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ON AN INTEGRAL OPERATOR WHICH PRESERVE THE UNIVALENCE

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1. Introduction and preliminaries

We denote by U_r the disc $\{z \in \mathbb{C} : |z| < r\}$, $0 < r \leq 1$, $U_1 = U$.

Let A be the class of functions f which are analytic in U and $f(0) = f'(0) - 1 = 0$.

Let S be the class of the functions $f \in A$ which are univalent in U.

Definition 1. Let f and g be two analytic functions in U. We say that f is subordinate to g, $f \prec g$, if there exists a function φ analytic in U, which satisfies $\varphi(0) = 0$, $|\varphi(z)| < 1$ and $f(z) = g(\varphi(z))$ in U.

Definition 2. A function $L : U \times I \to \mathbb{C}$, $I = [0, \infty)$, $L(z, t)$ is a Loewner chain, or a subordination chain if L is analytic and univalent in U for all $z \in U$ and for all $t_1, t_2 \in I$, $0 \leq t_1 < t_2$, $L(z, t_1) \prec L(z, t_2)$.

Lemma 1 [4]. Let $r_0 \in (0, 1]$ and let $L(z, t) = a_1(t) z + \ldots$, $a_1(t) \neq 0$ be analytic in U_{r_0} for all $t \in I$ and locally absolutely continuous in I, locally uniform with respect to U_{r_0}.

For almost all $t \in I$ suppose:

$$z \frac{\partial L(z,t)}{\partial z} = p(z,t) \frac{\partial L(z,t)}{\partial t}, \quad z \in U_{r_0}$$

where p is analytic in U and satisfies $\operatorname{Re} p(z, t) > 0$, $z \in U$, $t \in I$.

If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\frac{L(z,t)}{a_1(t)}$ forms a normal family in U_{r_0} then for each $t \in I$, L has an analytic and univalent extension to the whole disc U.

In the theory of univalent functions an interesting problem is to find those integral operators, which preserve the univalence, respectively certain classes of univalent functions. The integral operators which transform the class S into S are presented in the theorems A, B and C which follow.

The integral operators studied by Kim and Merkes is that from the theorem:

Theorem A [1]. If $f \in S$, then for $\alpha \in \mathbb{C}$, $|\alpha| \leq \frac{1}{4}$ the function F_α defined by:

$$(1) \qquad F_\alpha(z) = \int_0^z \left(\frac{f(u)}{u} \right)^\alpha \mathrm{d}u$$

belongs to the class S.

A similar result, for other integral operator has been obtained by Pfaltzgraff in:

Theorem B [3]. If $f \in S$, then for $\alpha \in \mathbb{C}$, $|\alpha| \leq \frac{1}{4}$, the function G_α defined by:

$$(2) \qquad G_\alpha(z) = \int_0^z [f'(u)]^\alpha \mathrm{d}u$$

belongs to the class S.

An integral operator different of (1) and (2) is obtained by Silvia Moldoveanu and N.N. Pascu in the next theorem:

Theorem C [2]. If $f \in S$, then for $\alpha \in \mathbb{C}$, $|\alpha - 1| \leq \frac{1}{4}$, the function I_α defined by:

$$(3) \qquad I_\alpha(z) = \left[\alpha \int_0^z f^{\alpha-1}(u)\, \mathrm{d}u \right]^{\frac{1}{\alpha}}$$

belongs to the class S.

In this note, using the subordination chains method, we obtain sufficient conditions for the regularity and univalence of the integral operator:

$$(4) \qquad H_\alpha(z) = \left[\alpha \int_0^z \left(f_1^{\lambda_1}(u) \cdot f_2^{\lambda_2}(u) \ \ldots \ f_n^{\lambda_n}(u) \right)^{\alpha-1} \mathrm{d}u \right]^{\frac{1}{\alpha}}$$

where $f_k \in A$, $k = \overline{1,n}$, $0 \leq \lambda_k \leq 1$, $\sum_{k=1}^{n} \lambda_k = 1$.

2. Main results

Theorem 1. Let $f_1, f_2, \ldots, f_n \in A$, $\alpha \in \mathbb{C}$, $|\alpha - 1| < 1$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$, $0 \le \lambda_k \le 1$, $k = \overline{1,n}$, $\sum_{k=1}^{n} \lambda_k = 1$. If:

$$(5) \qquad \left(1 - |z|^2\right) \left| (\alpha - 1) \frac{z f_k'(z)}{f_k(z)} \right| \le 1, \ (\forall)\, z \in U,\ k = \overline{1,n},$$

then the function H_α defined by (4) is analytic and univalent in U.

Proof. Because $f_1^{\lambda_1}(z) \cdot f_2^{\lambda_2}(z) \ldots f_n^{\lambda_n}(z) = z + a_2 z^2 + \ldots$ is analytic in U, there exists a number $r_1 \in (0, 1]$ such that $\frac{f_1^{\lambda_1}(z) \cdot f_2^{\lambda_2}(z) \ldots f_n^{\lambda_n}(z)}{z} \ne 0$ for any $z \in U_{r_1}$. Then for the function $\left[\left(\frac{f_1(z)}{z} \right)^{\lambda_1} \cdot \left(\frac{f_2(z)}{z} \right)^{\lambda_2} \cdot \ldots \cdot \left(\frac{f_n(z)}{z} \right)^{\lambda_n} \right]^{\alpha - 1}$ we can choose the uniform branch equal to 1 at the origin, analytic in U_{r_1}:

$$(6) \qquad g_1(z) = \left(\frac{f_1(z)}{z} \right)^{\lambda_1} \cdot \left(\frac{f_2(z)}{z} \right)^{\lambda_2} \cdot \ldots \cdot \left(\frac{f_n(z)}{z} \right)^{\lambda_n} = 1 + b_1 z + \ldots + b_n z^n + \ldots,$$

and we have:

$$(7) \qquad \int_0^{e^{-t} z} u^{\alpha - 1} g_1(u)\, \mathrm{d}u = z^\alpha g_2(z, t),$$

where:

$$(8) \qquad g_2(z, t) = \frac{1}{\alpha} e^{-t\alpha} + \ldots + \frac{b_n}{\alpha + n} e^{-(\alpha + n)t} z^n + \ldots$$

Let we consider the function:

$$(9) \qquad g_3(z, t) = \alpha\, f_2(z, t) + \alpha\, (e^{t} - e^{-t})\, e^{-t(\alpha - 1)} \cdot g_1(e^{-t} z)$$

Since $|\alpha - 1| < 1$ we have $g_3(0, t) = e^{-\alpha t}(1 - \alpha + \alpha\, e^{2t}) \ne 0$ for any $t \in I$ and it results that there exists $r_0 \in (0, r_1]$ such that $g_3(z, t) \ne 0$ in U_{r_0} for all $t \in I$. For the function $[g_3(z, t)]^{1/\alpha}$ we can choose an uniform branch, analytic in U_{r_0} for any $t \in I$. It results that the function:

(10) $$L(z, t) = z\,[g_3(z, t)]^{1/\alpha} = [g_4(z, t)]^{1/\alpha},$$

where:

(11) $$g_4(z,t) = \alpha \int_0^{e^{-t}z} \left(f_1^{\lambda_1}(u) \cdot f_2^{\lambda_2}(u)\ \ldots\ f_n^{\lambda_n}(u)\right)^{\alpha-1} \mathrm{d}u + {} + \alpha\,(e^{t} - e^{-t})\, z \left(f_1^{\lambda_1}(e^{-t}z)\ \ldots\ f_n^{\lambda_n}(e^{-t}z)\right)^{\alpha-1}$$

is analytic in U_{r_0}.

Using Lemma 1 we will prove that L is a subordination chain.

We observe that $L(z, t) = a_1(t)\, z + \ldots$, where:

(12) $$a_1(t) = e^{-t}\,(1 - \alpha + \alpha\, e^{2t})^{1/\alpha}.$$

Because $|\alpha - 1| < 1$ we have $a_1(t) \neq 0$ for all $t \in I$ and

(13) $$a_1(t) = e^{t\frac{2-\alpha}{\alpha}} \left[(1-\alpha)\, e^{-2t} + \alpha\right]^{\frac{1}{\alpha}}$$

and

(14) $$\lim_{t\to\infty} |\, a_1(t)\,| = \lim_{t\to\infty} e^{t \operatorname{Re}\frac{2-\alpha}{\alpha}}\, |\,\alpha\,|^{\frac{1}{\alpha}} = \infty \quad \text{if} \quad \operatorname{Re}\frac{2-\alpha}{\alpha} > 1 \text{ or}$$

$|\alpha - 1| < 1$.

It follows that $\dfrac{L(z,t)}{a_1(t)}$ forms a normal family of analytic functions in U_{r_2}, $r_2 = \dfrac{r_0}{2}$.

Let $p : U_{r_0} \times I,\ \ p(z,t) = \dfrac{z\,\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}}$.

In order to prove that p has an analytic extension with positive real part in U, for all $t \in I$ it is sufficient to prove that the function:

(15) $$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1}$$ is analytic in U for $t \in I$ and

(16) $|w(z, t)| < 1$, $(\forall)$ $z \in U$, $t \in I$.

But $\left| w(z,t) \right| < \max\limits_{|z|=1} \left| w(z,t) \right| = \left| w(e^{i\theta},t) \right|$, $\theta \in \mathbb{R}$ and it is sufficient that

(17) $\left| w(e^{i\theta},t) \right| \le 1$, $(\forall)$ $t > 0$, or

(18)
$$\left(1-|u|^2\right) \left| (\alpha-1)\, u \left[\lambda_1 \frac{f_1'(u)}{f_1(u)} + \lambda_2 \frac{f_2'(u)}{f_2(u)} + \ldots + \lambda_{n1} \frac{f_n'(u)}{f_n(u)} \right] \right| \le$$
$$\le \sum_1^n \lambda_k \left(1-|u|^2\right) \left| (\alpha-1) \frac{u \cdot f_k'(u)}{f_k(u)} \right| \le \sum_1^n \lambda_k = 1,$$

from (5) where $u = e^{-t} e^{i\theta}$, $u \in U$, $|u| = e^{-t}$. It results (16).

Hence the function L is a subordination chain and $L(z, t) = H_\alpha(z)$ from (4) is analytic and univalent in U.

Theorem 2. If $f_1, f_2, \ldots, f_n \in S$, $0 \le \lambda_k \le 1$, $\sum\limits_{k=1}^{n} \lambda_k = 1$, then for $\alpha \in \mathbb{C}$, $|\alpha - 1| \le \frac{1}{4}$, the function H_α defined by (4) belongs to the class S.

Proof. Because $f_k \in S$, $k = \overline{1,n}$ we have:

$$\left| \frac{z \cdot f_k'(z)}{f_k(z)} \right| \le \frac{1+|z|}{1-|z|}, \qquad (\forall)\ z \in U,$$

then

$$\left(1-|z|^2\right) \left| \frac{z \cdot f_k'(z)}{f_k(z)} \right| \le \left(1+|z|\right)^2 < 4$$

and

$$\left(1-|z|^2\right) \left| (\alpha-1) \frac{z \cdot f_k'(z)}{f_k(z)} \right| \le |\alpha-1| \left(1+|z|\right)^2 < 4\ |\alpha-1| \le 1$$

because $|\alpha - 1| \le \frac{1}{4}$. Then, from Theorem 1 we obtain that $H_\alpha \in S$.

Example. Let $f_1(z)=\dfrac{z}{(1-z)^2}$, $f_2(z)=\dfrac{z}{1-z}$ and $f_3(z)=z$.

Then, for $\lambda_1=\dfrac{1}{3}$, $\lambda_2=\dfrac{1}{4}$, $\lambda_3=\dfrac{5}{12}$ we have:

$$f_1^{\lambda_1}(z)\cdot f_2^{\lambda_2}(z)\cdot f_3^{\lambda_3}(z)=\frac{z^{\lambda_1}}{(1-z)^{2\lambda_1}}\cdot\frac{z^{\lambda_2}}{(1-z)^{\lambda_2}}\cdot z^{\lambda_3}=$$

$$=\frac{z^{\lambda_1+\lambda_2}}{(1-z)^{2\lambda_1+\lambda_2}}\cdot z^{1-(\lambda_1+\lambda_2)}=\frac{z}{(1-z)^{2\lambda_1+\lambda_2}}=\frac{z}{(1-z)^{\frac{11}{12}}}$$

$$\Rightarrow H_\alpha(z)=\left[\alpha\int_0^z\left(\frac{z}{(1-z)^{\frac{11}{12}}}\right)^{\alpha-1}\mathrm{d}u\right]^{\frac{1}{\alpha}}.$$

For $\alpha=\dfrac{3}{2}\Rightarrow H_{\frac{3}{2}}(z)=\left[\dfrac{3}{2}\displaystyle\int_0^z\left(\frac{z}{(1-z)^{\frac{11}{12}}}\right)^{\frac{1}{2}}\mathrm{d}u\right]^{\frac{2}{3}}.$

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