

NONCOMMUTATIVE GEOMETRY AND THE DIFRACTION ONE DIMENSIONAL NETWORK

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Abstract. Noncommutative geometry extend the notion from classical differential geometry from differential manifold to discrete spaces and even noncommutative spaces which are given by noncommutative algebra (over \mathbf{R} or \mathbf{C}). Such an algebra A replace the commutative algebra of function of class C^∞ over a smooth manifold.

In this work I present some aspects about the calculus of the distance in the noncommutative geometry case. An important role in the distance calculus is playing by the Dirac operator.

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1. Introduction

Definition 1.1_ Let V be a finite dimensional vector space over the scalar field \mathbf{K} , where $\mathbf{K}=\mathbf{R}$ or \mathbf{C} .

A *Quadratic form* on V is a mapping $Q : V \rightarrow \mathbf{K}$ such that:

1) $Q(\lambda v) = \lambda^2 Q(v)$

2) The associated form $B(v, w) = \frac{1}{2} \{Q(v) + Q(w) - Q(v - w)\} \quad v, w \in V$ is bilinear

In this case (V, Q) is a *Quadratic space*.

Definition 1.2 The pair (A, ν) is said to be a *Clifford algebra* for the quadratic space (V, Q) when :

1) A is generated as an algebra by $\{\nu(v) \mid v \in V\}$ and $\{\lambda 1 \mid \lambda \in \mathbf{K}\}$

2) $((\nu(v))^2 = -Q(v)1, \quad v \in V$

Example:1) The \mathbf{R} -algebra of complex numbers \mathbf{C} is generated by 1 and i , verifying the relation:

$i^2 = -1$, it is a Clifford algebra for the quadratic space (\mathbf{R}, Q) and the Clifford map c , where $Q : \mathbf{R} \rightarrow \mathbf{R}$ and $c : \mathbf{R} \rightarrow \mathbf{C}$ are given by:

$$Q(x) = -x^2, \quad c(x) = ix$$

2) When we take $Cl_{\mathbf{R}}(\mathbf{R}^{p+q}, Q)$ where Q is the quadratic form

$$Q(x) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2$$

we use the notation $Cl(p, q)$, we put $Cl(n) \equiv Cl(0, n)$ and $Cl^*(n) \equiv Cl(n, 0)$

Hence, for the universal real Clifford algebra $Cl_C(V, -Q)$ over the vector space $V = \mathbf{R}^n$, where Q comes from the bilinear form of the usual euclidian product in \mathbf{R}^n , we use the notation $Cl(n)$, that means that if we take an orthonormal basis e_1, e_2, \dots, e_n in \mathbf{R}^n , we have: $e_i e_j + e_j e_i = -2\delta_{ij} 1$

We have for instance:

$$Cl(1) = \mathbf{C}, \quad Cl(2) = \mathbf{H}, \quad Cl(3) = \mathbf{H} \oplus \mathbf{H}$$

3) Let be the Pauli matrices in $\mathbf{C}^{2 \times 2}$:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the associated matrices would be:

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We have:

$$\sigma_0^2 = \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbf{I}, \quad e_0^2 = \mathbf{I}, \quad e_1^2 = e_2^2 = e_3^2 = -\mathbf{I}$$

and $\sigma_j \sigma_k = -i \sigma_l$, $e_j e_k = e_l$, where $\{j, k, l\}$ are cyclic permutation of the set $\{1, 2, 3\}$.

$$\text{Let } A_{0,0} = \{\lambda \sigma_0, \lambda \in \mathbf{R}\}, \quad A_{1,0} = \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} \middle| x, y \in \mathbf{R} \right\},$$

$$A_{0,1} = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \middle| x, y \in \mathbf{R} \right\},$$

$$A_{0,2} = \left\{ \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \middle| x_j \in \mathbf{R}, j = \overline{0,3} \right\} = \left\{ \begin{pmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{pmatrix} \middle| z_j \in \mathbf{C} \right\}$$

Then we have: $A_{0,0} \cong R$, $A_{1,0} \cong R \oplus R$, $A_{0,1} \cong C$, $A_{0,2} \cong H$.

2. Dirac operator relative to a vector bundle

Given a riemannian manifold M , an operator $\Delta \in Diff^{(2)}(E, E)$ of order 2 is said to be a *generalized laplacian* if $\sigma_2(\Delta)(\xi_p) = -\|\xi_p\|_g^2 I_{E_p}$ where $\|\cdot\|_g$ is the metric norm of M , and an operator of order 1, $D \in Diff^{(1)}(E, E)$ is said to be a *Dirac operator* relative to the vector bundle (E, M, K^\wedge) whenever D^2 is a generalized laplacian. If D is a Dirac operator we can define for any $\xi_p \in T_p^*M$ an K -endomorphism $c(\xi_p)$ of E_p given by $c(\xi_p) = \sigma_1(D)(\xi_p)$ or, alternatively we define a map:

$$\begin{aligned} \gamma_p : T_p^*M \times E_p &\rightarrow E_p \\ (\xi_p, u_p) &\rightarrow c(\xi_p)(u_p) \equiv c(\xi_p)u_p \end{aligned}$$

for the tensorization of T_p^*M by \mathbf{C} we use the above relation $T_C^*M \equiv (T_p^*M) \otimes_R C$, so if we don't want to precise if $K = R$ or \mathbf{C} we will write $T_K^*M \equiv (T_p^*M) \otimes_R K$, and obviously $T_R^*M = T_p^*M$, note that the map γ_p could be defined using T_K^*M instead of T_p^*M .

The endomorphisms $c(\xi_p)$ verify the condition $c(\xi_p)^2 = -\|\xi_p\|_g^2 I_{E_p}$, equivalent to $c(\xi_p)c(\eta_p) + c(\eta_p)c(\xi_p) = -2g(\xi_p, \eta_p)I_{E_p}$, this means that we got for any $p \in M$ a K -bilinear map $\gamma_p : (T_K^*M) \times E_p \rightarrow E_p$ that is a K -linear map $(T_K^*M) \otimes_K E_p \rightarrow E_p$, with the preceding property.

The set of K -linear operators $c(\xi_p) : E_p \rightarrow E_p$ generates an associative and unital sub algebra A of the K -algebra $End_K(E_p)$ (with $1 \equiv I_{E_p}$ as element one) and there exist a K -linear map $c : T_K^*M \rightarrow A$ given by $\xi_p \rightarrow c(\xi_p)$ and verifying

$$c(\xi_p)c(\eta_p) + c(\eta_p)c(\xi_p) = -2g(\xi_p, \eta_p)1$$

This means that A is a Clifford algebra for the quadratic space $(T_K^*M, -g)$ and c is the Clifford mapping. The vector bundle E is a *Clifford vector bundle* and $\gamma : T_K^*M \times E \rightarrow E$, given by $\gamma(\xi, u)|_p = \gamma_p(\xi_p, u_p) = c(\xi_p)u_p$ is the corresponding *Clifford action*.

The following property exhibit the compatibility of D with the Clifford action:

$$D(sf) = c(df)s + D(s)f \quad (f \in F_K(M), \quad s \in \Gamma(M, E))$$

or alternatively, making $F_K(M)$ act multiplicatively in $\Gamma(M, E)$ through $\bar{f}(s) = sf$, we will have :

$$D(sf) - D(s)f = (D \circ \bar{f})(s) - (\bar{f} \circ D)(s) = ((D \circ \bar{f}) - (\bar{f} \circ D))(s) = [D, \bar{f}] = c(df)s,$$

that is:

$$[D, \bar{f}] = c(df)$$

or we can write using the previous conventions:

$$[D, \bar{f}] = c(df).$$

We can say that any Dirac operator, relative to the vector bundle (E, M, K^n) , with a riemannian manifold in the basis, determines in this vector bundle a structure of Clifford bundle compatible with D in the preceding sense.

Any Clifford vector bundle on a riemannian manifold endowed with a covariant derivative

$$\bar{\nabla} : \Gamma(M, E) \rightarrow \Gamma(M, E \otimes T_K^*M)$$

compatible with a Clifford action $\gamma : T_K^*M \otimes E \rightarrow E$ in a sense to be precise later, is associated to a Dirac operator acting on the sections of this vector bundle and compatible with the Clifford action, we will call such a vector bundle a *Clifford-Weyl bundle*.

The compatibility of $\bar{\nabla}$ with the Clifford action $\xi_p s = c(\xi_p)s$ means the following:

$$\bar{\nabla}_X(\omega \cdot s) = \nabla_X \omega \cdot s + \omega \cdot \bar{\nabla}_X s \quad (\omega \in \Omega^1(M, K))$$

or alternatively,

$$\bar{\nabla}_X(c(\omega)s) = c(\nabla_X \omega)s + c(\omega)\bar{\nabla}_X s \quad (\omega \in \Omega^1(M, K))$$

where $\nabla : \mathfrak{N}_K(M) \rightarrow \mathfrak{N}_K(M) \otimes \Omega^1(M, K)$ or else $\nabla_X : \mathfrak{N}_K(M) \rightarrow \mathfrak{N}_K(M)$

for any $X \in \mathfrak{N}_X(M)$, is the Levi Cevita covariant derivative in M , determining a covariant derivative $\nabla : \Omega^1(M, K) \rightarrow \Omega^1(M, K) \otimes \Omega^1(M, K)$ or actually, $\nabla_X : \Omega^1(M, K) \rightarrow \Omega^1(M, K)$ for any $X \in \mathfrak{N}_X(M)$.

The set $\Gamma(M, E)$ of the sections of a Clifford-Weyl vector bundle has a structure of $(Cl_K(T^*M), F_K(M))$ -bimodule, with the following properties of compatibility:

$$\bar{\nabla}(sf) = \bar{\nabla}(s)f + s \otimes df$$

$$\bar{\nabla}(\omega \cdot s) = (\nabla \omega) \cdot s + \omega \cdot \bar{\nabla}s$$

Next, I will prove the following result, which is the distance between two different points on the straight line, in noncommutative geometry case:

$$d_C(x, y) = \sup_{f \in A} \{|f(x) - f(y)| : \|[D, f]\| < 1\} = \sup_{f \in A} \{|f(x) - f(y)| : \|f'\| < 1\} = |x - y|$$

We know that $[D, \overline{f}] : A \rightarrow A$

$$\begin{aligned} g \rightarrow [D, \overline{f}](g) &= (D \circ \overline{f} - \overline{f} \circ D)(g) = \\ &= D(fg) - f(Dg) = (fg)' - fg' = f'g + fg' - fg' = \overline{f'} \end{aligned}$$

where I use the definition $\overline{f}(s) = sf$.

In general for an operator $T : E \rightarrow E$, we have : $\|T\| = \sup_{\|\xi\| \leq 1} \|T(\xi)\|$

In this case

$$\|[D, f]\| = \sup_{\|\xi\| \leq 1} \|f'\xi\| = \sup_{\|\xi\| \leq 1} \|f'\|\xi = \|f'\|$$

and using the inequality:

$$|f(x) - f(y)| \leq \|f'\| \cdot |x - y|$$

I prove the distance relation.

Now, I will use the Dirac operator to recover the distance between atoms in a periodical one dimensional diffraction network.

Let assume that we have a network in which, between two atom we have the same distance.

So, we can represent the network in this way:



Using this network we can construct the incidence matrix putting the element 1 when we have a link between two atom and putting 0 if we don't have link.

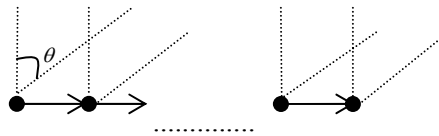
So, the incidence matrix would be:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

We define:

$$D_N = \begin{pmatrix} 0 & \frac{\sin \theta}{\lambda} & 0 & 0 & \dots & 0 \\ 0 & 0 & \frac{\sin \theta}{\lambda} & 0 & \dots & 0 \\ 0 & 0 & 0 & \frac{\sin \theta}{\lambda} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \frac{\sin \theta}{\lambda} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where θ is the angle between two wave and λ is the wave length.



Let be $N = \{1, 2, \dots, n\}$ the set of atoms.

We will denote with A , the algebra of all maps $f : N \rightarrow \mathbb{C}$.

The function f is represented as

$$f \rightarrow \tilde{f} = \begin{pmatrix} f_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & f_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & f_3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & f_n \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in C^n,$$

where $f(n) = f_n$, and with this representation we can construct the function

$$\bar{f} = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}.$$

We can associate to the complex function f , a real function F with the properties:

$$F_1 = 0 \text{ and } F_{k+1} = F_k + |f_{k+1} - f_k|.$$

We have the operator

$$\hat{D} = \begin{pmatrix} 0 & D_N'' \\ D_N & 0 \end{pmatrix}$$

We know that in general, the distance in noncommutative geometry is given by:

$$d_C(x, y) = \sup_{f \in A} \{|f(x) - f(y)| : \|[D, f]\| < 1\}$$

In our case we can compute, and is easy to prove that

$$\|[\hat{D}, \bar{f}]\| = \|[\hat{D}, F]\| \text{ for } \psi \in C^n.$$

So, after easy computation we will get:

$$\|[\hat{D}, \bar{f}]\| = \max \left\{ \frac{\sin \theta}{\lambda} |f_2 - f_1|, \dots, \frac{\sin \theta}{\lambda} |f_N - f_{N-1}| \right\},$$

and using the condition $\|[D, f]\| < 1$, we get:

$$d(i, i+n) = \frac{\lambda}{\sin \theta} + \dots + \frac{\lambda}{\sin \theta} = \frac{n\lambda}{\sin \theta}.$$

We can find the same result if we start from physics.

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