

## SOME SUMABILITY METHODS

by  
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Let  $A = \|\rho_k(n)\|_{n \in \mathbf{N}, k=0, \dots, n}$  be a matrix with  $\rho_k(n) \in \mathbf{R}$ . A sequence  $s = \{s_n\}_{n \in \mathbf{N}}$  is said to be  $A$ -summable to  $\rho$  if  $\lim_{n \rightarrow \infty} \sum_{k=0}^n \rho_k(n) \cdot s_k = \rho$ .

A sequence  $a = (a_n)_{n \geq 0}$  is called  $p, q$ -convex if

$$a_{n+2} - (p+q)a_{n+1} + pqa_n \geq 0, \quad (\forall) n \geq 0.$$

Let  $\mathbf{K}$  be set of all real sequences,  $\mathbf{K}_+$  the set of all real sequences positive,  $\mathbf{K}_{p,q}$  the set of all  $p, q$ -convex sequences and  $T : \mathbf{K}_{p,q} \rightarrow \mathbf{K}$  be a linear operator, defined as:

$$T(a; n) = \sum_{k=0}^n \rho_k(n) \cdot a_{s+k}$$

where  $\rho_k(n) \in \mathbf{R}$  ( $n \in \mathbf{N}$ ) and  $s \in \mathbf{N}$  are arbitrary.

The purpose of this work is to determine sufficient conditions for a real triangular matrix  $\|\rho_k(n)\|_{n \in \mathbf{N}, k=0, \dots, n}$  such that  $T(\mathbf{K}_{p,q}) \subseteq \mathbf{K}_+$ .

### Theorem 1

Let  $a = (a_n)_{n \geq 0} \in \mathbf{K}_{p,q}$  be given arbitrary.  $T(a; n) \in \mathbf{K}_+$  if

i)

$$\begin{cases} \sum_{i=0}^n \rho_i(n) \cdot q^i = 0 \\ \sum_{i=0}^n \rho_i(n) \cdot p^i = 0 \end{cases} \quad p \neq q, \quad p \neq 0, \quad q \neq 0$$

or

$$\begin{cases} \sum_{i=0}^n \rho_i(n) \cdot q^i = 0 \\ \sum_{i=0}^n \rho_i(n) \cdot i \cdot p^i = 0 \end{cases} \quad p = q \neq 0$$

ii)

$$\frac{1}{q \cdot p^{k+1}} \cdot \sum_{i=0}^k \rho_i(n) p^i \cdot \sum_{r=0}^{k-i} \left(\frac{p}{q}\right)^r \geq 0, \quad k = \overline{0, n-2}$$

**Proof.** For the computations we need the following notation:  $\rho_k(n) = \rho_k, (\forall) k = \overline{0, n}, n \in \mathbf{N}$ . Consider

$$T(a; n) = \sum_{k=0}^{n-2} c_k \cdot [a_{s+k+2} - (p+q)a_{s+k+1} + p \cdot qa_{s+k}],$$

$$T(a; n) = \sum_{k=0}^{n-2} c_k \cdot a_{s+k+2} - (p+q) \cdot \sum_{k=0}^{n-2} c_k \cdot a_{s+k+1} + p \cdot q \cdot \sum_{k=0}^{n-2} c_k \cdot a_{s+k}$$

$$T(a; n) = \sum_{k=0}^{n-2} a_{s+k} \cdot [c_{k-2} - (p+q)c_{k-1} + p \cdot q \cdot c_k] + a_{s+n-1} \cdot [c_{n-3} - (p+q)c_{n-2}] +$$

$$+ c_{n-2} \cdot a_{s+n} + a_{s+1} \cdot [p \cdot qc_1 - (p+q) \cdot c_0] + p \cdot q \cdot c_0 \cdot a_s.$$

Therefore

$$\begin{cases} pqc_0 = \rho_0 \\ pqc_1 - (p+q)c_0 = \rho_1 \\ c_{k-2} - (p+q)c_{k-1} + pqc_k = \rho_k, \quad k = \overline{2, n-2} \\ c_{n-3} - (p+q)c_{n-2} = \rho_{n-1} \\ c_{n-2} = \rho_n \end{cases} \quad (1)$$

From (1), it follows:

$$\begin{aligned} c_{r-2} - (p+q)c_{r-1} + pqc_r &= \rho_r, \quad r = \overline{2, n-2} \\ (c_{r-2} - pc_{r-1}) - q(c_{r-1} - pc_r) &= \rho_r \\ q^{r-2}(c_{r-2} - pc_{r-1}) - q^{r-1}(c_{r-1} - pc_r) &= \rho_r q^{r-2} \end{aligned}$$

Let add those equalities for  $r = \overline{2, k}$ . We obtain:

$$c_0 - pc_1 - q^{k-1} \cdot (c_{k-1} - p \cdot c_k) = \sum_{r=2}^k \rho_r \cdot q^{r-2}, \quad k = \overline{2, n-2}$$

$$p \cdot c_k - c_{k-1} = \frac{pc_1 - c_0}{q^{k-1}} + \frac{1}{q^{k-1}} \sum_{r=2}^k \rho_r \cdot q^{r-2},$$

$$p^k \cdot c_k - p^{k-1} \cdot c_{k-1} = (pc_1 - c_0) \cdot \left(\frac{p}{q}\right)^{k-1} + \left(\frac{p}{q}\right)^{k-1} \sum_{r=2}^k \rho_r \cdot q^{r-2},$$

$$p^i \cdot c_i - p^{i-1} \cdot c_{i-1} = (pc_1 - c_0) \cdot \left(\frac{p}{q}\right)^{i-1} + \left(\frac{p}{q}\right)^{i-1} \sum_{r=2}^i \rho_r \cdot q^{r-2}.$$

Let add those equalities for  $i = 2, \dots, k$ . We obtain:

$$p^k \cdot c_k - pc_1 = (pc_1 - c_0) \sum_{i=2}^k \left(\frac{p}{q}\right)^{i-1} + \sum_{i=2}^k \left(\frac{p}{q}\right)^{i-1} \cdot \sum_{r=2}^i \rho_r \cdot q^{r-2}$$

$$c_k = \frac{c^1}{q^{k-1}} + \frac{pc_1 - c_0}{p^k} \sum_{i=2}^k \left(\frac{p}{q}\right)^{i-1} + \frac{1}{qp^{k-1}} \sum_{i=2}^k \left(\frac{p}{q}\right)^i \cdot \sum_{r=2}^i \rho_r \cdot q^r$$

By the formula  $\sum_{i=2}^k a_i \cdot \sum_{r=2}^i b_r = \sum_{i=2}^k b_i \cdot \sum_{r=i}^k a_r$  we can write:

$$c_k = \frac{c^1}{p^{k-1}} + \frac{pc_1 - c_0}{p^k} \sum_{i=2}^k \left(\frac{p}{q}\right)^{i-1} + \frac{1}{qp^{k-1}} \sum_{i=2}^k \rho_i \cdot q^i \sum_{r=i}^k \left(\frac{p}{q}\right)^r$$

$$c_k = \frac{c^1}{p^{k-1}} + \frac{pc_1 - c_0}{p^k} \sum_{i=2}^k \left(\frac{p}{q}\right)^{i-1} + \frac{1}{qp^{k-1}} \sum_{i=2}^k \rho_i \cdot q^i \sum_{r=0}^{k-i} \left(\frac{p}{q}\right)^{r+i}, \quad k = \overline{2, n-2}$$

From (1) we have

$$c_0 = \frac{\rho_0}{p \cdot q}, \quad c_1 = \frac{\rho_1}{p \cdot q} + \frac{p+q}{p^2 \cdot q^2} \cdot \rho_0$$

Therefore

$$c_k = \frac{\rho^1}{q \cdot p^k} + \frac{p+q}{q^2 p^{k+1}} \cdot \rho_0 + \frac{1}{p^k} \left[ p \left( \frac{\rho_1}{pq} + \frac{p+q}{p^2 \cdot q^2} \cdot \rho_0 \right) - \frac{\rho_0}{pq} \right] \sum_{i=2}^k \left( \frac{p}{q} \right)^{i-1} +$$

$$+ \frac{1}{qp^{k+1}} \sum_{i=2}^k \rho_i \cdot q^i \sum_{r=0}^{k-i} \left( \frac{p}{q} \right)^{r+i},$$

$$c_k = \frac{1}{qp^{k+1}} \sum_{i=0}^k \rho_i \cdot q^i \cdot \sum_{r=0}^{k-i} \left( \frac{p}{q} \right)^{r+i}, \quad k = \overline{2, n-2} \quad (1')$$

From (1) and (1') it follows that:

$$\begin{cases} c_{n-2} = \frac{1}{qp^{n-1}} \sum_{i=0}^{n-2} \rho_i \cdot q^i \sum_{r=0}^{n-2-i} \left( \frac{p}{q} \right)^{r+i} = \rho_n \\ c_{n-3} = \frac{1}{qp^{n-2}} \sum_{i=0}^{n-3} \rho_i \cdot q^i \sum_{r=0}^{n-3-i} \left( \frac{p}{q} \right)^{r+i} = \rho_{n-1} + (p+q)\rho_n \end{cases} \quad (1'')$$

Because

$$c_{n-2} = \frac{1}{qp^{n-1}} \sum_{i=0}^{n-2} \rho_i \cdot q^i \cdot w_{n-1-i} = \rho_n$$

where

$$w_j = \begin{cases} \left( \frac{p}{q} \right)^j - 1 & , p \neq q \\ \frac{p}{q} - 1 & , p = q \end{cases} \quad j = \overline{1, n-1}$$

it results:

$$c_{n-2} = \frac{1}{qp^{n-1}} \sum_{i=0}^n \rho_i \cdot p^i \cdot w_{n-1-i} - \frac{\rho_n p_n}{qp^{n-1}} \cdot \frac{\frac{q}{p} - 1}{\frac{p}{q} - 1} = \rho_n.$$

Thus:

$$\sum_{i=0}^n \rho_i \cdot p^i \cdot w_{n-1-i} = 0 \quad (2)$$

Because

$$c_{n-3} = \frac{1}{qp^{n-2}} \sum_{i=0}^{n-3} \rho_i \cdot p^i \cdot w_{n-2-i} = \rho_{n-1} + (p+q)\rho_n$$

it results also

$$c_{n-3} = \frac{1}{qp^{n-2}} \sum_{i=0}^n \rho_i p^i \cdot w_{n-2-i} - \frac{\rho_{n-1} p}{q} \cdot \frac{\frac{q}{p} - 1}{\frac{p}{q} - 1} - \frac{\rho_n p^2}{q} \cdot \frac{\frac{q^2}{p^2} - 1}{\frac{p}{q} - 1}$$

and thus:

$$\sum_{i=0}^n \rho_i \cdot p^i \cdot w_{n-2-i} = 0 \quad (3)$$

From (2) and (3) we obtain:

$$\begin{cases} \sum_{i=0}^n \rho_i \cdot q^i = 0 \\ \sum_{i=0}^n \rho_i \cdot p^i = 0 \end{cases} \quad p \neq q$$

or

$$\begin{cases} \sum_{i=0}^n \rho_i \cdot q^i = 0 \\ \sum_{i=0}^n \rho_i \cdot i \cdot p^i = 0 \end{cases} \quad p = q$$

this means i).

From condition  $c_k \geq 0$  for all  $k$  from 0 to  $n-2$  we obtain ii).

### References

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