution generated by η.

An immediate consequence of the definition of almost paracontact structure is that the endomorphism φ has rank $2n$.

If a $(2n+1)$-dimensional manifold M with (φ, ξ, η) structure admits a pseudo-

Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2}$$

then we say that M has an almost paracontact metric structure and g is called compatible. Any compatible metric g with a given almost paracontact structure is necessarily of signature $(n+1, n)$. Note that setting $Y = \xi$, we have $\eta(X) = g(X, \xi)$. Further, any almost paracontact structure admits a compatible metric.

Definition 1. *If* $(g(X, \varphi Y) = d\eta(X, Y)$, *where* $d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y])$, *then* η *is a paracontact form and the almost paracontact metric manifold* $(M, \varphi, \eta, \xi, g)$ *is said to be a paracontact metric manifold.*

A paracontact structure on $M^{(2n+1)}$ naturally give rise to an almost paracomplex structure on the product $M^{(2n+1)} \times \mathbb{R}$. If this almsot paracomplex structure is integrable, then the given paracontact metric manifold is said to a para-Sasakian. Equivalently, (see [22]) a paracontact metric manifold is a para-Sasakian if and only if

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{3}$$

the manifold $(M, \varphi, \xi, \eta, g)$ of dimension $(2n + 1)$ is said to be trans-para-Sasakian manifolds if and only if

$$(\nabla_X \varphi)Y = \alpha(-g(X, Y)\xi + \eta(Y)X) + \beta(g(X, \varphi Y)\xi + \eta(Y)\varphi X), \tag{4}$$

from (4), we also have

$$\nabla_X \xi = -\alpha\varphi X - \beta(X - \eta(X)\xi). \tag{5}$$

Now, we have the following lemma [15]

Lemma 1. *[15] Let* $(M^{(2n+1)}, \varphi, \eta, \xi, g)$ *be a trans-para-Sasakian manifold. Then we have*

$$R(X, Y)\xi = -(\alpha^2 + \beta^2)[\eta(Y)X - \eta(X)Y], \tag{6}$$

$$R(\xi, Y)Z = -(\alpha^2 + \beta^2)[g(Y, Z)\xi - \eta(Z)X], \tag{7}$$

$$S(X, \xi) = -2n(\alpha^2 + \beta^2)\eta(X), \tag{8}$$

$$(\nabla_X \eta)Y = \alpha g(X, \varphi Y) - \beta(g(X, Y) - \eta(X)\eta(Y)), \tag{9}$$

for all $X, Y, Z \in T(M)$.

3. Null hypersurfaces

Let $\bar{M}$ be a semi-Riemannian manifold with index r, $0 < r < 2n+1$ and M be a hypersurface of $\bar{M}$, with induced metric $g = \bar{g}\,|_{\bar{M}}$. M is a null hypersurface of $\bar{M}$ if the metric g is of rank $2n-1$ and the orthogonal complement $TM^{\perp}$ of tangent space TM , given as

$$TM^{\perp} = \Big\{ X_p \in T_pM^{\perp} : g_p(X_p, Y_p) = 0, \forall \quad Y_p \in \Gamma(T_pM) \Big\}$$

is a distribution of rank 1 on M. $TM^{\perp} \subset TM$ and then coincides with the *radical distribution* $Rad(TM)$ such that

$$Rad(TM) = TM \cap TM. \tag{10}$$

A complementary bundle of $TM^{\perp}$ in TM is a non-degenerate distribution of constant rank $2n-1$ over M. It is known as *screen distribution* and denoted by $S(TM)$.

Let $(M, g, S(TM))$ be a null hypersurface of a semi-Riemannian manifold $\bar{M}$. Then there exists a unique rank over subbundle $tr(TM)$ called the null *transversal vector* bundle of of M with respect to $S(TM)$, such that for any non-zero section E of $Rad(TM)$ on coordinate neighborhood of M, there exists a unique section $\mathcal{N}$ of $tr(TM)$ on U satisfying

$$g(\mathcal{N}, X) = 0, \quad g(\mathcal{N}, \mathcal{N}) = 0, \quad g(\mathcal{N}, \mathcal{R}) = 1, \quad \forall X \in \Gamma(S(TM))\,|_U. \tag{11}$$

Then, we have the decomposition on the tangent bundle [8]

$$TM = S(TM) \perp Rad(T) \tag{12}$$

$$T\bar{M} = TM \oplus tr(TM) = S(TM) \perp \{Rad(T) \oplus tr(TM)\}. \tag{13}$$

Let $P : TM \longrightarrow S(TM)$ be the projection morphism. Then, we have the local Gauss-Weingarten formulas of M and $S(TM)$ as follows:

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{14}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{tr}_X N, \tag{15}$$

$$\nabla_X PT = \nabla^*_X PY + C(X, PY)E, \tag{16}$$

$$\bar{\nabla}_X E = -A^*_E X - \tau(X)E \tag{17}$$

for ant $X, Y \in \Gamma(TM)$, where ∇ is a linear connection on M and ∇^* is a linear connection on $S(TM)$ and B, A_N and τ are called the local second fundamental form on $T(M)$respectively. It is well know that the induced connection ∇ is semi-symmetric non-metric connection and we get

$$(\nabla_X g) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \tag{18}$$

$$T(X,Y) = \eta(X)Y - \eta(Y)X. \tag{19}$$

B is symmetric on $T(M)$, where T is the torsion tensor with respect to the induced connection ∇ on M and $\eta(X) = g(X,N)$ is a differential 1-form on TM.
Also the second fundamental form B is independent of the choice of $S(TM)$ and

$$B(X,E) = 0. \tag{20}$$

The local second fundamental forms are related to their shape operators by

$$B(X,PY) = g(A^*_E X, PY), \qquad g(A^*_E X, N) = 0, \tag{21}$$

$$C(X,PY) = g(A_N X, PY), \qquad g(A_N X, N) = 0. \tag{22}$$

From (21), A^*_E is a $S(TM)$-valued real self-adjoint operator and satisfies

$$A^*_E E = 0. \tag{23}$$

4. Screen Semi-invariant Null hypersurfaces

In this segment, we have discuss screen semi-invariant null hypersurfaces of a trans-para Sasakian manifold.

Let M be a null hypersurface of a trans-para Sasakian manifold $\bar{M}$ with $\xi \in \Gamma(TM)$. If E is a local section of $\Gamma Rad(TM)$, the

$$g(\varphi E, E) = 0, \tag{24}$$

and φE is tangent to M. Therefore, we obtain a distribution $\varphi(Rad(TM))$ of dimension 1 on M.
If

$$\varphi((tr(TM)) \subset S(TM), \quad and \quad \varphi(Rad(TM)), \tag{25}$$

then null hypersurface M is called a screen semi-invariant null hypersurface of $\bar{M}$ [1].

Since M is a screen semi-invariant null hypersurface then we can state:

$$g(\varphi N, N) = 0 \tag{26}$$

$$g(\varphi N, E) = -g(N, \varphi E) = 0. \tag{27}$$

$$g(N, E) = 1 \tag{28}$$

from (2), we obtain

$$g(\varphi E, \varphi N) = -1. \tag{29}$$

Therefore, $\varphi(Rad(TM)) \oplus \varphi(tr(TM))$ is a non-degenerate vector sub-bundle of screen distributions $S(TM)$.
Now, since $S(TM)$ and $\varphi(Rad(TM)) \oplus \varphi(tr(TM))$ are non-degenerate distribution $\bar{D}_0$ such that

$$S(TM) = D_0 \perp \{\varphi(Rad(TM)) \oplus \varphi(tr(TM))\}. \tag{30}$$

Therefore, in this case $\varphi(D_0) = D_0$ and $\xi \in D_0$. In view of (12), (14) and (30) we obtain the followings

$$TM = D_0 \perp \{\varphi(Rad(TM)) \oplus \varphi(tr(TM))\} \perp Rad(TM) \tag{31}$$

$$T\bar{M} = D_0 \perp \{\varphi(Rad(TM)) \oplus \varphi(tr(TM))\} \perp \{Rad(TM)(TM)\}. \tag{32}$$

Now, we take $D_1 = Rad(TM) \perp \varphi(Rad(TM)) \perp D_0$ and $D_2 = \varphi(tr(TM))$ on M, we get

$$TM = D_1 \oplus D_2. \tag{33}$$

Let the local null vector fields $V = \varphi E$ and $U = \varphi N$ and denote the projection morphism of TM into D_1 and D_2 by P_1 and P_2, respectively. Therefore , for $X \in \Gamma(TM)$,we have

$$X = P_1 X + P_2 X, \qquad P_2 X = u(X)U, \tag{34}$$

where u is a differential 1-form locally defined by

$$u(X) = -g(\varphi E, X), \quad and \quad v(X) = -g(\varphi N, X). \tag{35}$$

Operating φ on X, we get

$$\varphi X = \varphi(P_1 X) + u(X)N. \tag{36}$$

If we put $\varphi X = \varphi(P_1 X)$ in above relation, we obtain the following:

$$\varphi X = \omega X + u(X)N, \tag{37}$$

where ω is a tensor field defined as $\omega = \varphi \circ P_1$ of type $(1,1)$.
Again operating ω to (37), we get

$$\omega^2 X = X - \eta(X)\xi - u(X)(U), \qquad u(U) = 1. \tag{38}$$

Now, from (12) comparing the different components, we get

$$(\nabla_X \omega)Y = \alpha(-g(X,Y)\xi + \eta(Y)X) + \beta(g(X,\varphi Y)\xi + \eta(Y)\omega X) \tag{39}$$

$$+B(X,Y)\bar{U} + u(Y)A_N X,$$

$$(\nabla_X u)Y = u(Y)\tau(X) - B(X,\omega X) + \beta\eta(Y)u(X), \tag{40}$$

$$(\nabla_X v)Y = v(Y)\tau(X) + g(A_N X, \omega Y) + (\beta - \alpha)\eta(X)\eta(Y), \tag{41}$$

$$\nabla_X \bar{U} = \omega(A_N X - \tau(X)\bar{U} - \alpha\eta(X)\xi + \beta v(X)\xi, \tag{42}$$

$$\nabla_X \bar{V} = \omega(A^*_E X) - \tau(X)U + \beta u(X)\xi, \tag{43}$$

$$B(X,\bar{U}) = C(X,\bar{V}). \tag{44}$$

5. Re-current screen semi-invariant null hypersurface

Now, we give the following definition

Definition 2. *Let M be a screen semi-invariant null hypersurface of trans-para Sasakian manifold $\bar{M}$ and μ be a 1-form on M. If M admits a re-current tensor field ω such that*

$$(\nabla_X \omega)Y = \mu(X)\omega Y \tag{45}$$

then it is called recurrent [8].

Theorem 2. *Let M is a re-current screen semi-invariant null hypersurface of a trans-para Sasakian manifold $\bar{M}$. Then*

1. $\alpha = \beta = 0$ *i.e., $\bar{M}$ is a para cosympletic manifold,*

2. *ω is parallel with respect to the induced connection ∇ on M,*

3. $A_N X = -\mu(X)\bar{U} - v(X)\xi$

4. $A^*_E X = -\mu(X)\bar{V} - u(X)\xi$.

Proof. (1)-From (39), we have

$$\mu(X)\omega Y = \alpha(-g(X,Y)\xi + \eta(Y)X) + \beta(g(X,\varphi Y)\xi + \eta(Y)\omega X) \tag{46}$$

$$+B(X,Y)\bar{U} + u(Y)A_N X.$$

Setting $Y = \xi$ in (46) and using (1), we obtained that

$$\alpha\{X - \eta(X)\xi + u(X)U\} + \beta\omega X = 0. \tag{47}$$

Putting $X = \xi$ in (47) and using the fact that $\omega\xi = V$, we have

$$\alpha\xi + \beta V = 0. \tag{48}$$

Taking the scalar product with N and $\bar{U}$ to the above equation, we get

$$\alpha = \beta = 0. \tag{49}$$

Therefore, $\bar{M}$ is a para-cosympletic manifold and we arrive at (1).
(2)- Taking $Y = \xi$ to (46) and in view (21) and (35), we get

$$\mu(X)V = -g(X,E)\xi. \tag{50}$$

Taking inner product of $\bar{U}$ it follows that $\mu = 0$. Thus, ω is parallel with respect to the connection ∇ and we arrive at (2).
(3)- Now taking $Y = \bar{U}$ in (46) and using the fact that $\mu(X) = 0$, we obtain (3). Similarly taking inner product $\bar{V}$ to (46), we get (4).

Theorem 3. *Let M be a re-current screen semi-invariant null hypersurface of a trans-para Sasakian manifold $\bar{M}$. Then D_1 and D_2 are parallel distributions on M.*

Proof. Taking inner product with $\bar{V}$ to (39) and in view of (45), we can write as

$$B(X, Y) = u(Y)u(A_N X). \tag{51}$$

Putting $Y = \bar{V}$ and $Y = \omega Z$ in (51), we get

$$B(X, Y) = 0, \quad and \quad B(X, \omega Z = 0. \tag{52}$$

Now, from (37) and (43), we find for all $Z \in \Gamma(D_0)$,

$$g(\nabla_X E, \bar{V}) = B(X, \bar{V}), \tag{53}$$

$$g(\nabla_X Z, \bar{V}) = B(X, \omega Z), \quad g(\nabla_X \bar{V}, \bar{V}) = 0. \tag{54}$$

From these equations and (52), we see that

$$\nabla_X Y \in \Gamma(D_1), \quad \forall X \in \Gamma(TM), \quad \forall Y \in \Gamma(D_1).$$

and hence D_1 is a parallel distribution on M.

On the other hand, setting $Y = \bar{U}$ in (46, we have

$$B(X, \bar{U})\bar{U} = A_N X. \tag{55}$$

Using $\omega\bar{U} = 0$ in (55), it is obtained that

$$\omega(A_N X) = 0. \tag{56}$$

Using this result and equation (42) reduced to

$$\nabla_X \bar{U} = \tau(X)\bar{U}. \tag{57}$$

It follows that

$$\nabla_X \bar{U} \in \Gamma(D_2), \quad \forall X \in \Gamma(TM),$$

and hence D_2 is a parallel distribution on M.

Therefore immediate consequence of the above theorem and from equation (33), we have the following theorem

Theorem 4. *Let M be a re-current screen semi-invariant null hypersurface of a trans-para Sasakian manifold $\bar{M}$. Then M is locally a product manifolds $C_{\bar{U}} \times M$, where $C_{\bar{U}}$ is a null curve tangent to D_2 and M is a leaf of the distribution D_1.*

Now, we have following

Definition 3. *[8] A null hypersurface of semi-Riemannian manifold is said to be screen conformal if there exists a non-zero smooth function* λ *such that*

$$A_N X = \lambda A_N^* X \quad or \quad C(X, PY) = \lambda B(X, Y). \tag{58}$$

Theorem 5. *Let* M *be a re-current screen semi-invariant null hypersurface of a trans-para Sasakian manifold* $\bar{M}$*. Consider that* M *is a screen conformal null hypersurface. Then* M *is either geodesic or screen totally geodesic if and only if* $X \in \Gamma(D_0)$*.*

Proof. Since M is screen conformal, from Theorem (2) using relations (3) and (4), we get

$$\mu(X)U + v(X)\xi = \lambda(\mu(X)\bar{V} + u(X)\xi). \tag{59}$$

Taking inner product with $\bar{V}$ to (59), we have

$$\mu(X) = 0. \tag{60}$$

So, by using relation (3) and (4) from Theorem (2), we get the required assertion.

6. Lie re-current screen semi-invariant null hypersurface

This section starts with the following definition:

Definition 4. *[8] Let* M *be a screen semi-invariant null hypersurface of a trans-para Sasakian manifold* $\bar{M}$ *and* ρ *be a 1-form on* M*. Then* M *is called Lie re-current if it is admits a Lie re-current tensor field* ω *such that*

$$(\mathcal{L}_X \omega)Y = \rho(X)\omega Y, \tag{61}$$

where $\mathcal{L}_X$ *denotes the Lie derivative on* M *with respect to* X*,, that is*

$$(\mathcal{L}_X \omega)Y = [X, \omega Y] - \omega[X, Y]. \tag{62}$$

If the structure tensor field ω *satisfies the condition*

$$\mathcal{L}_X \omega = 0, \tag{63}$$

then ω *is called Lie parallel. A screen semi-invariant null hypersurface* M *of a trans-para Sasakian manifold* $\bar{M}$ *is called Lie re-current if its structure tensor field* ω *is Lie re-current.*

Theorem 6. *Let M be a Lie re-current screen semi-invariant null hypersurface of a tans-para Sasakian manifold $\bar{M}$. Then the structure tensor field ω is Lie parallel.*

Proof. In view of (62), (63) and (39), we get

$$\rho(X)\omega Y = -\nabla_{\omega Y}X + \omega\nabla_Y X + u(Y)A_N X - B(X,Y)\bar{U} \tag{64}$$

$$+\alpha(-g(X,Y)\xi + \eta(Y)X) + \beta g(X,\varphi Y)\xi + \beta\eta(Y)\omega X.$$

Putting $Y = E$ in (64) and by the use of (20), we have

$$\rho(X)\bar{V} = -\nabla_{\bar{V}}X + \omega\nabla_E X - \beta u(X)\xi. \tag{65}$$

Taking inner product with $\bar{V}$ to (65), we obtain

$$g(\nabla_{\bar{V}}X,\bar{V}) = u(\nabla_{\bar{V}}X) = 0, \qquad \textit{and} \quad \eta(\nabla_{\bar{V}}X) = \beta u(X). \tag{66}$$

Replacing Y by $\bar{V}$ in (64) and using the fact that $\eta(Y) = 0$, we have

$$\rho(X)E = -\nabla_{\bar{E}}X + \omega\nabla_{\bar{V}}X + B(X,\bar{V}) + \bar{U} + \alpha u(X)\xi. \tag{67}$$

Applying ω to the above equation, using (38) with (66), it is obtained that

$$\rho(X)E = -\nabla_{\bar{E}}X + \omega\nabla_{\bar{V}}X + \bar{U} + \beta u(X)\xi. \tag{68}$$

Comparing the above equation with (65), we get $\rho = 0$. Therefore we arrive at ω is Lie-parallel.

Theorem 7. *Let M be a Lie re-current screen semi-invariant null hypersurface of a tans-para Sasakian manifold $\bar{M}$. Then $\alpha = \beta = 0$ and $\bar{M}$ is a para-cosympletic manifold.*

Proof. Replacing X by U in (65) and using (21), (22), (35), (39)-(42) and $\omega\bar{U} = 0$ and $\omega\xi = 0$, it is obtained that

$$u(Y)A_N\bar{U} - \omega(A_N\omega Y) - A_N Y - \tau(\omega Y)\bar{U} \tag{69}$$

$$-\alpha v(Y)\xi + \beta\eta(Y)\xi - \alpha\eta(Y)\bar{U} = 0.$$

Taking inner product with ξ into (69) and using the fact that $C(X,\xi) = -\alpha v(X) + \beta\eta(X)$, it is obtained that $\alpha v(Y) = 0$ and $\beta\eta(Y) = 0$, and hence $\alpha = \beta = 0$. That is, $\bar{M}$ is a para-cosympletic manifold.

Theorem 8. *Let M be a Lie re-current screen semi-invariant null hypersurface of a tans-para Sasakian manifold $\bar{M}$. Then the following statements are holds:*

1. $\tau = \beta\eta$ *on* TM, *and* (2) $A^*_E\bar{U} = 0$, *and* $A^*_E\bar{V}$=0.

Proof. Taking inner product with N to (65) and using, (22), we have

$$-g(\nabla - \omega Y X, N) + g(\nabla_Y X, \bar{U}) = \beta\eta(Y)u(X), \tag{70}$$

since $\alpha = 0$ in (70). Replacing X by ξ in (70) and using (17) and (21), we get

$$B(X, \bar{U}) = \tau(\omega X). \tag{71}$$

Taking $X = \bar{U}$ and using 44) and $\omega\bar{U} = 0$, we have

$$C(\bar{U}, \bar{V}) = B(\bar{U}, \bar{U} = 0. \tag{72}$$

Adopting the inner product with V in (68) and using (21), (44), (72), and $\alpha = 0$, it is obtained that

$$B(X, \bar{U}) = -\tau(\omega X). \tag{73}$$

Comparing the above equation with (67), it is obtained that $\tau(\omega X) = 0$.
Replacing X by $\bar{V}$ in (69) and using (43), we have

$$B(\omega Y, \bar{U}) + \beta\eta(Y) = \tau(Y). \tag{74}$$

Taking $Y = \bar{U}$ and $Y = \xi$ and using $\omega\bar{U} = \omega\xi = 0$, it is obtained that

$$\tau(\bar{U}) = 0, \qquad \tau(\xi) = -\beta. \tag{75}$$

Setting $X = \omega Y$ to $\tau\omega X) = 0$ and using (38) and (75), we get $\tau(X) = -\beta\eta(X)$. Thus we have (1).
As $\tau(\omega X) = 0$, from (21) and (70), we have $g(A^*_E\bar{U}, X) = 0$. The non-degeneracy of $S(TM)$ implies that $A^*_E\bar{U} = 0$. Putting X by E to (66) and using (23) and $\tau(\omega X) = 0$, then we obtained $A^*_E\bar{V} = 0$, thus we arrive at (2).

7. Screen semi-invariant Hopf null hypersurface

Definition 5. *Let M be a screen semi-invariant null hypersurface of a trans-para Sasakian manifold $\bar{M}$ and $\bar{U}$ be a structure tensor field on M. The structure tensor field $\bar{U}$ is called principal if there exists a smooth function σ such that*

$$A^*_E X = \sigma\bar{U}. \tag{76}$$

A screen semi-invariant null hypersurface M of a trans-para Sasakian manifold $\bar{M}$ is called Hopf null hypersurface if it admits principal vector field $\bar{U}$ [8].

If we consider equation (76), from (21) and (35), we obtain

$$B(X,\bar{U}) = -\sigma v(X), \quad and \quad C(X,\bar{V}) = -\sigma u(X). \tag{77}$$

Now, we have the following theorems:

Theorem 9. *Let M be a screen semi-invariant Hopf null hypersurface of a trans-para Sasakian manifold $\bar{M}$. If M is screen totally umbilical then $\kappa = 0$ and M is a screen totally geodesic null hypersurface.*

Proof. We know that, M is screen totally umbilical null hypersurface if there exists a smooth function f such that $A_N X = fg(X,Y)$ or

$$C(X,PY) = fg(X,Y), \tag{78}$$

and $f = 0$, we say that M is a screen totally geodesic null hypersurface.
Therefore, in (78) replacing PY with $\bar{V}$ and use of (35) and (77), we find

$$fv(X) = fu(X). \tag{79}$$

Putting $X = \bar{U}$ in (79) we obtain $f = 0$. So, we get $A_N = 0 = C$ and $\kappa = 0 = g(A_N X, \bar{V})$. Therefore $\kappa = 0$ and M is a screen totally geodesic null hypersurface.

Theorem 10. *Let M be a screen semi-invariant Hopf null hypersurface of a trans-para Sasakian manifold $\bar{M}$. If $\bar{V}$ is a parallel null vector field then M is a Hopf null hypersurface such that $\kappa = 0$.*

Proof. Let us consider $\bar{V}$ is parallel null vector field, from (36) and (43), we find

$$\varphi(A_E^* X) - \beta u(A_E^* X)N + \tau(X)\bar{V}. \tag{80}$$

Applying φ to (80) and in view of (1), we have

$$A_E^* X - \beta u(A_E^* X)\bar{U} + \tau(X)E = 0. \tag{81}$$

Taking inner product with N to (81), we get at $\tau = 0$, which yields

$$A_E^* X = \beta u(A_E^* X)\bar{U}. \tag{82}$$

Therefore, we can say that M is a Hopf null hypersurface. If we take inner product with $\bar{U}$ to (82), we find $\kappa(X) = 0 = B(X,\bar{U})$.

8. Integrability of screen semi-invariant null hypersurface

This, section explores the integrability conditions for the distributions engage with the screen semi-invariant hypersurface of a trans-para Sasakian manifold:

We note that $X \in D_1$ if and only if $u(X) = 0$. Now from (refs17), we have for all $X, Y \in \Gamma(TM)$.

$$u(\nabla_Y X) = \nabla_X u(Y) + u(Y)\tau(X) - B(X, \omega Y) + \beta\eta(Y)u(X) \tag{83}$$

from which we get

$$u([X,Y]) = B(X, \omega Y) - B(\omega X, Y) +_X u(Y) - \nabla_Y u(X) \tag{84}$$

$$+u(Y)\tau(X) - u(X)\tau(Y) + \beta\eta(Y)u(X) - \beta\eta(X)u(Y).$$

Let $X, Y \in D_1$. Then $u(X) = 0 = u(Y)$, and from the equation (84) we get

$$u([X,Y]) = B(X, \omega Y) - B(\omega X, Y),$$

for all $X, Y \in D_1$. Thus we obtain a necessary and sufficient condition for the integrability of the distribution D_1 in the following:

Theorem 11. *Let M be a screen semi-invariant null hypersurface of a trans-para Sasakian manifold $\bar{M}$. Then the distribution D_1 is integrable if and only if*

$$B(X, \omega Y) = B(\omega X, Y), \qquad X, Y \in \Gamma(D_1). \tag{85}$$

Now, we find a necessary and sufficient condition for the distribution D_2 to be integrable.

Theorem 12. *Let M be a screen semi-invariant null hypersurface of a trans-para Sasakian manifold $\bar{M}$. Then the distribution D_2 is integrable if and only if*

$$A_N\xi + \alpha\bar{U} + \beta\omega\bar{U} \tag{86}$$

Proof. It is Noted here that $X \in D_2$ if if and only if $\varphi X = \omega X = 0$. Now for all $X, Y \in \Gamma(TM)$, in view of (39), we arrive at

$$\omega(\nabla_X Y) = \nabla_X \omega(Y) - u(Y)A_N X - B(X,Y)\bar{U} \tag{87}$$

$$-\alpha(-g(X,Y)\xi + \eta(Y)X) - \beta(g(X, \varphi Y)\xi + \eta(Y)\omega X).$$

From (87), we get

$$\omega([X,Y]) = \nabla_X \omega(Y) - \nabla_Y \omega(X) + u(X)A_N Y - u(Y)A_N X \tag{88}$$

$$+\alpha(\eta(Y)X - \eta(X)Y) + \beta(\eta(Y)\omega X - \eta(X)\omega Y).$$

In particular for $X, Y \in D_2$, we get

$$\omega([X,Y]) = +u(X)A_N Y - u(Y)A_N X + \alpha(\eta(Y)X - \eta(X)Y) \tag{89}$$

$$+\beta(\eta(Y)\omega X - \eta(X)\omega Y).$$

Setting $X = \bar{U}$ and $Y = \xi$ and hence, D_2 is integrable if and only if

$$\omega[\bar{U}, \xi] = 0 \tag{90}$$

which, in view of (90), is equivalent to (86).

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