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SOME VARIATIONS OF MULTICOLOR RADO NUMBERS

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Abstract. Let E_m denote a system of equations/inequalities in m variables. Generalizing the concept of a Schur number, the Rado number $R^t(E_m)$ is defined to be the least natural number such that every t-coloring of the elements of the set $\{1, 2, \ldots, R^t(E_m)\}$ contains a monochromatic solution to E_m. In this paper, we review the relevant background for $R^t(E_m)$ and consider two variations. First, we define the weakened Rado number, in which solutions to E_m are sought that use at most $s < t$ colors. We compute a few special cases, then turn our attention to the rainbow Rado number $RR(E_m, n)$, defined to be the minimum number of colors such that every coloring of $\{1, 2, \ldots, n\}$ contains a rainbow solution to E_m (a solution in which all m variables receive distinct colors). We evaluate rainbow Rado numbers for two specific systems of inequalities, then conclude with some directions for future work on this topic.

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1. Introduction

A classic problem in combinatorial number theory is the determination of Schur numbers. Introduced as a tool for investigating the finite version of Fermat's Last Theorem, Schur [16] proved their existence in 1916. To define a (generalized) Schur number (see [3]), consider the equation

$$\mathcal{L}_m: \; x_1 + x_2 + \cdots + x_{m-1} = x_m,$$

and define a *t-coloring* of a set A to be a map $c : A \longrightarrow \{1, 2, \ldots, t\}$. For $m \geq 3$, the Schur number $S_t(m)$ is defined to be the least natural number such that every t-coloring of $\{1, 2, \ldots, S_t(m)\}$ contains a monochromatic solution to $\mathcal{L}_m$. It is currently known that $S_2(m) = m^2 - m - 1$ [3], $S_3(3) = 14$ [11], $S_3(4) = 43$ [2], $S_3(5) = 94$ [2], $S_3(6) = 173$ [2], $S_4(3) = 45$ [11], and $S_5(3) = 161$ [12].

One variation of a Schur number is the strict Schur number (see Section 8.3 of [14]), called a weak Schur number in [9]. We denote it by $\widehat{S}_t(m)$ and define it analogously to $S_t(m)$, but replace $\mathcal{L}_m$ by the system

$$\mathcal{L}_m^*: \quad \begin{array}{l} x_1 + x_2 + \cdots + x_{m-1} = x_m \\ x_1 < x_2 < \cdots < x_m. \end{array}$$

It is known that $\widehat{S}_2(3) = 9$ [1], $\widehat{S}_3(3) = 24$ [5], $\widehat{S}_4(3) = 67$ [5], $\widehat{S}_5(3) \geq 196$ [9], and $\widehat{S}_6(3) \geq 572$ [9].

In [4], Bialostocki and Schaal considered another variation of Schur numbers by defining the systems of inequalities

$$\mathcal{M}_m: \ x_1 + x_2 + \cdots + x_{m-1} < x_m \qquad \text{and} \qquad \mathcal{M}_m^*: \ \begin{array}{l} x_1 + x_2 + \cdots + x_{m-1} < x_m \\ x_1 < x_2 < \cdots < x_m, \end{array}$$

and calling the resulting numbers Rado numbers, due to their close connection with Rado's theorem. In Section 8.4 of Landman and Robertson's book [14], $\mathcal{M}_m$ is called a *Schur inequality,* which is considered in a *strict* sense in $\mathcal{M}_m^*$. Following the terminology introduced in [4], let E_m denote a system of equations/inequalities in m variables and define the *Rado number* $R^t(E_m)$ to be the least natural number such that every t-coloring of $\{1, 2, \ldots, R^t(E_m)\}$ contains a monochromatic solution to E_m. This notation generalizes Schur numbers and their strict variation since $R^t(\mathcal{L}_m) = S_t(m)$ and $R^t(\mathcal{L}_m^*) = \widehat{S}_t(m)$. In [4], it was shown that for $m \geq 3$, $R^2(\mathcal{M}_m) = m^2 - m + 1$ and

$$R^2(\mathcal{M}_m^*) = \begin{cases} \frac{9}{16}m^3 - m^2 + m + 1 & \text{if} \quad m \equiv 0 \pmod 4 \\ \frac{9}{16}m^3 - m^2 + \frac{13}{16}m + \frac{13}{8} & \text{if} \quad m \equiv 1 \pmod 4 \\ \frac{9}{16}m^3 - m^2 + m + \frac{1}{2} & \text{if} \quad m \equiv 2 \pmod 4 \\ \frac{9}{16}m^3 - m^2 + \frac{17}{16}m + \frac{5}{8} & \text{if} \quad m \equiv 3 \pmod 4. \end{cases}$$

In this paper, we consider two variations of Rado numbers. The first variation is considered in Section 2 and is called a weakened Rado number. It is defined in a similar manner to the weakened Schur numbers considered in [7]. Rather than seeking out monochromatic solutions to the system $\mathcal{M}_m^*$, the weakened Rado numbers we consider seek out solutions that use at most $s < t$ colors. We evaluate several classes of weakened Rado numbers when $s > 1$.

The second variation is considered in Section 3 and is called a rainbow Rado number (similar the the rainbow Schur number considered in [6]). For these numbers, we consider how many colors are necessary to force $\{1, 2, \ldots, n\}$ to have a rainbow solution to $\mathcal{M}_m^*$ (i.e., a solution that uses m distinct colors). We provide explicit evaluations for a couple of rainbow Ramsey numbers and conclude in Section 4 with topics for future research in this area.

2. Weakened Rado Numbers

For $t \geq 3$ and $1 \leq s < \min\{m, t\}$, define the *s-weakened t-colored Rado number* $R_s^t(E_m)$ to be the least natural number such that every t-coloring of $\{1, 2, \ldots, R_s^t(E_m)\}$ contains a solution to E_m that uses at most s colors. Hence, $R_1^t(E_m)$ is the usual t-colored Rado number for E_m.

In [7], the numbers $R_s^t(\mathcal{L}_m^*)$ were considered, where they were denoted by $WS_s^m(t)$. It was shown that for all $t \geq 3$,

$$R_2^t(\mathcal{L}_3^*) = \begin{cases} \frac{6t+5}{5} & \text{if} \quad t \equiv 0 \pmod 5 \\ \frac{6t+9}{5} & \text{if} \quad t \equiv 1 \pmod 5 \\ \frac{6t+3}{5} & \text{if} \quad t \equiv 2 \pmod 5 \\ \frac{6t+7}{5} & \text{if} \quad t \equiv 3 \pmod 5 \\ \frac{6t+6}{5} & \text{if} \quad t \equiv 4 \pmod 5 \end{cases}$$

and

$$R_{t-1}^t(\mathcal{L}_t^*) = \frac{t(t-1)}{2} + 2.$$

The special cases $R_2^3(\mathcal{L}_4^*) = 11$ and $R_3^4(\mathcal{L}_5^*) = 14$ were also proved.

For the remainder of this section, we focus on the evaluation of $R_s^t(\mathcal{M}_m^*)$, beginning with the case where $m = 3$ and $s = 2$. If a 2-colored $x_1 + x_2 < x_3$ is avoided, then every such inequality must be rainbow colored. So, the problem of determining $R_2^t(\mathcal{M}_3^*)$ is equivalent to determining the maximum natural number p such that there exists a t-coloring of $\{1, 2, \ldots, p\}$ in which every solution to E_3 is rainbow.

It is easily checked that for

$$\{1, 2, 3, 4, 5\},$$

every solution to $\mathcal{M}_3^*$ is rainbow (i.e., uses three distinct colors). It follows that $R_2^3(\mathcal{M}_3^*) > 5$. Now consider a 3-coloring of $\{1, 2, \ldots, 6\}$, using say, red, blue, and green. If we wish to avoid a solution to $\mathcal{M}_3^*$ that uses at most 2 colors, then

$$1 + 2 < 6, \; 1 + 3 < 6, \; \text{and } 2 + 3 < 6$$

force 1, 2, 3, and 6 to receive different colors. Hence, $R_2^3(\mathcal{M}_3^*) \leq 6$, from which we obtain the initial case $R_2^3(\mathcal{M}_3^*) = 6$.

The values $R_2^4(\mathcal{M}_3^*) = 7$ and $R_2^5(\mathcal{M}_3^*) = 8$ can be proved in a similar manner, using

$$\{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad \{1, 2, 3, 4, 5, 6, 7\}$$

to obtain the lower bounds. In all of these cases so far, observe that all elements in $\{1, 2, \dots \lceil \frac{n}{2} \rceil\}$ must receive distinct colors if $\{1, 2, \dots, n\}$ avoids a solution to $x_1 + x_2 < n$, where $x_1 < x_2$, that uses at most 2 colors. Also, the inequality $1 + x_2 < x_3$ prevents any two elements in $\{1, 2, \dots, n\}$ that differ by more than 1 from receiving the same color. These observations lead us to the following theorem.

Theorem 1. *For $t \geq 3$,*

$$R_2^t(\mathcal{M}_3^*) = \begin{cases} \frac{1}{3}(4t+6) & \text{if} \quad t \equiv 0 \pmod 3 \\ \frac{1}{3}(4t+5) & \text{if} \quad t \equiv 1 \pmod 3 \\ \frac{1}{3}(4t+4) & \text{if} \quad t \equiv 2 \pmod 3. \end{cases}$$

Proof. Let $t \geq 3$ and consider three cases.
<u>Case 1</u> Suppose that $t \equiv 0 \pmod 3$. Then the set $\{1, 2, \dots, \frac{1}{3}(4t+6) - 1\}$ can be colored according to the partition

$$\{1\} \cup \{2\} \cup \cdots \cup \left\{\frac{2t-3}{3}\right\} \cup \left\{\frac{2t}{3}, \frac{2t+3}{3}\right\} \cup \left\{\frac{2t+6}{3}, \frac{2t+9}{3}\right\} \cup \cdots \cup \left\{\frac{4t}{3}, \frac{4t+3}{3}\right\}$$

without producing a 2-colored solution to $\mathcal{M}_3^*$. Since this partition consists of $\frac{2t-3}{3}$ sets of cardinality 1 and $\frac{t+3}{3}$ sets of cardinality 2, we have a total of t subsets in this partition. It follows that $R_2^t(\mathcal{M}_3^*) \geq \frac{1}{3}(4t+6)$. Now consider an arbitrary t-coloring of $\{1, 2, \dots, \frac{1}{3}(4t+6)\}$. As we observed before the statement of this theorem, avoiding a solution to $\mathcal{M}_3^*$ that uses at most 2 colors forces $1, 2, \dots, \frac{2t+3}{3}$ to receive distinct colors. Also, at most two of $\frac{2t+6}{3}, \frac{2t+9}{3}, \dots, \frac{4t+6}{3}$ can receive the same color. It follows that at least $t+1$ colors are required to avoid a solution to $\mathcal{M}_3^*$ that uses at most 2 colors, and hence, $R_2^t(\mathcal{M}_3^*) \leq \frac{1}{3}(4t+6)$ in this case.
<u>Case 2</u> Suppose that $t \equiv 1 \pmod 3$. Then the set $\{1, 2, \dots, \frac{1}{3}(4t+5) - 1\}$ can be colored according to the partition

$$\{1\} \cup \{2\} \cup \cdots \cup \left\{\frac{2t-2}{3}\right\} \cup \left\{\frac{2t+1}{3}, \frac{2t+4}{3}\right\} \cup \left\{\frac{2t+7}{3}, \frac{2t+10}{3}\right\} \cup \cdots \cup \left\{\frac{4t-1}{3}, \frac{4t+2}{3}\right\}$$

without producing a 2-colored solution to $\mathcal{M}_3^*$. Since this partition consists of $\frac{2t-2}{3}$ sets of cardinality 1 and $\frac{t+2}{3}$ sets of cardinality 2, we have a total of t subsets in this partition. It follows that $R_2^t(\mathcal{M}_3^*) \geq \frac{1}{3}(4t+5)$. Now consider an arbitrary t-coloring of $\{1, 2, \ldots, \frac{1}{3}(4t+5)\}$. Avoiding a solution to $\mathcal{M}_3^*$ that uses at most 2 colors forces $1, 2, \ldots, \frac{2t+4}{3}$ to receive distinct colors. Also, at most two of $\frac{2t+7}{3}, \frac{2t+10}{3}, \ldots, \frac{4t+5}{3}$ can receive the same color. It follows that at least $t+1$ colors are required to avoid a solution to $\mathcal{M}_3^*$ that uses at most 2 colors, and hence, $R_2^t(\mathcal{M}_3^*) \leq \frac{1}{3}(4t+5)$ in this case.

Case 3 Suppose that $t \equiv 2 \pmod 3$.Then the set $\{1, 2, \ldots, \frac{1}{3}(4t+4)-1\}$ can be colored according to the partition

$$\{1\} \cup \{2\} \cup \cdots \cup \left\{\frac{2t-1}{3}\right\} \cup \left\{\frac{2t+2}{3}, \frac{2t+5}{3}\right\} \cup \left\{\frac{2t+8}{3}, \frac{2t+11}{3}\right\} \cup \cdots \cup \left\{\frac{4t-2}{3}, \frac{4t+1}{3}\right\}$$

without producing a 2-colored solution to $\mathcal{M}_3^*$. Since this partition consists of $\frac{2t-1}{3}$ sets of cardinality 1 and $\frac{t+1}{3}$ sets of cardinality 2, we have a total of t subsets in this partition. It follows that $R_2^t(\mathcal{M}_3^*) \geq \frac{1}{3}(4t+4)$. Now consider an arbitrary t-coloring of $\{1, 2, \ldots, \frac{1}{3}(4t+4)\}$. Avoiding a solution to $\mathcal{M}_3^*$ that uses at most 2 colors forces $1, 2, \ldots, \frac{2t+5}{3}$ to receive distinct colors. Also, at most two of $\frac{2t+8}{3}, \frac{2t+11}{3}, \ldots, \frac{4t+4}{3}$ can receive the same color. It follows that at least $t+1$ colors are required to avoid a solution to $\mathcal{M}_3^*$ that uses at most 2 colors, and hence, $R_2^t(\mathcal{M}_3^*) \leq \frac{1}{3}(4t+4)$ in this case.

Next, we turn our attention to the case where $m = 4$ and $s = 3$.

Theorem 2. *For $t \geq 4$,*

$$R_3^t(\mathcal{M}_4^*) = \begin{cases} 8\lfloor \frac{t}{5} \rfloor + 3 & \text{if } t \equiv 0 \pmod 5 \\ 8\lfloor \frac{t}{5} \rfloor + 5 & \text{if } t \equiv 1 \pmod 5 \\ 8\lfloor \frac{t}{5} \rfloor + 6 & \text{if } t \equiv 2 \pmod 5 \\ 8\lfloor \frac{t}{5} \rfloor + 7 & \text{if } t \equiv 3 \pmod 5 \\ 8\lfloor \frac{t}{5} \rfloor + 9 & \text{if } t \equiv 4 \pmod 5. \end{cases}$$

Proof. Let $t \geq 4$ and apply the Division Algorithm to obtain $t = 5q + r$, with $0 \leq r \leq 4$. Note that $q = \lfloor \frac{t}{5} \rfloor$. In $\mathcal{M}_4^*$, we consider the smallest difference between x_3 and x_4 that can be obtained. If we consider the inequality $1 + 2 + x_3 < x_4$, then we find that avoiding a 3-colored $\mathcal{M}_4^*$ implies that at most 4 consecutive numbers may receive the same color. We divide the remainder of the proof into cases.

<u>Case 1</u> Assume that $t \equiv 0 \pmod 5$ and write $t = 5q$. To prove the lower bounds, consider the partition

$$\{1\} \cup \{2\} \cup \cdots \{4q-1\} \cup \{4q, 4q+1, 4q+2\} \cup$$
$$\{4q+3, 4q+4, 4q+5, 4q+6\} \cup \{4q+7, 4q+8, 4q+9, 4q+10\} \cup \cdots \cup \{8q-1, 8q, 8q+1, 8q+2\}.$$

This partition consists of exactly t colors and every solution to $\mathcal{M}_4^*$ is rainbow. This follows from the previous observation that at most 4 consecutive numbers can receive the same color and the inequality $1+x_2+x_3 < 8q+2$ forces $1, 2, \ldots, 4q-1$ to receive their own colors. Taking that inequality, it can be shown that x_2+x_3 must be less than $8q+1$. From this, we can see that, at the point where $x_2+x_3 \geq 8q+1$, they no longer have to be their own color, as it is impossible for them to show up in inequalities together. It follows that $R_3^t(\mathcal{M}_4^*) \geq 8q+3$ in this case. To prove the upper bound, consider an arbitrary t-coloring of $\{1, 2, \ldots, 8q+3\}$. The t-coloring of this set would give a similar result as $8q+2$, but the additional number in the set would have to be added to the last set of numbers, but this would exceed the maximum of four numbers grouped together and force a non-rainbow solution.
<u>Case 2</u> Assume that $t \equiv 1 \pmod 5$ and write $t = 5q+1$. To prove the lower bounds, consider the partition

$$\{1\} \cup \{2\} \cup \cdots \{4q\} \cup \{4q+1, 4q+2, 4q+3, 4q+4\} \cup$$
$$\{4q+5, 4q+6, 4q+7, 4q+8\} \cup \cdots \cup \{8q+1, 8q+2, 8q+3, 8q+4\}.$$

This partition consists of exactly t colors and every solution to $\mathcal{M}_4^*$ is rainbow. This follows from the previous observation that at most 4 consecutive numbers can receive the same color and the inequality $1+x_2+x_3 < 8q+4$ forces $1, 2, \ldots, 4q$ to receive their own colors. Taking that inequality, it can be shown that x_2+x_3 must be less than $8q+3$. From this, we can see that, at the point where $x_2+x_3 \geq 8q+3$, they no longer have to be their own color, as it is impossible for them to show up in inequalities together. It follows that $R_3^t(\mathcal{M}_4^*) \geq 8q+5$ in this case. To prove the upper bound, consider an arbitrary t-coloring of $\{1, 2, \ldots, 8q+5\}$. The t-coloring of this set would give a similar result as $8q+4$, but the additional number in the set would have to be added to the last set of numbers, but this would exceed the maximum of four numbers grouped together and force a non-rainbow solution.
<u>Case 3</u> Assume that $t \equiv 2 \pmod 5$ and write $t = 5q+2$. To prove the lower bounds, consider the partition

$$\{1\} \cup \{2\} \cup \cdots \{4q+1\} \cup \{4q+2, 4q+3, 4q+4, 4q+5\} \cup$$
$$\{4q+6, 4q+7, 4q+8, 4q+9\} \cup \cdots \cup \{8q+2, 8q+3, 8q+4, 8q+5\}.$$

This partition consists of exactly t colors and every solution to $\mathcal{M}_4^*$ is rainbow. This follows from the previous observation that at most 4 consecutive numbers can receive the same color and the inequality $1+x_2+x_3 < 8q+5$ forces $1, 2, \ldots, 4q+1$ to receive their own colors. Taking that inequality, it can be shown that $x_2 + x_3$ must be less than $8q + 4$. From this, we can see that, at the point where $x_2 + x_3 \geq 8q + 4$, they no longer have to be their own color, as it is impossible for them to show up in inequalities together. It follows that $R_3^t(\mathcal{M}_4^*) \geq 8q + 6$ in this case. To prove the upper bound, consider an arbitrary t-coloring of $\{1, 2, \ldots, 8q + 6\}$. The t-coloring of this set would give a similar result as $8q + 5$, but the additional number in the set would have to be added to the last set of numbers, but this would exceed the maximum of four numbers grouped together and force a non-rainbow solution.
Case 4 Assume that $t \equiv 3 \pmod 5$ and write $t = 5q+3$. To prove the lower bounds, consider the partition

$$\{1\} \cup \{2\} \cup \cdots \{4q + 1\} \cup \{4q + 2, 4q + 3, 4q + 4, 4q + 5\}\cup$$
$$\{4q + 6, 4q + 7, 4q + 8, 4q + 9\} \cup \cdots \cup \{8q + 3, 8q + 4, 8q + 5, 8q + 6\}.$$

This partition consists of exactly t colors and every solution to $\mathcal{M}_4^*$ is rainbow. This follows from the previous observation that at most 4 consecutive numbers can receive the same color and the inequality $1+x_2+x_3 < 8q+6$ forces $1, 2, \ldots, 4q+1$ to receive their own colors. Taking that inequality, it can be shown that $x_2 + x_3$ must be less than $8q + 5$. From this, we can see that, at the point where $x_2 + x_3 \geq 8q + 5$, they no longer have to be their own color, as it is impossible for them to show up in inequalities together. It follows that $R_3^t(\mathcal{M}_4^*) \geq 8q + 7$ in this case. To prove the upper bound, consider an arbitrary t-coloring of $\{1, 2, \ldots, 8q + 7\}$. The t-coloring of this set would give a similar result as $8q + 6$, but the additional number in the set would have to be added to the last set of numbers, but this would exceed the maximum of four numbers grouped together and force a non-rainbow solution.
Case 5 Assume that $t \equiv 4 \pmod 5$ and write $t = 5q+4$. To prove the lower bounds, consider the partition

$$\{1\} \cup \{2\} \cup \cdots \{4q + 2\} \cup \{4q + 3, 4q + 4\}\cup$$
$$\{4q+5, 4q+6, 4q+7, 4q+8\} \cup \{4q+9, 4q+10, 4q+11, 4q+12\} \cup \cdots \cup \{8q+5, 8q+6, 8q+7, 8q+8\}.$$

This partition consists of exactly t colors and every solution to $\mathcal{M}_4^*$ is rainbow. This follows from the previous observation that at most 4 consecutive numbers can receive the same color and the inequality $1+x_2+x_3 < 8q+8$ forces $1, 2, \ldots, 4q+2$ to receive their own colors. Taking that inequality, it can be shown that $x_2 + x_3$ must be less than $8q + 7$. From this, we can see that, at the point where $x_2 + x_3 \geq 8q + 7$, they no longer have to be their own color, as it is impossible for them to show up in inequalities together. It follows that $R_3^t(\mathcal{M}_4^*) \geq 8q+9$ in this case. To prove the upper bound, consider an arbitrary t-coloring of $\{1,2,\dots,8q+9\}$. The t-coloring of this set would give a similar result as $8q+8$, but the additional number in the set would have to be added to the last set of numbers, but this would exceed the maximum of four numbers grouped together and force a non-rainbow solution.

Finally, consider the case where $s=t-1$ and $m=t$. We obtain the following theorem.

Theorem 3. *For all $t \geq 3$, $R_{t-1}^t(\mathcal{M}_t^*) = \frac{1}{2}t(t-1)+3$.*

Proof. The lower bound for $R_{t-1}^t(\mathcal{M}_t^*)$ is proved by giving a t-coloring of

$$\left\{1,2,\dots,\frac{1}{2}t(t-1)+2\right\}$$

that lacks a $(t-1)$-colored solution to $\mathcal{M}_t^*$. Observe that the only inequalities that arise from $\mathcal{M}_t^*$ for this set are

$$1+2+\cdots+(t-1) < \frac{1}{2}t(t-1)+1$$
$$1+2+\cdots+(t-1) < \frac{1}{2}t(t-1)+2$$
$$1+2+\cdots+(t-2)+t < \frac{1}{2}t(t-1)+2$$

If a $(t-1)$-colored solution to $\mathcal{M}_t^*$ is avoided, then $1,2,3,\dots,t-1,\frac{1}{2}t(t-1)+2$ all have distinct colors, t can receive the same color as that of $t-1$, and the remaining numbers can be given any color. One such partition is

$$\{1\}\cup\{2\}\cup\cdots\cup\{t-2\}\cup\{t-1,t\}\cup\left\{t+1,\cdots\frac{1}{2}t(t-1)+2\right\}.$$

It follows that $R_{t-1}^t(\mathcal{M}_t^*) \geq \frac{1}{2}t(t-1)+3$. Now, we consider an arbitrary t-coloring of $\{1,2,\dots,\frac{1}{2}t(t-1)+3\}$. Consider the inequalities

$$1+2+\cdots+(t-1) < \frac{1}{2}t(t-1)+2, \tag{1}$$
$$1+2+\cdots+(t-2)+t < \frac{1}{2}t(t-1)+2, \tag{2}$$
$$1+2+\cdots+(t-3)+(t-1)+t < \frac{1}{2}t(t-1)+3. \tag{3}$$

As there are exactly t numbers in Inequality (1), they must each receive different colors. From Inequality (2), t must be the same color as $t-1$. Finally, in Inequality (3), we see that $t-1$ and t are the same color, implying $R_{t-1}^t(\mathcal{M}_t^*) \leq \frac{1}{2}t(t-1)+3$. Combining these two bounds results in the statement of the theorem.

3. Rainbow Rado Numbers

Define the *rainbow Rado number* $RR(E_m, n)$ to be the minimum number of colors necessary such that every coloring of $\{1, 2, \ldots, n\}$ contains a rainbow solution to E_m. Such numbers generalize the rainbow Schur numbers introduced in [6], which were themselves, variations of rainbow numbers for edge colorings of complete graphs (see Section 11.4 of [8]). Equivalently, one could define the *anti-Rado number* $AR(E_m, n)$ to be the maximum number of colors that can be used to color $\{1, 2, \ldots, n\}$ such that no rainbow solution to E_m exists (generalizing concepts first introduced in [10]). It is clear from these definitions that

$$RR(E_m, n) = AR(E_m, n) + 1,$$

for all $m \geq 3$ and $n \geq 3$.

Observe that a coloring of $\{1, 2, \ldots, n\}$ produces a rainbow solution to $\mathcal{L}_m$ if and only if it produces a rainbow solution to $\mathcal{L}_m^*$. This is because a solution to $x_1+x_2+\cdots+x_{m-1} = x_m$ will never be rainbow when $x_i = x_j$ for some $1 < i < j < m$. The same is true regarding rainbow solutions to $\mathcal{M}_m$ and $\mathcal{M}_m^*$. Thus, we only consider the systems $\mathcal{L}_m^*$ and $\mathcal{M}_m^*$. When $m = 3$, it was proved in Section 2 of [6] that $RR(\mathcal{L}_3^*, n) = \lfloor \log_2(n) \rfloor + 2$, for all $n \geq 3$. Our attention in this section will focus on the evaluation of $RR(\mathcal{M}_3^*, n)$ and $RR(\mathcal{M}_4^*, n)$.

Since $1 + 2 < 4$ is the first nontrivial solution to $\mathcal{M}_3^*$, the number $RR(\mathcal{M}_3^*, n)$ is only defined when $n \geq 4$.

Theorem 4. *For all $n \geq 4$, $RR(\mathcal{M}_3^*, n) = 4$.*

Proof. First note that the 3-coloring

$$\{1, 2, \ldots, n-1, n, n+1\}$$

avoids a rainbow solution to $\mathcal{M}_3^*$ since no inequality in $\mathcal{M}_3^*$ contains both n and $n+1$. It follows that $RR(\mathcal{M}_3^*, n) \geq 4$. To prove that 4 is also an upper bound, we proceed by induction on $n \geq 4$. For the set $\{1, 2, 3, 4\}$, there is only one inequality in $\mathcal{M}_3^*$: $1 + 2 < 4$. It is easily checked that $RR(\mathcal{M}_3^*, 4) \leq 4$ since whenever all four numbers receive unique colors, a rainbow solution to $1 + 2 < 4$ is forced. Now that we have proved the base case, we state the inductive hypothesis: suppose that $RR(\mathcal{M}_3^*, n) \leq 4$ for some $n \geq 3$. We must prove that $RR(\mathcal{M}_3^*, n + 1) \leq 4$. Consider a 4-coloring of $\{1, 2, \ldots, n + 1\}$. If the color assigned to $n + 1$ was already used for some i such that $1 \leq i \leq n$, then this coloring induces a 4-coloring of $\{1, 2, \ldots, n\}$, which produces a rainbow solution to $\mathcal{M}_3^*$ by the inductive hypothesis. If the color used for $n + 1$ is unique, then the equation $1 + x_2 < n + 1$ only avoids a rainbow solution when 1 receives the same color as every $2 \leq x_2 \leq n - 1$. Even if n also receives a unique color, at most 3 colors are used. Hence, we have shown that in all cases, 4-coloring $\{1, 2, \ldots, n+1\}$ produces a rainbow solution to $\mathcal{M}_3^*$, completing the proof that $RR(\mathcal{M}_3^*, n+1) \leq 4$. It follows that $RR(\mathcal{M}_3^*, 4) = 4$.

Since $1+2+3 < 7$ is the first nontrivial solution to $\mathcal{M}_4^*$, the number $RR(\mathcal{M}_4^*, n)$ is only defined when $n \geq 7$.

Theorem 5. *For all $n \geq 7$,*

$$RR(\mathcal{M}_4^*, n) = \begin{cases} \frac{n+7}{2} & \text{if } n \text{ is odd} \\ \frac{n+8}{2} & \text{if } n \text{ is even.} \end{cases}$$

Proof. For the lower bounds, we consider the cases where n is odd or even separately. First, suppose that $n \geq 7$ is odd and consider the partition

$$\left\{1, 2, 3, \cdots, \frac{n-3}{2}\right\} \cup \left\{\frac{n-1}{2}\right\} \cup \left\{\frac{n+1}{2}\right\} \cup \cdots \cup \{n\}.$$

The first set in this partition includes at least 2 numbers from every inequality in $\mathcal{M}_4^*$ since

$$1 + \frac{n-1}{2} + \frac{n+1}{2} < n \quad \Longrightarrow \quad n+1 < n.$$

This partition has $\frac{n+5}{2}$ colors, from which it follows that $RR(\mathcal{M}_4^*, n) \geq \frac{n+7}{2}$. When $n \geq 8$ is even, consider the partition

$$\left\{1, 2, 3, \cdots, \frac{n-4}{2}\right\} \cup \left\{\frac{n-2}{2}\right\} \cup \left\{\frac{n}{2}\right\} \cup \cdots \cup \{n\}.$$

The first set of this partition includes at least 2 numbers from every inequality in $\mathcal{M}_4^*$ since

$$1 + \frac{n-2}{2} + \frac{n}{2} < n \quad \Longrightarrow \quad n < n.$$

This partition has $\frac{n+6}{2}$ colors, from which it follows that $RR(\mathcal{M}_4^*, n) \geq \frac{n+8}{2}$. Now we turn our attention to proving the upper bounds for $RR(\mathcal{M}_4^*, n)$, which we do by induction on $n \geq 7$. First consider the base cases. When $n = 7$ or $n = 8$, an n-coloring of $\{1, 2, \ldots, n\}$ necessarily assigns to each number its own color. Then $1+2+3 < n$ is a rainbow solution to $\mathcal{M}_4^*$, from which it follows that $RR(\mathcal{M}_4^*, 7) \leq 7$ and $RR(\mathcal{M}_4^*, 8) \leq 8$. Now, suppose that

$$RR(\mathcal{M}_4^*, n) \leq \begin{cases} \frac{n+7}{2} & \text{if } n \text{ is odd} \\ \frac{n+8}{2} & \text{if } n \text{ is even,} \end{cases}$$

for some $n \geq 7$. If $n+1$ is even, and we consider an $\frac{(n+1)+8}{2}$-coloring of $1, 2, \ldots, n+1\}$, then regardless of whether or not $n+1$ receives its own color or not, the set $\{1, 2, \ldots, n\}$ uses at least $\frac{n+7}{2}$ colors, from which the inductive hypothesis implies the existence of a rainbow solution to $\mathcal{M}_4^*$. If $n+1$ is odd, then consider whether or not $n+1$ receives a unique color. If not, then $\{1, 2, \ldots, n\}$ is colored using $\frac{n+8}{2}$ colors and we have a rainbow solution to $\mathcal{M}_4^*$ by the inductive hypothesis. Otherwise, $n+1$ receives a unique color, which is clearly different from the color that 1 receives. Consider the inequality

$$1 + x_2 + x_3 < n + 1,$$

where $1 < x_2 < x_3 < n+1$. If no such rainbow solution exists, then the numbers $2, 3, \ldots, \frac{n-2}{2}$ must all receive the same color as 1. Even if all of the numbers $\frac{n}{2}, \frac{n+2}{2}, \ldots, n+1$ receive distinct colors, this results in at most $\frac{n+6}{2}$ colors being used, giving a contradiction. It follows that using $\frac{n+8}{2}$ colors produces a rainbow solution to $\mathcal{M}_4^*$ in this case.

4. Conclusion

Besides the evaluation of $R_s^t(\mathcal{M}_m^*)$ and $RR(\mathcal{M}_m^*, n)$ for values of m, n, s, and t not considered here, we conclude by listing a few other avenues for continued research.

1. Consider other systems of equations/inequalities than the ones considered here. For example, equations of the form $x + ay = z$, where $a \geq 2$, were considered in [17]. As far as we are aware, the analogous inequality (and its strict counterpart) have not yet been considered.

2. Consider Gallai-Rado numbers for various systems of equations/inequalities. Such numbers will avoid rainbow solutions to a given system while determining when a monochromatic solution exists. The Schur number analogue of this concept was considered for $\mathcal{L}_3$ in Section 3 of [6].

3. In [15], Robertson and Zeilberger considered the number of monochromatic solutions to $\mathcal{L}_3$ and in [13], the authors focused on the analogous problem for $\mathcal{M}_3$. For systems with more than 3 variables, this problem remains open.

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