ON THE MAXIMUM PROBABILITY CRITERIA CONCERNING THE SEQUENCIAL DECISION - MAKING PROBLEM

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Abstract. This paper presents, originally,some results about an important criterion used in the theory of the decisions. Here are treated the partial and the total co-operative cases and, finally, will be presented an application for a market competitional problem who will finish with a ruining problem.

INTRODUCTION

A problem of sequencial decision is described by the ensemble ([1],[3]):

$$S = \left\{ X, X_0, \bar{X}, M, u_i, i \in M, D^i_{x_n}, i \in M, n \in N, x_n \in X, f_n, n \in N \right\}$$

Where the significance of the elements is as follows:

1) The set X represents **the space of the positions** and is a topological linear space (real) and B_X is the $\sigma -$algebra generated generated by the topology of the space X .

To be measurable space (X, B_X),the set $\mu(B_X)$of all measures of probability defined in B_X is associated. The measure of probability $P_X \in \mu(B_X)$ is associated which each state $x \in X$.

2) $X_0, \bar{X}$ represents **the set of initial states and final states** respectively and they
are supposed to be compact sets in X

3) The set $M = \{1,2,..., m\}$represents **the set of deciders** taking part in the decision-making process and $\mu_i : \bar{X} \to R$ represents **the utility function** of the deciders $i \in M$ (u_i is supposed to be continuous)

Each decider $i \in M$ is associated with the value $a_i \in R$called **ceiling**, which signifies that the participation of the decider $i \in M$ in the decision-making process is connected with the intention of obtaining a profit, which increases the ceiling.

The set $\bar{X}_i = \left\{ \bar{x} \in \bar{X} : u_i(\bar{x}) \geq a_i \right\}$ is called **the target set** of the decider $i \in M$.

4) The evolution of the decision-making process is described with the help of the recurrence relations:

$$x_{n+1} = f_n(x_n, d_n), x_0 \in X_0, \forall n \in N$$

where $d_n \in D(x_n) = \prod_{i=1}^{n} D^i(x_n)$ and $D^i(x_n)$ represents **the set of the decision** that can be made in the state $x_n \in X$ by the decider $i \in M (D^i(x)$ is supposed to be a topological linear space $x \in X, i \in M$).

The application f $f_n : XxD_X \to X, n \in N$ are called **transition functions** and they are supposed to be continuous and bounded ($D_X = \bigcup_{x \in X} D(x)$).

If $x_n \in \bar{X}$, then $f_n(x_n, d_n) = x_n, \forall d_n \in D(x_n)$. . When there is no risk of confusion $D^i(x_n)$ is written as D_n^i and $D(x_n)$ as D_n .

The notion of inferior semi-continuity(i.s.c.), higher semi-continuity(h.s.c.) and continuity in the Haussdorf sense, both for the univocal and multivocal application will be the basic elements in the proves of some theorems.

2. THE EXISTENCE OF THE GUARANTEED OPTIMAL STRATEGIES

We shall put ourself in the position of decider in the case of the problem of sequencial decision described in introduction. Two situations will be analyzed:

a) the partial co-operative case;
b) the total co-operative case;

The purpose of this paragraph is to specify the margins of the interval within which the maximum profit of deciders1 lies, as well as the strategies(simple or mixed) throught which these margins are reached.

If the decision-making process has evolved to the state $x_n \in X \setminus \bar{X}$, the adoption by decider 1 of the criterion of maximum probability implies the adoption of the problem([3]):

$$\left(P_1^n\right): \sup_{d_n} \left\{ \bar{p} \in R : P_{f_n\left(x_n, d_n^1, \tilde{d}_n^1\right)} \left\{ \bar{x} \in \bar{X} : u_i\left(\bar{x}\right) \geq a_i \right\} \geq \bar{p} \right\}$$

$$d_n = \left(d_n^1, \tilde{d}_n^1 \right) \in D_n^1 x \prod_{j=2}^{m} D_n^j$$

As decider 1 will decide first and deciders $j \in M \setminus \{1\}$ adopt,simultaneously, the following notations will be considerated:

$$D_n^1 = D_1, \prod_{j=2}^{m} D_n^j = D_2$$

The following functionals are introduced:

$$F_n : D_1 x D_2 \to R, F_n(d_1, d_2) = P_{f_n(x_n, d_1, d_2)} \left\{ \bar{x} \in \bar{X} : u_1\left(\bar{x}\right) \geq a_1 \right\}, (d_1, d_2) \in D_1 x D_2$$

$$g_n : D_1 x D_2 \to R, g_n(d_1, d_2) = P_{f_n(x_n, d_1, d_2)} \left\{ \bar{x} \in \bar{X} : \sum_{i \in M} u_i(\bar{x}^*) \geq \sum_{i \in M} a_i \right\} -$$

$$- P_{x_n} \left\{ \bar{x}^* \in \bar{X} : \sum_{i \in M} u_i(\bar{x}) \geq \sum_{i \in M} a_i \right\}, (d_1, d_2) \in D_1 x D_2$$

and the multivocal application:

$$B_n : D_1 \to P(D_2), B_n(d_1) = \{ d_2 \in D_2, g_n(d_1, d_2) \geq 0 \}$$

For greater convenience we shall write F, g, B instead of $F_n, g_n, B_n, (D_1, d_{D_1}), (D_2, d_{D_2})$ are assumed to be compact spaces.

The following hipothesis are made:

1) the forming of a coalition in the sense of maximum probability is allowed;
2) if the first decider has adopted the strategy $d_1 \in D_1$, the other decider will adopt only strategies from $B(d_1)$.

Remark 2.1 Hypothesis 2 is based on the following argument:if the choice of the pair of the strategies $(d_1, d_2) \in D_1 x B(d_1)$ increases in the state x_{n+1} the value:

$$P_{f_n(x_n, d_1, d_2)} \left\{ \bar{x} \in \bar{X} : \sum_{j \in M \setminus \{1\}} u_j(\bar{x}) \geq \sum_{j \in M \setminus \{1\}} a_j \right\}$$

then this choice will suit deciders $j \in M \setminus \{1\}$; if in the state x_{n+1} the value:

$$P_{f_n(x_n, d_1, d_2)} \left\{ \bar{x} \in \bar{X} : \sum_{j \in M \setminus \{1\}} u_j(\bar{x}) \geq \sum_{j \in M \setminus \{1\}} a_j \right\}$$

doesn't increase (as against the value: $P_{x_n} \left\{ \bar{x} \in \bar{X} : \sum_{j \in M \setminus \{1\}} u_j(\bar{x}) \geq \sum_{j \in M \setminus \{1\}} a_j \right\}$), but $g(d_1, d_2) \geq 0$ then forming a coalition in the sense of maximum probability, deciders

$j \in M \setminus \{1\}$ will be favored again. In his turn, decider 1 will be favored as he has the possibility of improving his control over deciders $j \in M \setminus \{1\}$.

Having introduced these notations, we can formulate problem (P_1^n) in the following way:

$\left(P_1^n\right)$: determine $d^* = \left(d_1^*, d_2^*\right) \in D_1 x D_2$ which verifies the equality:

$$F\left(d_1^*, d_2^*\right) = \sup_{(d_1, d_2) \in D_1 x D_2} F\left(d_1, d_2\right).$$

The solving of the problem $\left(P_1^n\right)$ represents however the ideal case for decider 1 as in concrete situations it hardly ever happens for all the deciders of the set M to have the same target set (in other words, all the deciders of the set M have the same target).

As for any $\bar{d}_2 \in D_2$ the following inequalities occur:

$$\sup_{d_1 \in D_1} \inf_{d_2 \in B(d_1)} F\left(d_1, d_2\right) \le \sup_{d_1 \in D_1} F\left(d_1, \bar{d}_2\right) \le \sup_{d_1 \in D_1} \sup_{d_2 \in B(d_1)} F\left(d_1, d_2\right).$$

The following functionals will be introduced naturally

$$f_1 : D_1 \to R,\ f_1\left(d_1\right) = \sup_{d_2 \in B(d_1)} F\left(d_1, d_2\right)$$

$$f_2 : D_1 \to R,\ f_2\left(d_1\right) = \inf_{d_2 \in B(d_1)} F\left(d_1, d_2\right).$$

Theorem 2.1. The following results occur:

1) If F is h.s.c., g is continuous(as an univocal application in the topology generated by $d_{D_1 x D_2}$), $B\left(d_1\right)$ is closed in the metric space $\left(P\left(D_2\right), \bar{d}\right), \forall d_1 \in D_1$ ($\bar{d}$ is the Haussdorf metric drawn with the help of the d_{D_2} metric) and B is closed(as a multivocal application), then there is $d_1^* \in D_1$ so the following equality takes place:

$$f_1\left(d_1^*\right) = \max_{d_1 \in D_1} \max_{d_2 \in B(d_1)} F\left(d_1, d_2\right);$$

2) If F is h.s.c.(as an univocal application in the topology generated by $d_{D_1 x D_2}$), B is continuous(as a multivocal application in the topology generated by the Hauswsdorf metric d drawn with the help of d_{D_2} metric), then there is $d_1^{**} \in D_1$ so the following equality occurs:

$$f_2\left(d_1^{**}\right) = \max_{d_1 \in D_1} \inf_{d_2 \in B(d_1)} F\left(d_1, d_2\right)$$

Proof

1)We first prove that if g is continuous, then B is h.s.c.(as a multivocal application). Let us consider:

$$\left(d_n^1\right)_n \subset D_1, d_n^1 \underset{n}{\rightarrow} d_*^1, \left(d_n^2\right) \in B\left(d_n^1\right), d_n^2 \underset{n}{\rightarrow} d_*^2 .$$

Because g is continuous, we obtain that $g\left(d_*^1, d_*^2\right) = \lim_n g\left(d_n^1, d_n^2\right)$., As $g\left(d_n^1, d_n^2\right) \geq 0$, $\forall n \in N$, it means that $g\left(d_n^1, d_n^2\right) \geq 0$, therefore $d_*^2 \in B\left(d_1\right)$ and consequentely B is h.s.c..

If $d_1 (\in D_1) \underset{n}{\rightarrow} d_*^1$, from the fact that F is h.s.c. and $B\left(d_1\right)$ is closed $\forall n \in N$, it follows that there is $d_n^2 \in B\left(d_n^1\right)$ so that the following conditions occur:

$$f_1\left(d_n^1\right) = \sup_{d_2 \in B\left(d_n^1\right)} F\left(d_n^1, d_2\right) = \max_{d_2 \in B\left(d_n^1\right)} F\left(d_n^1, d_2\right) = F\left(d_n^1, d_n^2\right).$$

From the fact that B is h.s.c. it follows that there is $d_*^2 \in B\left(d_*^1\right)$ so that $d_*^2 = \lim_n d_n^2$. As
F is h.s.c. we shall have:

$$\overline{\lim_n} f_1\left(d_n^1\right) = \overline{\lim_n} F\left(d_n^1, d_n^2\right) \leq F\left(d_*^1, d_*^2\right) \leq \max_{d_2 \in B\left(d_1\right)} F\left(d_*^1, d_2\right) = f_1\left(d_*^1\right)$$

From $\lim_n f_1\left(d_n^1\right) \leq f_1\left(d_*^1\right)$ it follows that f_1 is h.s.c.; as D_1 is compact it means that there is $d_*^1 \in D_1$ so that the below equality is verified:

$$f\left(d_*^1\right) = \max_{d_1 \in D_1} \max_{d_2 \in B\left(d_1\right)} F\left(d_1, d_2\right).$$

2).In order to demonstrate the existence of d_1^{**} it is sufficient to prove that f_2 is h.s.c.;let us consider any $d_1^0 \in D_1$. Also, let us consider any sufficiently small $\varepsilon > 0$. As

$$f_2\left(d_1^0\right) = \inf_{d_2 \in B\left(d_1^0\right)} F\left(d_1^0, d_2\right)$$

there is $d_2^0 \in B\left(d_1^0\right)$ so that $F\left(d_1^0, d_2^0\right) \leq f_2\left(d_1^0\right) + \varepsilon$. F beeing h.s.c. for the closen ε there will be δ so that:

$$F\left(d_1^0, d_2^0\right) \geq F\left(d_1, d_2\right) - \varepsilon/2, \forall \left(d_1, d_2\right) \in D_1 x D_2, d_{D_1 x D_2}\left(\left(d_1, d_2\right), \left(d_1^0, d_2^0\right)\right) < \varepsilon .$$

As B is continuous, there is $\gamma > 0$ so that $d\left(B\left(d_1\right), B\left(d_2\right)\right) \leq \delta$. Let us consider

$$V_{d_1^0} = \left\{d_1 \in D_1 : d_{D_1}\left(d_1, d_1^0\right) \leq \min(\delta, \gamma)\right\}.$$

For any $d_1 \in V_{d_1^0}$, there is a $d_2 \in B\left(d_1^0\right)$ with $d_{D_2}\left(d_2, d_2^0\right) \leq \delta$. Henceforth, for any $d_1 \in V_{d_1^0}$ we have the equalities:

$$f_2\left(d_1^0\right)+\varepsilon/2 \geq F\left(d_1^0, d_2^0\right) \geq F\left(d_1, d_2\right)-\varepsilon/2 \geq f_2\left(d_1\right)-\varepsilon/2$$

and so $f_2\left(d_1^0\right) \geq f_2\left(d_1\right)-\varepsilon, \forall d_1 \in V_{d_1}$. This means that f_2 is h.s.c. and so there is $d_1^{**} \in D_1$ so that the following equalities occur:

$$f_2\left(d_1^{**}\right)=\max_{d_1 \in D_1} \inf_{d_2 \in B(d_1)} F\left(d_1, d_2\right).$$

Remark 2.2 Theorem 2.1 specifies the existence of the strategies where the boarders of the interval in which decider 1 will obtain his maximum profit can be reached(and which is the maximum of the probability of realization of the target set in the state x_{n+1}).

Theorem 2.2 If B is h.s.c. and closed(as a multivocal application), F is h.s.c.(as an univocal application) and $B(d)$ is compact for every $d \in d_1$, then there is $d_* \in D_1$ so that the following equality occurs:

$$f_1\left(d_*\right)=\max_{d_1 \in D_1} \max_{d_2 \in B(d_1)} F\left(d_1, d_2\right)$$

Proof In order to prove the existence of d^* having the required property it is sufficient to show that f is h.s.c. Let us consider $\left(d_n^1\right)_n \subset D_1, \lim_n d_n^1 = d_*^1$. From the fact that F is h.s.c. and $B(D)$ is compact, it follows that there is $d_n^0 \in B\left(d_n^1\right)$ so that:

$$f_1\left(d_n^1\right)=\sup_{d_n^2 \in B\left(d_n^1\right)} F\left(d_n^1, d_n^2\right)=\max_{d_n^2 \in B\left(D_N^1\right)} F\left(d_n^1, d_n^2\right)=F\left(d_n^1, d_n^0\right) \quad (1)$$

From the fact that B is h.s.c. (as multivocal application) we obtain that there is $d_*^2 \in B\left(d_n^1\right)$ so that $d_*^2=\lim_n d_n^0$. Because B is also closed(as a multivocal application), we get that $d_*^2 \in B\left(d_*^1\right)$. F is h.s.c. so:

$$\overline{\lim_n} F\left(d_n^1, d_n^0\right) \leq F\left(d_*^1, d_*^2\right) \leq \max_{d_2 \in B\left(d_*^1\right)} F\left(d_*^1, d_2\right)= f_1\left(d_*^1\right) \quad (2)$$

which means that f_1 is h.s.c. and consequently there is $d_* \in D_1$ so that the below equalities occur:

$$f_1\left(d_*\right)=\max_{d_1 \in D_1} \max_{d_2 \in B(d_1)} F\left(d_1, d_2\right) \quad (3)$$

Remark 2.3 From the etheorem 2.2 it follows when the conditions from the enunciation of the theorem are satisfied, decider 1 has the possibility of knowing the maximum profit he can obtain which is a very important result for concrete problems. In cases when this result doesn’t satisfy these conditions,it is p-ossible for decider 1 to change his strategic behavior(he changes the criterion ofr optimality or he may try to form a coalition etc.).

Corollary 1 If F is continuous, B is continuous and $B(D)$ is compact, $\forall d \in D_1$, then there are d_1^*, d_2^{**} so that for every $\bar{d}_2 \in B(d_1)$ we have:

$$f_2\left(d_1^{**}\right) = \max_{d_1 \in D_1} \min_{d_2 \in B(d_1)} F(d_1, d_2) \le \max_{d_1 \in D_1} F\left(d_1, \bar{d}_2\right) \le \max_{d_1 \in D_1} \max_{d_2 \in B(d_1)} F(d_1, d_2) = f_1\left(d_1^*\right)$$

There are the following two cases:

a) The partial co-operative case. It corresponds with the situation when $p_1^n > f_1\left(d_1^*\right)$. In this case the decider 1 must give up the coalition idea (because in the next stage x_{n+1} is led through an inferior gain to the gain p_1^n according to the x_n stage).

Remark 2.4 The term ”partial co-operative” comes from the fact that the deciders of the set $M \setminus \{1\}$ can form a coalition in this case(forming the total coalition or distinct coalitions), even though the decider 1 might not belong to any coalition.

The decider 1 has the following alternatives:

i) The deciders of the set $M \setminus \{1\}$ adopt a prudent strategic behavior(whichmeans that they adopt maxmin or minmax strategies). In this case, according to ”the equalization criterion” [4], the decider 1 owns a strategy $\tilde{d}_1^n$ which, if he adopts it, he will obtain $p_1^{n+1} > p_1^n$ (that means that the maximum probability criterion is equivalent with the equalization criterion).

ii) The decider 1 has no information regarding to the strategic behaviorof the other deciders. In this case, if the following conditions are fulfilled:

a) the target sets $\bar{X}_j, j \in M$, realize un unfolding of $\bar{X}$;
b) the strategy sets D_n^1 are compact sets in $R^k, \forall n \in N$;
c) application F is continuous in both arguments and convex in the second, there is a finite subset $\bar{D}_n^1 \subset D_1$ so that a necessary condition for solving the problem $\left(P_1^n\right)$ is the solving of the following problem:

$$\left(\tilde{P}_1^n\right) : \max_{d_n \in D_n}\left(H_{n+1}^{d_n} + I_{n,n+1}^{d_n}\right)$$

where

$$H_{n+1}^{d_n} = -\sum_{i=1}^{n} P_{n,j}^{d_n} \ln P_{n,j}^{d_n}$$

represents the **undeterminancy(** Shannon entropy) provided the choice of the strategy:

$$d_n \in \bar{D}_n = \bar{D}_n^1 x \prod_{j=2}^{n} D_n^j, P_{f_n(x_n,d_n)}\left\{\bar{x} \in \bar{X} : u_j(\bar{x}) \geq a_j\right\} = P_{n,j}^{d_n}$$

$$I_{n,n+1}^{d_n} = \sum_{j=1}^{n} P_{n+1,j}^{d_n} \ln \frac{P_{n+1,j}^{d_n}}{P_n^j}$$

represents **the mean information profit**(in the Renyi sense) obtained through passing on form the state x_n to the state x_{n+1} as a result of the adoption of the strategy d_n .

Remark 2.5 Let us consider d_n^* the solution of $(\tilde{P}_1^n)$. Not always $P_{n,1}^{d_n^*} > p_1^n$. In [3] is proved that if the inequality:

$$H_{n+1}^{d_n^*} + I_{n,n+1}^{d_n^*} \geq -p_1^n \ln p_1^n - \ln\left(1 - p_1^n\right)$$

holds, then $P_{n,1}^{d_n^*} > p_1^n$(which means that the reached probability in the state x_{n+1} of the target set $\bar{X}_1$ is greater than the reached probability of the target set $\bar{X}_1$ by the decider 1).

b). The total co-operative case. It corresponds with the situation when $p_1^n < f_1\left(d_1^*\right)$.Therefore, forming a coalition and adopting the strategy d_1^*, at the state x_{n+1} the decider 1 increases his own reached probability of the fixed target set $\bar{X}_1$ (that is $P_{n,1}^{d_n^*} \geq p_1^n$).

It appears the natural problem of finding what are the conditions for the decider 1 to obtain $\lim_n p_1^n = 1$. That is, if the decider 1 forms a coalition with the other deciders and it is formed the total coalition, which are the conditions for this coalition to reach the target set at the end of the decisional process. In order to do this, we first introduce some important notations and notions.

The sequence of multivocal applications $\left(B_n\right)_n$ is defined through reccurence. The sequence is defined as follows:

$$B_1 : X_0 \to P(X), B_1(x_0) = \{x_1 \in X : x_1 = f_0(x_0, d_0), d_0 \in D_0\}$$

$$B_n : X_0 \to P(X), B_n(x_0) = \{x_n \in X : x_n = f_{n-1}(x_{n-1}, d_{n-1}), d_{n-1} \in D_{n-1}, x_{n-1} \in B_{n-1}(x_0)\},$$

$n > 1$.

The set $B_n(x_0)$ represents the set of the states which can be reached in n stages starting from the initial stage $x_0 \in X_0$. Let us consider the functionals:

$$F^n : X_0 x(X \setminus X_0) \to R, F^n(x_0, x) = P_{x \in B_n(x_0)}\left\{\bar{x} \in \bar{X} : u_m(\bar{x}) \geq a_m\right\}$$

$$R^n : X_0 \to R, R^n(x_0) = \sup_{x \in B_n(x_0)} F^n(x_0, x), n \in N$$

(we wrote $u_M = \sum_{i \in M} u_i, a_M = \sum_{i \in M} a_i$). We denote: $T^n(x_0) = \bigcup_{k=1}^{n} B_k(x_0)$, $T(x_0) = \bigcup_{n=1}^{\infty} T^n(x_0), \forall x_0 \in X_0$. The sets $T^n(x_0), T(x_0)$ represent the sets of trajectories of n duration that start from x_0 and the sets of trajectories that start from x_0, respectively.

The transition functions f_n are supposed to be Lipschitzian with the same Lipschitz constant $M_0, \forall n \in N$ .We further on attempt to prove that the problem of sequencial decision under consideration there are convergent trajectories and optimum trajectories.

Theorem 2.3 We have the next results:

1) If for every $n \in N, F^n$ is h.s.c., then there are $x_0^{*,n} \in B_n\left(x_0^{*,n}\right)$ so that:

$$F^n\left(x_0^{*,n}, x_n^*\right) = \max_{x_0 \in X_0} \max_{x \in B_n(x_0)} F^n(x_0, x).$$

2) If F^n is h.s.c. and $x_0^{*,n} = x_0, \forall n \in N$ (i.e. all of the optimum trajectories start from the same end), then all of the optimum trajectories converge towards $\bar{X}_M$.

3) Let us take $\left(x_n^m\right)_{n,m} \subset T(x_0), x_0^m = x_0, x_0$ fixed. There is always $\left(x_n^{m_k}\right)_{n,k} \subset \left(x_n^m\right)_{n,m}$ and $\left(x_n\right)_n \in T(x_0)$ so that $\lim_k \left\|x_n^{m_k} - x_n\right\|_X = 0$.

Remark 2.6 1) From the teorem 2.3 it results that for any $n \in N$ there is $x_0^{*,n} \in X_0, x_n^* \in B_n\left(x_0^{*,n}\right)$ so that whatever the trajectories from $T^n\left(x_0^{*,n}\right)$ of ends $x_0^{*,n}, x_n^*$, maximum profit is guaranteed in the end x_n^*.

2) The conditions in which the existence of the optimum trajectories has been demonstrated, in the case of the finite horizon and infinite horizon (theorem 2.3 and 2.4), are very hard. If these conditions are loosen , it is only the existence of convergent trajectories that can be demonstrated (without securing their conditions of optimality) on the basis of the following theorem:

Conclusions Theorems 2.4 and 2.5 shows that if the deciders form a coalition, achieve the total coalition and the initial state is the same for all the deciders, then:

1) there is a trajectory which converges in the target set $\bar{X}_M$;
2) there is a sequence of trajectories which converge at this trajectory.

3. APPLICATION IN A RUINING PROBLEM

Below it will be given an market competitional problem which leads at the end to a ruin problem

Let us consider a sequencial decision problem in which the deciders are formed a coalition in two coalition C_1, C_2 having the final state sets $\bar{X}_1, \bar{X}_2$ which form a partition for $\bar{X} : \bar{X} = \bar{X}_1 \cup \bar{X}_2, \bar{X}_1 \cap \bar{X}_2 = \Phi$

We also consider the model of the following market phenomen:in their struggle for supremacy in taking hold of a certain commodity market, the deciders from C_1, intending to eliminate the deciders from C_2 which control the market, want in a first stage to take hold of at least one strategic point of the existing k in this market. Having one penetrated the commodity market, the deciders in C_1 will try the complete elimination of the deciders in C_2 by ruining them.

We interpret **the decision process of first stage** as a game made up of k-simultaneous periods. We assume that in this stage the capitals of the two coalitions from A and B, each banking unit of a decider from C_2 can ruin m_j monetary units of the A capital in the game $j, j = 1,...,k$.

Let D_1^j, D_2^j be the set of the strategies of C_1 and C_2 respectively, in the game $j, j = 1,...,k$ for any $d_1 = \left(d_1^1, d_1^2,..., d_1^k\right) \in \coprod_{j=1}^{k} D_1^j, d_2 = \left(d_2^1, d_2^2,..., d_2^k\right) \in \coprod_{j=1}^{k} D_2^j$. We shall have $d_1^j, d_2^j \geq 0, j = 1,...,k, \sum_{j=1}^{k} d_1^j = A, \sum_{j=1}^{k} d_2^j = B$.

We introduce the utility function $u : \coprod_{j=1}^{k} D_1^j x D_2^j \to R,$

$$u\left(d_1^1, d_1^2,..., d_1^k, d_2^1, d_2^2,..., d_2^k\right) = \sum_{j=1}^{k} \min\left(m_j d_2^j - d_1^j, 0\right).$$

Let us calculate the guaranteed optimum strategy for C_2 (maxmin strategy) as well as the maxmin value of the noncooperative game between C_1 and C_2. We will use the result ([3]):

$$V_1 = \max_{d_2} \min_{d_1} \left(\sum_{i=1}^{k} \min\left(m_i d_2^i - d_1^i, 0\right) \right) = \max_{d_2} \min_{d_1} \sum_{i=1}^{k} m_i d_2^i - d_1^i = \max_{d_2} \min_{d_1} \left(m_i d_2^i - A\right)$$

By introducing the partial utility function $\tilde{u}_i : D_1^i \to R, \ \tilde{u}_i\left(d_2^i\right) = m_i d_2^i - A$

we shall have $\tilde{u}_i(0) = -A = \tilde{u}_1(0)$ and hence it results (from equalization principle) that among the optimum strategy will be strategies of the form $\left(d_2^j, 0, ..., 0\right)$ so that $V_1 = \tilde{u}_j\left(d_2^j\right)$ (where j is determinated from the condition: $m_j d_2^j - A = \min_{1 \le i \le k}\left(m_i d_2^i - A\right)$.

It will results directly that the guaranteed optimum (simple) strategy for C_2 will be:

$$d_2^j = \frac{B}{m_j \sum_{i=1}^{k} \frac{1}{m_j}},$$

the maximum value beeing:

$$V_1 = \min\left(\frac{B}{\sum_{i=1}^{k} \frac{1}{m_i} - A}, 0 \right).$$

For the determining of the guaranteed optimum for C_1 (minmax strategy) as well as og the minmax value we shall first observe that the u efficiency function is concave in $d_2 = \left(d_2^1, d_2^2, ..., d_2^k\right)$ and so the V_1 maxmin value for C_2 will be equal to the value V of the game ([3],[4]): $V = V_1$. The minmax value is:

$$V_2 = \min\left\{ \min_{1 \le i \le k}\left(m_i B - A\right), 0 \right\}.$$

The minmax (mixed) strategy will be:

$$d_1^j = \frac{1}{m_j \sum_{i=1}^{k} \frac{1}{m_i}}, \ j = 1, ..., k$$

as for any $d_2 \in \prod_{i=1}^{k} D_2^i$ we have:

$$\sum_{i=1}^{k}\frac{1}{m_i\sum_{j=1}^{k}\frac{1}{m_j}}\max\left(A-m_i d_2^i,0\right)\geq \max\left(\sum_{i=1}^{k}\frac{A-m_i d_2^i}{m_i\sum_{j=1}^{k}\frac{1}{m_j}}\right)=\max\left(A-\frac{B}{\sum_{j=1}^{k}\frac{1}{m_j}},0\right)=$$

$$=\min\left(\frac{B}{\sum_{j=1}^{k}\frac{1}{m_j}-A},0\right)=V_1$$

Remark 3.1 As a result of the concavity of the u functional in relation to $d_2=\left(d_2^1,d_2^2,...,d_2^k\right)$, the value of the game between the two coalitions will be equal to V_2 and consequently a decision-making behavior for C_2 which is based on keeping decisions does not favor this coalition. It is very important for C_2 to obtain additional informations on the strategic behavior of C_1.

Remark 3.2 The optimum solution of C_1 consists in the concentration of the forces in a single game (in the j_0 game in which the condition $m_{j_0}=\min\limits_{1\leq j\leq k}\{m_j\}$ is realized), keeping the secret about the game in which it concentrates its forces. If C_2 has no information on C_1, it has to distribute its forces uniformly.

After the first stage, the remaining capital reserves beeing $A_1, B_1 \in N$, the second stage, **the ruining stage** proper, takes place as a particular sequencial process:

$X=\{a,b\}, a,b\in N, a+b=A_1+B_1; X_0=\{A_1,B_1\}; \bar{X}=\{A_1+B_1,0\}\cup\{0,A_1+B_1\}$.

If $x_n\in X, x_n=\{a_1^n,a_2^n\}$, we will have:

$$x_{n+1}=f_n\left(x_n,d_1^n,d_2^n\right)=\left(a_1^{n+1},a_2^{n+1}\right)$$

where:

$$\left(a_1^{n+1},a_2^{n+1}\right)\in\left\{\left(a_1^n+1,a_2^n-1\right),a_1^n-1,a_2^n+1\right\},\forall\left(d_1^n,d_2^n\right)\in D_1^n x D_2^n$$

Remark 3.3 The sequencial process described before consists on a series of null sum games, the loss of the game in the state x_n by a coalition means its having to concede to the winning colaition a monetary unit aut of the available capital.

At this stage ,there arises the problem of determining the mean duration of thje decision-making process as well as the probabilities of getting ruined for the two coalition if it is known that the probability of winning the game for the C_1 coalition in the state x_n is $p =$constant, $n \in N$. It results that $\left(a_1^n\right)_n$ is a homogeneous Markov chain with the state $0,1,2,...,C = A_1 + B_1$, and with the passing matrix:

$$M = \begin{pmatrix} 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & p & ... & 0 & 0 & 0 \\ - & - & - & - & - & - & - \\ 0 & 0 & 0 & ... & q & 0 & p \\ 0 & 0 & 0 & ... & 0 & 0 & 1 \end{pmatrix}, q = 1 - p$$

The potential matrix R is given by:

$$R = \left[I - \begin{pmatrix} 0 & p & 0 & ... & 0 & 0 & 0 \\ q & 0 & p & ... & 0 & 0 & 0 \\ - & - & - & - & - & - & - \\ 0 & 0 & 0 & ... & 0 & 0 & p \\ 0 & 0 & 0 & ... & 0 & q & 0 \end{pmatrix} \right]^{-1}$$

the elements $r(i, j)$ of this matrix beeing:

$$r(i, j) = \begin{cases} \dfrac{1}{(2p-1)\left[\left(\dfrac{p}{q}\right)^{A_1} - 1\right]} \begin{cases} \left[\left(\dfrac{p}{q}\right)^{j} - 1\right]\left[\left(\dfrac{p}{q}\right)^{A_1 - i} - 1\right], & j \le i \\ \left[\left(\dfrac{p}{q}\right)^{i} - 1\right]\left[\left(\dfrac{p}{q}\right)^{C-i} - \left(\dfrac{p}{q}\right)^{j-i}\right], & j > i \end{cases} & , p \ne \dfrac{1}{2} \\ \dfrac{2}{A_1} \begin{cases} j(C-i), & j \le i \\ i(C-j), & j > i \end{cases} & , p = \dfrac{1}{2} \end{cases}$$

The mean duration D_m of the decision-making process will be:

$$D_m = \sum_{l=1}^{c-1} r(A, l) = \begin{cases} \dfrac{1}{2p-1}\left[\dfrac{\left(\dfrac{p}{q}\right)^{C} - \left(\dfrac{p}{q}\right)^{B_1}}{\left(\dfrac{p}{q}\right)^{C} - 1} - A_1 \right] & , p \ne \dfrac{1}{2} \\ A_1 B_1, & p = \dfrac{1}{2} \end{cases}$$

The ruining probability of the C_1 coalition is given by $P_r^{C_1}$, ,where:

$$P_r^{C_1} = \begin{cases} \dfrac{\left(\dfrac{p}{q}\right)^{B_1} - 1}{\left(\dfrac{p}{q}\right)^{C} - 1}, & p \neq \dfrac{1}{2} \\ 1 - \dfrac{A_1}{C}, & p = \dfrac{1}{2} \end{cases}$$

and the runing probability of the C_2 coalition will be:

$$P_r^{C_2} = 1 - P_r^{C_1}$$

REFERENCES

[1] Mitran,I. – **Co-coperative and partial co-operative decisional models, monographs**, AMSE, France, 1990;
[2] Mitran,I. – **Optimalite minmax**, monographs, AMSE, France, 1992.
[3] Mitran, I.- **On the maximum probability**, Proceedings of the International Conference on Approximation and Optimization, ICAOR, Cluj-Napoca, vol. II, pp169-178

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