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A common fixed point theorem for four maps using a Lipschitz type condition

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Abstract

In this paper, we prove a general common fixed point theorem for two pairs of weakly compatible self-mappings of a (possibly non complete) metric space such that one of them satisfies the property (E.A) under a contractive condition of Lipschitz type. Our result provides a generalization and some improvements to a result obtained by K. Jha, R.P. Pant and S.L. Singh in 2003 and a recent result obtained by H. Bouhadjera and A. Djoudi in 2008.

1 Introduction

In metric fixed point theory, many papers were devoted to the study of common fixed points of four self-mappings of a metric space.

Let (X, d) be a metric space and let A, B, S and T be four self-mappings of (X, d).

To simplify notations, for all $x, y \in X$, we set

$$N(x,y) := \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), d(Sx,By), d(Ax,Ty)\},$$

$$m(x,y) := \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \frac{d(Sx,By)+d(Ax,Ty)}{2}\}$$

and

$$\sigma(x,y) := d(Sx,Ty) + d(Ax,Sx) + d(By,Ty) + d(Sx,By) + d(Ax,Ty).$$

Key Words: common fixed point for four mappings, noncompatible mappings, property (E.A), weakly compatible mappings, Meir-Keeler type contractive condition, contractive condition of Lipschitz type, metric spaces

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A Meir-Keeler type (ϵ, δ)-contractive condition for the mappings A, B, S and T may be given in the form:

given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \leq m(x, y) < \epsilon + \delta \Longrightarrow d(Ax, By) < \epsilon. \tag{1.1}$$

In connection to the Meir-Keeler type (ϵ, δ)-contractive condition, we consider the following two conditions:

given $\epsilon > 0$ there exists $\delta > 0$ such that for all x, y in X

$$\epsilon < m(x, y) < \epsilon + \delta \Longrightarrow d(Ax, By) \leq \epsilon, \tag{1.2}$$

and

$$d(Ax, By) < m(x, y), \quad \text{whenever} \quad m(x, y) > 0 \tag{1.3}$$

Jachymski [3] has shown that contractive condition (1.1) implies (1.2) but contractive condition (1.2) does not imply the contractive condition (1.1). Also, it is easy to see that the contractive condition (1.1) implies (1.3).

Condition (1.1) is not sufficient to ensure the existence of common fixed points of the maps A, B, S and T. Some kinds of commutativity or compatibility between the maps are always required. Also, other topological conditions on the maps or on their ranges are invoked.

Two self-mappings A and S of a metric space (X, d) are called compatible (see Jungck [6]) if,

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for some t in X.

This concept was frequently used to prove existence theorems in common fixed point theory.

In [4], K. Jha, R.P. Pant and S.L. Singh have established the following theorem.

Theorem 1.1. *([4]) Let (A, S) and (B, T) be two compatible pairs of self-mappings of a complete metric space (X, d) such that*

(i) $AX \subset TX$, $BX \subset SX$,

(ii) given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \leq m(x, y) < \epsilon + \delta \Longrightarrow d(Ax, By) < \epsilon, \quad \textit{and}$$

(iii) $d(Ax, By) < k\sigma(x, y)$ for all $x, y \in X$, for $0 \leq k \leq \frac{1}{3}$.

If one of the mappings A, B, S and T is continuous then A, B, S and T have a unique common fixed point.

Definition 1.1. *([7]). Two self mappings S and T of a metric space (X, d) are said to be weakly compatible if $Tu = Su$, for some $u \in X$, then $STu = TSu$.*

It is obvious that compatibility implies weak compatibility. Examples exist to show that the converse is not true.

In [2], Theorem 1.1 was generalized to the case of two pairs of weakly compatible maps by the following result.

Theorem 1.2. *([2]) Let (A, S) and (B, T) be two weakly compatible pairs of self-mappings of a complete metric space (X, d) such that*
(a) $AX \subseteq TX$ and $BX \subseteq SX$,
(b) one of AX, BX, SX or TX is closed,
(c) given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon < m(x, y) < \epsilon + \delta \Longrightarrow d(Ax, By) \leq \epsilon, \quad \text{and}$$

(c') $x, y \in X$, $m(x, y) > 0 \Longrightarrow d(Ax, By) < m(x, y)$,
(d) $d(Ax, By) \leq k[d(Sx, Ty)+d(Ax, Sx)+d(By, Ty)+d(Sx, By)+d(Ax, Ty)]$, for $0 \leq k < \frac{1}{3}$.
Then A, B, S and T have a unique common fixed point.

Other related results to these theorems are published in [11], [12] and [5].

The study on common fixed point theory for noncompatible mappings is also interesting. Work along these lines has been recently initiated by Pant [8], [9], [10].

In 2002, Aamri and Moutawakil [1] introduced a generalization of the concept of noncompatible mappings.

Definition 1.2. *Let S and T be two self mappings of a metric space (X, d). We say that S and T satisfy property (E.A) if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$.

Remark 1. *It is clear that two self-mappings of a metric space (X, d) will be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$ but

$$\lim_{n\to\infty} d(STx_n, TSx_n)$$

is either non-zero or not exists.

Therefore two noncompatible self-mappings of a metric space (X, d) satisfy property (E.A).

In this paper, we establish a common fixed point theorem for two weakly compatible pairs (A, S) and (B, T) of self-mappings of a (possibly non complete) metric space (X, d) such that one of them satisfies the property (E.A) under conditions which are weaker than the conditions (a), (b), (c), (c') and (d) used in Theorem 1.2.

Indeed, in the main result of this paper (see Theorem 2.1), we (can) drop the completeness of the whole metric space (X, d), we drop the condition (c), we replace the condition (c') by the condition

$$x, y \in X,\ N(x, y) > 0 \Longrightarrow d(Ax, By) < N(x, y),$$

which is weaker than (c') and keep (d) but with a Lipschitz constant k taking values in the interval $[0, \frac{1}{2})$ instead of the interval $[0, \frac{1}{3})$.

So, our main result provides a generalization and some improvements to the main results of [4] and [2].

2 Main result

Let (X, d) be a metric space. Let A, B, S and T be self-mappings of X. We recall the notations:

$$N(x, y) := \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ax, Ty)\}$$

and

$$\sigma(x, y) := d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) + d(Sx, By) + d(Ax, Ty).$$

The main result of this paper reads as follows.

Theorem 2.1. *Let (A, S) and (B, T) be two weakly compatible pairs of self-mappings of a metric space (X, d) such that*
$(H1)$: $AX \subseteq TX$ and $BX \subseteq SX$,
$(H2)$: one of AX, BX, SX or TX is a closed subspace of (X, d),
$(H3)$: $x, y \in X$, $N(x, y) > 0 \Longrightarrow d(Ax, By) < N(x, y)$, and
$(H4)$: $d(Ax, By) \le k\,\sigma(x, y)$, for all $x, y \in X$, where k is such that $0 \le k < \frac{1}{2}$.

If one of the pairs $\{A, S\}$ or $\{B, T\}$ satisfies the property (E.A), then A, B, S and T have a unique common fixed point.

Proof. (I) Suppose that the pair $\{A, S\}$ satisfies the property (E.A). Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \tag{2.1}$$

for some $z \in X$. Since $AX \subseteq TX$, then for each integer n, there exists y_n in X such that $Ax_n = Ty_n$. By using $(H4)$, we have

$$d(Ax_n, By_n) \leq k[d(Sx_n, Ty_n) + d(Ax_n, Sx_n) + d(By_n, Ty_n) + d(Sx_n, By_n) + d(Ax_n, Ty_n)],$$

which implies

$$d(Ax_n, By_n) \leq \frac{3k}{1-2k} d(Ax_n, Sx_n). \tag{2.2}$$

By letting n to infinity in (2.2), we obtain

$$\lim_{n\to\infty} d(Ax_n, By_n) = 0. \tag{2.3}$$

By (2.1) and (2.3), we get

$$z = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n. \tag{2.4}$$

(1) Suppose that $A(X)$ is a closed subspace of (X, d). Then $z \in A(X)$. Since $AX \subseteq TX$, then there exists $u \in X$ such that $z = Tu$. By (H 4), we get

$$d(Ax_n, Bu) \leq k[d(Sx_n, Tu) + d(Ax_n, Sx_n) + d(Bu, Tu) + d(Sx_n, Bu) + d(Ax_n, Tu)],$$

which, by letting $n \to \infty$, implies that

$$d(z, Bu) \leq 2kd(z, Bu). \tag{2.5}$$

Since $\leq k < \frac{1}{2}$, then it follows from (2.5) that $z = Bu$. Thus, we have $z = Tu = Bu$.

Since $B(X) \subset S(X)$, then there exists $v \in X$ such that $Bu = Sv$. Then $z = Tu = Bu = Sv$. By applying the inequality (H 4), we get

$$\begin{aligned} d(Av, Sv) &= d(Av, Bu) \\ &\leq k[d(Sv, Tu) + d(Av, Sv) + d(Bu, Tu) + d(Sv, Bu) + d(Av, Tu)] \\ &= 2kd(Av, Sv), \end{aligned}$$

which implies that $Av = Sv$. Hence, we obtain

$$z = Tu = Bu = Sv = Av. \tag{2.6}$$

The conclusions in (2.6) will be obtained by similar arguments, if we suppose that $T(X)$, $B(X)$ or $S(X)$ is a closed subspace of X.

(2) Since $\{A, S\}$ and $\{B, T\}$ are weakly compatible, it follows

$$Bz = Tz \quad \text{and} \quad Az = Sz \tag{2.7}$$

Now, we show that $z = Az$. To get a contradiction, let us suppose the contrary. Then we have

$$N(z,u) = \max\{d(Sz,Tu), d(Az,Sz), d(Bu,Tu), d(Sz,Bu), d(Az,Tu)\} = \\ = d(Az,z) > 0.$$

So, by virtue of the assumption (H 3), we get

$$d(Az,z) = d(Az,Bu) < N(z,u) = d(Az,z),$$

which is a contradiction. Thus we get $z = Az$. Hence, we obtain $z = Az = Sz$.
Now, we show that $z = Bz$. To get a contradiction, let us suppose the contrary. Then we have

$$N(z,z) = \max\{d(Sz,Tz), d(Az,Sz), d(Bz,Tz), d(Sz,Bz), d(Az,Tz)\} = \\ = d(Bz,z) > 0.$$

By virtue of the assumption (H 3), we get

$$d(z,Bz) = d(Av,Bz) < N(z,z) = d(Bz,z),$$

which is a contradiction. Thus we get $z = Bz = Tz$. Hence, we have

$$z = Bz = Tz = Az = Sz.$$

We conclude that z is a common fixed point for A, B, S and T.

(II) If we suppose that the pair $\{B, T\}$ satisfies the property (E.A), then by similar arguments we obtain the same conclusions as in the part (I). So, in all cases, the mappings A, B, S and T have at least a common fixed point z in X.

(III) It remains to show the uniqueness of the fixed common fixed point z. Suppose that w is another common fixed point for the mappings A, B, S

and T, such that $w \neq z$. Obviously we have $N(w,z) = d(w,z) > 0$. Then, by applying the condition (H 3), we obtain

$$d(w,z) = d(Aw, Bz) < N(w,z) = d(w,z),$$

which is a contradiction. So the mappings A, B, S and T have a unique common fixed point. This completes the proof. □

As a consequence, we have the following corollary.

Corollary 2.1. *Let (A,S) and (B,T) be two weakly compatible pairs of self-mappings of a metric space (X,d) such that*
$(H1)$: $AX \subseteq TX$ and $BX \subseteq SX$,
$(H2)$: one of AX, BX, SX or TX is a closed subspace of (X,d),
$(H3)$: $x, y \in X$, $N(x,y) > 0 \Longrightarrow d(Ax, By) < N(x,y)$, and
$(H4)$: $d(Ax,By) \leq k\,\sigma(x,y)$, for all $x, y \in X$, where k is such that $0 \leq k < \frac{1}{2}$.

If one of the following two conditions is satisfied.
(i) A and S are noncompatible, or
(ii) B and T are noncompatible.

Then the mappings A, B, S and T have a unique common fixed point.

We observe that in Theorem 2.1, we do not need the completeness of the whole space (X,d).

When the space (X,d) is complete then we have the following corollaries.

Corollary 2.2. *Let (A,S) and (B,T) be two compatible pairs of self-mappings of a complete metric space (X,d) such that*
(i) $AX \subset TX$, $BX \subset SX$,
(ii) one of AX, BX, SX or TX is closed,
(iii) given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \leq m(x,y) < \epsilon + \delta \Longrightarrow d(Ax,By) < \epsilon, \quad \textit{and}$$

(iv) $d(Ax,By) \leq k\sigma(x,y)$ for all $x, y \in X$, for $0 \leq k < \frac{1}{2}$.
Then the mappings A, B, S and T have a unique common fixed point.

Corollary 2.3. *Let (A,S) and (B,T) be two weakly compatible pairs of self-mappings of a complete metric space (X,d) such that*
(a) $AX \subseteq TX$ and $BX \subseteq SX$,
(b) one of AX, BX, SX or TX is closed,
(c) given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon < m(x,y) < \epsilon + \delta \Longrightarrow d(Ax,By) \leq \epsilon,$$

(c') $x, y \in X$, $m(x,y) > 0 \Longrightarrow d(Ax, By) < m(x,y)$, and

(d) $d(Ax, By) \leq k[d(Sx,Ty)+d(Ax,Sx)+d(By,Ty)+d(Sx,By)+d(Ax,Ty)]$, for $0 \leq k < \frac{1}{2}$.

Then A, B, S and T have a unique common fixed point.

To see that Corollary 2.2 and Corollary 2.3 are consequences of Theorem 2.1, we need to recall the following lemma which is proved by Jachymski in [3].

Lemma 2.1. *(2.2 of [3]): Let A, B, S and T be self mappings of a metric space (X, d) such that $AX \subset TX$, $BX \subset SX$. Assume further that given $\epsilon > 0$ there exists $\delta > 0$ such that for all x, y in X*

$$\epsilon < m(x,y) < \epsilon + \delta \Longrightarrow d(Ax, By) \leq \epsilon, \tag{c}$$

and

$$d(Ax, By) < m(x,y), \quad \textit{whenever} \quad m(x,y) > 0 \tag{c'}$$

Then for each x_0 in X, the sequence $\{y_n\}$ in X defined by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \quad \forall n \in \mathbb{N}$$

is a Cauchy sequence.

Proofs. To prove Corollary 2.2 and Corollary 2.3, let x_0 be an arbitrary point in X. Since $AX \subseteq TX$ and $BX \subseteq SX$, we can define inductively two sequences $\{x_n\}$ and $\{y_n\}$ in X by the rule:

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}, \tag{2.8}$$

for each nonnegative integer n. By Lemma 2.1, it follows that the sequence $\{y_n\}$ is a Cauchy sequence. Since (X, d) is complete, then there exists a point (say) z in X such that

$$z = \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} \tag{2.9}$$

Since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, then by (2.8) and (2.9) it follows that we have

$$z = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Bx_{2n+1}. \tag{2.10}$$

So the pairs $\{A, S\}$ and $\{B, T\}$ enjoy the property (E.A). Since for all $x, y \in X$, we have

$$m(x,y) > 0 \Longleftrightarrow N(x,y) > 0 \quad \text{and} \quad m(x,y) \leq N(x,y),$$

then the conditions of Corollary 2.2 (resp. Corollary 2.3) imply the conditions $(H1)$, $(H2)$, $(H3)$ and $(H4)$ of Theorem 2.1. It follows that the conclusions of Corollaries 2.2 and 2.3 are obtained by application of Theorem 2.1.

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