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GALOIS-TYPE CONNECTIONS AND CLOSURE OPERATIONS ON PREORDERED SETS

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Abstract. For a function f of one preordered set X to another Y, we shall establish several consequences of the following two definitions:

(a) f is increasingly φ–regular, for some function φ of X to itself, if for any $x_1, x_2 \in X$ we have $x_1 \leq \varphi(x_2)$ if and only if $f(x_1) \leq f(x_2)$;

(b) f is increasingly g–normal, for some function g of Y to X, if for any $x \in X$ and $y \in Y$ we have $f(x) \leq y$ if and only if $x \leq g(y)$.

These definitions have been mainly suggested to us by a recent theory of relators (families of relations) worked out by Á. Száz and G. Pataki and the extensive literature on Galois connections and residuated mappings.

Introduction

In this paper, for a function f of one preordered set X to another Y, we shall establish several consequences of the following two definitions:

(a) f is *increasingly φ-regular*, for some function φ of X to itself, if for any $x_1, x_2 \in X$ we have $x_1 \leq \varphi(x_2)$ if and only if $f(x_1) \leq f(x_2)$;

(b) f is *increasingly g-normal*, for some function g of Y to X, if for any $x \in X$ and $y \in Y$ we have $f(x) \leq y$ if and only if $x \leq g(y)$.

These definitions have been mainly suggested to us by a recent theory of relators (families of binary relations) worked out by Száz [**28**] and Pataki [**20**] and the extensive literature on Galois connections [**1**, p. 124] and residuated mappings [**2**, p. 11].

For instance, we shall show that if f is an increasingly g-normal function of X to Y, then

(1) f is increasing;

(2) $f[\sup(A)] \subset \sup(f[A])$ for all $A \subset X$;

(3) $g(y) \in \max\{x \in X : f(x) \leq y\}$ for all $y \in Y$.

Moreover, if $\varphi = g \circ f$, then

(4) f is increasingly φ-regular;

(5) φ is a closure operation on X;

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(6) $\varphi(x) \in \max\{\, u \in X : f(u) \leq f(x) \,\}$ for all $x \in X$.

On the other hand, as a partial converse to (4), we shall also show that if f is an increasingly φ–regular function of X onto Y, then there exists a function g of Y to X such that f is increasingly g-normal. Moreover, if in particular Y is partially ordered, then g is injective. Therefore, in spite of (4), the increasingly normal functions are more general objects than the increasingly regular ones. However, the latter ones, being more closely to closure operations, are sometimes more convenient.

The results obtained extend and supplement some basic theorems on Galois connections and residuated mappings. Moreover, they can be immediately applied to almost all topological and order theoretic structures derived from relators in [**26**], [**27**] and [**32**].

For instance, if

$$F(\mathcal{R}) = \operatorname{int}_{\mathcal{R}} = \{(A,\, x) \in \mathcal{P}(X) \times X :\ \exists\, R \in \mathcal{R} : R(x) \subset A\,\}$$

for any relator $\mathcal{R}$ on X and

$$G(\operatorname{int}) = \mathcal{R}_{\operatorname{int}} = \left\{ S \subset X^2 :\ \operatorname{int}_S \subset \operatorname{int} \right\}$$

for any relation int on $\mathcal{P}(X)$ to X, then we can see that F is an increasingly G-normal function of $\mathcal{P}^2\left(X^2\right)$ to $\mathcal{P}\left(\mathcal{P}(X) \times X\right)$. Moreover,

$$\Phi(\mathcal{R}) = (G \circ F)(\mathcal{R}) = \mathcal{R}^{\wedge} = \left\{ S \subset X^2 :\ \forall x \in X : x \in \operatorname{int}_{\mathcal{R}}\left(S(x)\right) \right\}.$$

Thus, $\wedge$ is a closure operation on $\mathcal{P}^2\left(X^2\right)$. Moreover, for any relator $\mathcal{R}$ on X, $\mathcal{R}^{\wedge}$ is the largest relator on X such that $\operatorname{int}_{\mathcal{R}^{\wedge}} \subset \operatorname{int}_{\mathcal{R}}$ (resp. $\operatorname{int}_{\mathcal{R}^{\wedge}} = \operatorname{int}_{\mathcal{R}}$).

However, if for instance

$$F(\mathcal{R}) = \mathcal{T}_{\mathcal{R}} = \{A \subset X :\ A \subset \operatorname{int}_{\mathcal{R}}(A)\}$$

for any relator $\mathcal{R}$ on X, then by [**15**, Example 5.3] there does not, in general, exist a largest relator $\mathcal{R}^{\square}$ on X such that $\mathcal{T}_{\mathcal{R}^{\square}} \subset \mathcal{T}_{\mathcal{R}}$ (resp. $\mathcal{T}_{\mathcal{R}^{\square}} = \mathcal{T}_{\mathcal{R}}$). Thus, the function F is not even regular. Therefore, it is rather curious that topology and analysis have been mostly based on open sets.

In this respect, it is also worth mentioning that all reasonable generalizations of the usual topological structures (such as proximities, closures, topologies, filters and convergences) can be easily derived from relators (according to the results of [**27**] and [**25**]). Thus, they need not be studied separately.

Moreover, the various operations on relators can be used to put the basic concepts of topology and analysis in a proper perspective. For instance, it has become completely clear that compactness and connectedness are particular cases of precompactness and well-chainedness, respectively. (See [**29**] and [**21**].) Moreover, several continuity properties of relations can be briefly expressed in terms of these operations and compositions of relations and relators. (See [**30**] and [**38**].)

1. A few basic facts on relations

A subset F of a product set $X \times Y$ is called a *relation on X to Y*. If in particular $F \subset X^2$, then we may simply say that F is a relation on X. Thus, $\Delta_X = \{(x,x) : x \in X\}$ is a relation on X.

If F is a relation on X to Y, then for any $x \in X$ the set $F(x) = \{y \in Y : (x,y) \in F\}$ is called the *image of x under F*. And the set $D_F = \{x \in X : F(x) \neq \emptyset\}$ is called the *domain of F*.

In particular, a relation F on X to Y is called a *function* if for each $x \in D_F$ there exists $y \in Y$ such that $F(x) = \{y\}$. In this case, by identifying singletons with their elements, we may usually write $F(x) = y$ in place of $F(x) = \{y\}$.

More generally, if F is a relation on X to Y, then for any $A \subset X$ the set $F[A] = \bigcup_{x \in A} F(x)$ is called the image of A under F. And the set $R_F = F[D_F]$ is called the range of F.

If F is a relation on X to Y such that $D_F = X$, then we say that F is a *relation of X to Y*. While, if F is a relation X to Y such that $R_F = Y$, then we say that F is a *relation on X onto Y*.

If F is a relation on X to Y, then a function f of D_F to Y is called a *selection of F* if $f \subset F$, i.e., $f(x) \in F(x)$ for all $x \in D_F$. Thus, the Axiom of Choice can be briefly expressed by saying that every relation has a selection.

If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse F^{-1} of F can be defined such that $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for all $y \in Y$.

Moreover, if F is a relation on X to Y and G is a relation on Y to Z, then the composition $G \circ F$ of G and F can be defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G[F[A]]$ for all $A \subset X$.

A relation R on X is called *reflexive, symmetric*, *antisymmetric*, and *transitive* if $\Delta_X \subset R$, $R^{-1} \subset R$, $R \cap R^{-1} \subset \Delta_X$, and $R \circ R \subset R$, respectively. Now, a reflexive relation may be called a *tolerance* (*preorder*) if it is symmetric (transitive).

If R is a relation on X, then we write $R^n = R \circ R^{n-1}$ for all $n \in \mathbb{N}$ by agreeing that $R^0 = \Delta_X$. Moreover, we also write $R^\infty = \bigcup_{n=0}^\infty R^n$. Thus, R^∞ is the smallest preorder on X such that $R \subset R^\infty$. Therefore, $R^{\infty\infty} = R^\infty$.

A family $\mathcal{R}$ of relations on one nonvoid set X to another Y is called a *relator on X to Y*. Moreover, the ordered pair $(X,Y)(\mathcal{R}) = ((X,Y),\mathcal{R})$ is called a *relator space*. (For the origins, see [**26**] and the references therein.)

If in particular $\mathcal{R}$ is a relator on X to itself, then we may simply say that $\mathcal{R}$ is a *relator on X*. Moreover, by identifying singletons with their element, we may naturally write $X(\mathcal{R})$ in place of $(X,X)(\mathcal{R})$. Namely, $(X,X) = \{\{X\}\}$.

A relator $\mathcal{R}$ on X to Y is called *simple* if $\mathcal{R} = \{R\}$ for some relation R on X to Y. In this case, we may simply write $(X,Y)(R)$ in place of $(X,Y)(\mathcal{R}) = (X,Y)(\{R\})$.

Ordered sets and formal contexts [**11**, p. 17] are simple relator spaces. While, uniform spaces [**10**] are non-simple relator spaces. Note that the ordinary uniformities are tolerance relators. While, topologies and filters correspond to preorder relators by [**40**].

2. A few basic facts on ordered sets

If $\leq$ is a relation on a nonvoid set X, then having in mind the terminology of Birkhoff [**1**, p. 2] the simple relator space $X(\leq)$ is called a *goset* (*generalized ordered* set). And we usually write X in place of $X(\leq)$.

If $X(\leq)$ is a goset, then by taking $X^* = X$ and $\leq^* = \leq^{-1}$ we can form a new goset $X^*(\leq^*)$. This is called the *dual of* $X(\leq)$. And we usually write $\geq$ in place of $\leq^*$.

The goset X is called *reflexive*, *transitive*, and *antisymmetric* if the inequality relation $\leq$ in it has the corresponding property. Moreover, for instance, X is called *preordered* if it is reflexive and transitive.

In particular, a preordered set will be called a *proset* and a partially ordered set will be called a *poset*. The usual definitions on posets can be naturally extended to gosets [**33**] (or even to arbitrary relator spaces [**32**]).

For instance, for any subset A of a goset X, the members of the families

$$\mathrm{lb}(A) = \{x \in X : \forall a \in A : x \leq a\}$$

and

$$\mathrm{ub}(A) = \{x \in X : \forall a \in A : a \leq x\}$$

are called the *lower* and *upper bounds of* A *in* X, respectively.

Moreover, the members of the families

$$\min(A) = A \cap \mathrm{lb}(A), \qquad \max(A) = A \cap \mathrm{ub}(A),$$
$$\inf(A) = \max(\mathrm{lb}(A)), \qquad \sup(A) = \min(\mathrm{ub}(A)),$$

are called the *minima, maxima, infima* and *suprema of* A *in* X, respectively.

Thus, we can can easily prove the following theorems.

Theorem 2.1. *If* $A \subset X$*, then*

(1) $\mathrm{lb}(A) = \bigcap_{a \in A} \mathrm{lb}(a)$*;* (2) $\mathrm{ub}(A) = \bigcap_{a \in A} \mathrm{ub}(a)$.

Remark 2.2. Hence, it is clear

(1) $\mathrm{lb}(\emptyset) = X$ and $\mathrm{ub}(\emptyset) = X$;

(2) $\mathrm{lb}(B) \subset \mathrm{lb}(A)$ and $\mathrm{ub}(B) \subset \mathrm{ub}(A)$ for all $A \subset B \subset X$.

Theorem 2.3. *For any* $A, B \subset X$*, we have*

$$B \subset \mathrm{ub}(A) \iff A \subset \mathrm{lb}(B).$$

Remark 2.4. Hence, it will follow that

(1) $\mathrm{lb}(A) = \mathrm{lb}(\mathrm{ub}(\mathrm{lb}(A)))$; (2) $\mathrm{ub}(A) = \mathrm{ub}(\mathrm{lb}(\mathrm{ub}(A)))$.

Theorem 2.5. *If* $A \subset X$*, then*

(1) $\min(A) = \{x \in A : A \subset \mathrm{ub}(x)\}$*;*

(2) $\max(A) = \{x \in A : A \subset \mathrm{lb}(x)\}$.

Remark 2.6. By this theorem, for any $A \subset X$, we may also naturally define

$$\mathrm{lb}^*(A) = \{x \in X : A \cap \mathrm{lb}(x) \subset \mathrm{ub}(x)\} .$$

Thus, $\min^*(A) = A \cap \mathrm{lb}^*(A)$ is just the family of all minimal elements of A.

Theorem 2.7. *If $A \subset X$, then*

(1) $\inf(A) = \mathrm{lb}(A) \cap \mathrm{ub}\,(\mathrm{lb}(A))$; (2) $\sup(A) = \mathrm{ub}(A) \cap \mathrm{lb}\,(\mathrm{ub}(A))$.

Theorem 2.8. *If $A \subset X$, then*

(1) $\inf(A) = \sup\,(\mathrm{lb}(A))$; (2) $\sup(A) = \inf\,(\mathrm{ub}(A))$;

(3) $\min(A) = A \cap \inf(A)$; (4) $\max(A) = A \cap \sup(A)$.

Theorem 2.9. *If X is reflexive, then the following assertions are equivalent:*

(1) *X is antisymmetric:*

(2) $\mathrm{card}\,(\min(A)) \leq 1$ *for all $A \subset X$;*

(3) $\mathrm{card}\,(\max(A)) \leq 1$ *for all $A \subset X$.*

Remark 2.10. In [**34**], by taking $\mathcal{L} = \{A \subset X : A \subset \mathrm{lb}(A)\}$, we have first proved that (1) holds if and only if $\mathrm{card}(A) \leq 1$ for all $A \in \mathcal{L}$.

Moreover, we have observed that, because of Theorem 2.8, we may write infimum and supremum instead of minimum and maximum in the above theorem.

3. Increasingly regular and normal structures

Definition 3.1. A function φ of a proset X to itself will be called a *unary operation* on X.

More generally, a function f of one proset X to another Y will be called a *structure on X*.

Remark 3.2. This terminology and the following definitions have been mainly motivated by the various structures derived from relators and their induced operations. (See [**28**] and [**20**].)

A structure f on X to Y may be called *increasing* if for any $x_1, x_2 \in X$, with $x_1 \leq x_2$, we have $f(x_1) \leq f(x_2)$. Moreover, f may be called *decreasing* if it is increasing as a structure on X to Y^*.

Now, somewhat differently, we shall introduce the following

Definition 3.3. A structure f on X will be called *increasingly φ-regular*, for some operation φ on X, if for any $x_1, x_2 \in X$ we have

$$x_1 \leq \varphi(x_2) \iff f(x_1) \leq f(x_2).$$

Remark 3.4. Now, a structure f on X to Y may be naturally called *decreasingly φ-regular* if it is an increasingly φ-regular structure on X to Y^*. That is, for any $x_1, x_2 \in X$, we have $x_1 \leq \varphi(x_2)$ if and only if $f(x_2) \leq f(x_1)$.

The above definition closely resembles to a brief characterization of Galois connections established by Schmidt [**23**, p. 205]. (See also Pickert [**22**] and Davey and Pristley [**6**, p. 155].)

However, instead of Galois connections, it is now more convenient to use the residuated mappings of Derdérian [**7**] and Blyth and Janowitz [**2**, p. 11] in the following modified form.

Definition 3.5. A structure f on X to Y will be called *increasingly g-normal*, for some structure g on Y to X, if for any $x \in X$ and $y \in Y$ we have

$$f(x) \leq y \iff x \leq g(y).$$

Remark 3.6. Now, a structure f on X to Y may be naturally called decreasingly g-normal if it is an *increasingly g-normal* structure on X to Y^*. That is, for any $x \in X$ and $y \in Y$, we have $y \leq f(x)$ if and only if $x \leq g(y)$.

To establish the relationships between increasingly regular and normal structures, we first prove the following

Theorem 3.7. *If f is an increasingly g-normal structure on X to Y and φ is an operation on X such that $\varphi \leq g \circ f \leq \varphi$, then f is increasingly φ-regular.*

Proof. By using above inequalities, the transitivity of X, and Definition 3.5, we can easily see that for any $x_1, x_2 \in X$, we have

$$\begin{aligned} x_1 \leq \varphi(x_2) &\iff x_1 \leq (g \circ f)(x_2) \\ &\iff x_1 \leq g\left(f(x_2)\right) \iff f(x_1) \leq f(x_2). \end{aligned}$$

Therefore, f is increasingly φ-regular. □

Now, as an immediate consequence of the above theorem and the reflexivity of X, we can also state

Corollary 3.8. *If f is an increasingly g-normal structure on X to Y, then $\varphi = g \circ f$ is an operation on X such that f is increasingly φ-regular.*

Remark 3.9. Hence, it is clear that several properties of increasingly normal structures can be immediately derived from those of the increasingly regular ones.

However, the following partial converse to Theorem 3.7 indicates that the increasingly normal structures are still more general objects than the increasingly regular ones.

Theorem 3.10. *If f is an increasingly φ-regular structure on X onto Y and g is a structure on Y to X such that $\varphi \leq g \circ f \leq \varphi$, then f is increasingly g-normal.*

Proof. Suppose that $x \in X$ and $y \in Y$. Then, since $Y = f[X]$, there exists $u \in X$ such that $y = f(u)$. Hence, by Definition 3.3, the above inequalities and the transitivity of X, it is clear that

$$\begin{aligned} f(x) \leq y \iff f(x) \leq f(u) &\iff x \leq \varphi(u) \\ \iff x \leq (g \circ f)(u)) &\iff x \leq g\left(f(u)\right) \iff x \leq g(y). \end{aligned}$$

Therefore, f is an increasingly g-normal. □

Now, as an immediate consequence of the above theorem and the reflexivity of X, we can also state

Corollary 3.11. *If f is an increasingly φ-regular structure on X onto Y and g is a structure on Y to X such $\varphi = g \circ f$, then f is increasingly g-normal.*

Remark 3.12. Note that Definitions 3.3 and 3.5 and Corollaries 3.8 and 3.11 do not need the assumed reflexivity and transitivity of X and Y.

In the sequel, we shall also need the following

Theorem 3.13. *If f is an increasingly g-normal structure on X to Y, then g is an increasingly f–normal structure on Y^* to X^*.*

Proof. By the corresponding definitions, for any $y \in Y$ and $x \in X$, we have

$$g(y) \leq^* x \iff x \leq g(y) \iff f(x) \leq y \iff y \leq^* f(x).$$

Therefore, the required assertion is true. □

Remark 3.14. From the above theorem or its proof, it is clear that the converse of Theorem 3.13 is also true.

For a preliminary illustration of the above notions and the forthcoming results, we shall only mention here the following straightforward extension of Theorem 2.3.

Example 3.15. Let R be a relation on one set X to another Y. For any $A \subset X$ and $B \subset Y$, define

$$F(A) = \mathrm{ub}_R(A) = \{y \in Y : \forall x \in A : xRy\}$$

and

$$G(A) = \mathrm{lb}_R(B) = \{x \in X : \forall y \in B : xRy\}.$$

Then, it can be easily seen that F is a decreasingly G–normal structure on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$.

Remark 3.16. The above construction was first considered by Birkhoff [**1**, p. 122] in 1940 under the name polarities. (For some further studies, see also Ore [**19**] and Everett [**9**].)

4. Closure operations and increasingly regular structures

Definition 4.1. An operation φ on a proset X will be called

(1) *expansive* if $\Delta_X \leq \varphi$; (2) *semi-idempotent* if $\varphi^2 \leq \varphi$.

Remark 4.2. Note that if (1) holds, then we also have $\varphi = \Delta_X \circ \varphi \leq \varphi \circ \varphi = \varphi^2$. Therefore, if both (1) and (2) hold and X is a poset, then φ is actually idempotent in the sense that $\varphi^2 = \varphi$.

Now, as a straightforward extension of the corresponding definition of [**2**, p. 111], we may also have the following

Definition 4.3. An increasing and expansive operation will be called a *preclosure operation*. And a semi-idempotent preclosure operation will be called a *closure operation*.

Moreover, an expansive and semi-idempotent operation will be called a *semi-closure operation*. And an increasing and idempotent operation will be called a *modification operation*.

Remark 4.4. Now, an operation φ on X may be naturally called an *interior operation* on X if it is a closure operation on X^*. That is, it is increasing, $\varphi \leq \Delta_X$ and $\varphi \leq \varphi^2$.

To feel the importance of modification operations, note that if

$$\Phi(\mathcal{R}) = \mathcal{R}^\infty = \{R^\infty : R \in \mathcal{R}\} \quad \text{and} \quad \Psi(\mathcal{R}) = \mathcal{R}^\partial = \big\{S \subset X^2 : S^\infty \in \mathcal{R}\big\}$$

for any relator $\mathcal{R}$ on a set X, then Φ and Ψ are only modification operations on the poset $\mathcal{P}^2(X^2)$.

We can also note that Φ is an increasingly Ψ-normal structure on $\mathcal{P}^2(X^2)$ to itself. Moreover, by [16], $\mathcal{R}^{\wedge\infty}$ is the largest preorder relator on X such that $\mathcal{T}_{\mathcal{R}^{\wedge\infty}} \subset \mathcal{T}_{\mathcal{R}}$ (resp. $\mathcal{T}_{\mathcal{R}^{\wedge\infty}} = \mathcal{T}_{\mathcal{R}}$).

From Corollaries 3.8 and 3.11, it is clear that increasingly regular structures are also closely related to closure operations. To easily establish this relationship, we shall start with the following

Theorem 4.5. *If f is an increasingly φ-regular structure on X to Y, then*

(1) *φ is expansive;* (2) *f is increasing;* (3) *$f \leq f \circ \varphi \leq f$.*

Proof. If $x \in X$, then by the reflexivities of Y and X we have $f(x) \leq f(x)$ and $\varphi(x) \leq \varphi(x)$. Hence, by using the assumed regularity of f, we can infer that $x \leq \varphi(x)$ and $f(\varphi(x)) \leq f(x)$. Therefore, (1) and the second part of (3) are true.

Moreover, if $x_1, x_2 \in X$ such that $x_1 \leq x_2$, then by the inequality $x_2 \leq \varphi(x_2)$ and the transitivity of X we also have $x_1 \leq \varphi(x_2)$. Hence, by using the assumed regularity of f, we can infer that $f(x_1) \leq f(x_2)$. Therefore, (2) is also true. Now, if $x \in X$, then from the inequality $x \leq \varphi(x)$, we can also see that $f(x) \leq f(\varphi(x))$. Therefore, the first part of (3) is also true. □

Now, as an immediate consequence of assertion (3), we can also state

Corollary 4.6. *If f is an increasingly φ-regular structure on X to a poset Y, then $f = f \circ \varphi$.*

Moreover, as a straightforward extension of [**20**, Theorem 1.9], we can also prove the following

Theorem 4.7. *If φ is an operation on X, then the following assertions are equivalent:*

(1) *φ is a closure operation;*

(2) *φ is increasingly φ-regular;*

(3) *there exists an increasingly φ-regular structure f on X.*

Proof. Suppose that (1) holds and $x_1, x_2 \in X$. If $x_1 \leq \varphi(x_2)$, then by the increasingness of φ we also have $\varphi(x_1) \leq \varphi(\varphi(x_2))$. Moreover, by the semi-idempotency of φ, we also have $\varphi(\varphi(x_2)) \leq \varphi(x_2)$. Hence, by the transitivity of X, it follows that $\varphi(x_1) \leq \varphi(x_2)$.

On the other hand, by the expansivity of φ, we have $x_1 \leq \varphi(x_1)$. Therefore, if $\varphi(x_1) \leq \varphi(x_2)$, then by the transitivity of X we also have $x_1 \leq \varphi(x_2)$. Thus, (2) also holds. Moreover, if (2) holds, then by taking $f = \varphi$ we can at once see that (3) also holds.

Therefore, to complete the proof, we need only show that (3) also implies (1). For this, assume that f is an increasingly φ-regular structure on X to another proset Y. Then, by Theorem 4.5, the operation φ is expansive and $f \circ \varphi \leq f$. Hence, we can infer that $f \circ \varphi^2 = f \circ \varphi \circ \varphi \leq f \circ \varphi$. Therefore, by the transitivity of Y, we also have $f \circ \varphi^2 \leq f$. Thus, for any $x \in X$, we have $f\left(\varphi^2(x)\right) \leq f(x)$. Hence, by using the assumed regularity of f, we can infer that $\varphi^2(x) \leq \varphi(x)$. Therefore, $\varphi^2 \leq \varphi$, and thus φ is semi-idempotent.

Finally, if $x_1, x_2 \in X$ such that $x_1 \leq x_2$, then by Theorem 4.5 we have $f(x_1) \leq f(x_2)$. Moreover, by Theorem 4.5, we also have $f(\varphi(x_1)) \leq f(x_1)$. Hence, by the transitivity of Y, it follows that $f(\varphi(x_1)) \leq f(x_2)$. Hence, by using the assumed regularity of f, we can already infer that $\varphi(x_1) \leq \varphi(x_2)$. Therefore, φ is increasing, and thus (1) also holds. □

Remark 4.8. According to Erné [**8**, p. 50], the origins of (2) go back to R. Dedekind. For similar observations, see also Everett [**9**] and Meyer and Nieger [**17**, p. 343].

A simple application of Theorem 4.7 yields the following characterization of increasingly regular structures.

Corollary 4.9. *If f is a structure and φ is an operation on X, then the following assertions are equivalent:*

(1) *f is increasingly φ-regular;*

(2) *φ is a closure operation and, for any $x_1, x_2 \in X$, we have*

$$\varphi(x_1) \leq \varphi(x_2) \iff f(x_1) \leq f(x_2).$$

Proof. From Theorem 4.7, we know that if (1) holds, then φ is a closure operation. Moreover, if φ is a closure operation, then for any $x_1, x_2 \in X$ we have

$$x_1 \leq \varphi(x_2) \iff \varphi(x_1) \leq \varphi(x_2).$$

Therefore, by Definition 3.3, assertions (1) and (2) are equivalent. □

Finally, we note that from Theorem 4.5 and 4.7, by using Theorems 3.7 and 3.10, we can easily derive some theorems on increasingly normal structures.

Theorem 4.10. *If f is an increasingly g-normal structure on X to Y, then f is increasing. Moreover, if φ is an operation on X such that $\varphi \leq g \circ f \leq \varphi$, then φ is a closure operation on X such that $f \leq f \circ \varphi \leq f$.*

Proof. By Theorem 3.7, the structure f is increasingly φ-regular, and thus by Theorems 4.5 and 4.7 the required assertions are true. $\square$

From the above theorem, by Theorem 3.13, it is clear that we also have

Corollary 4.11. *If f is an increasingly g-normal structure on X to Y, then g is increasing. Moreover, if ψ is an operation on Y such that $\psi \leq f \circ g \leq \psi$, then ψ is an interior operation on Y such that $g \leq g \circ \psi \leq g$.*

Remark 4.12. Note that if in particular X and Y are posets, then we may write equalities instead of the above inequalities.

Now, as a slight extension of [**6**, Lemma 7.26, p. 159], we can also prove the following counterpart of Corollary 4.9.

Theorem 4.13. *For any structures f on X to Y and g on Y to X, the following assertions are equivalent:*

(1) *f is increasingly g-normal;*

(2) *f and g are increasing and $\Delta_X \leq g \circ f$ and $f \circ g \leq \Delta_Y$.*

Proof. If (1) holds, then by Theorem 4.10 and Corollary 4.11 the structures f and g are increasing. Moreover, $g \circ f$ is a closure operation on X and $f \circ g$ is an interior operation on Y. Thus, in particular $\Delta_X \leq g \circ f$ and $f \circ g \leq \Delta_Y$. Therefore, (2) also holds.

Suppose now that (2) holds and $x \in X$ and $y \in Y$. If $f(x) \leq y$, then by the increasingness of g we also have $g\left(f(x)\right) \leq g(y)$. Hence, since $x = \Delta_X(x) \leq (g \circ f)(x) = g\left(f(x)\right)$, it is clear that $x \leq g(y)$. Conversely, if $x \leq g(y)$, then by the increasingness of f we also have $f(x) \leq f\left(g(y)\right)$. Hence, since $f\left(g(y)\right) = (f \circ g)(y) \leq \Delta_Y(y) = y$, it is clear that $f(x) \leq y$. Therefore, (1) also holds. $\square$

Remark 4.14. This theorem shows that increasingly normal structures are also natural generalizations of residuated mappings [**2**, p. 11].

5. Characterizations of increasingly normal structures

Definition 5.1. For any structure f on one proset X to another Y, we define two relations Γ_f and g_f on Y to X such that

$$\Gamma_f(y) = \{x \in X : f(x) \leq y\} \qquad \text{and} \qquad g_f(y) = \max\left(\Gamma_f(y)\right)$$

for all $y \in Y$.

Remark 5.2. Note that thus we have $\Gamma_f(y) = f^{-1}\left[\mathrm{lb}(y)\right] = \left(f^{-1} \circ \mathrm{lb}\right)(y)$ for all $y \in Y$. Moreover, note that if in particular X is a poset, then by Theorem 2.9 the relation g_f is already a function.

By Definition 3.5 and Theorem 4.10, the above definitions could only be naturally applied to increasing structures. Therefore, it is not surprising that most of the forthcoming statements need increasing structures.

Theorem 5.3. *If f is a structure on X to Y, then*

(1) $\Gamma_f(y_1) \subset \Gamma_f(y_2)$ *for all* $y_1, y_2 \in Y$ *with* $y_1 \leq y_2$;

(2) *if f is increasing, then* $\Gamma_f(y)$ *is descending in* X *for all* $y \in Y$.

Proof. To check (2), note that if $y \in Y$ and $x \in \Gamma_f(y)$, then by the defintion of $\Gamma_f(y)$ we have $f(x) \leq y$. Moreover, if $u \in X$ such that $u \leq x$, then by the increasingness of f we also have $f(u) \leq f(x)$. Hence, by the transitivity of Y, it follows that $f(u) \leq y$. Therefore, $u \in \Gamma_f(y)$, and thus the required assertion is also true. □

Theorem 5.4. *If f is a structure on X to Y, then*

(1) $g_f(y) = \Gamma_f(y) \cap \operatorname{ub}(\Gamma_f(y))$ *for all* $y \in Y$;

(2) *if f is increasing, then* $g_f(y) = \{x \in X : \operatorname{lb}(x) = \Gamma_f(y)\}$ *for all* $y \in Y$.

Proof. To check (2), note that, by Definition 5.1 and Theorem 2.5, for any $x \in X$ and $y \in Y$, we have

$$x \in g_f(y) \iff x \in \max(\Gamma_f(y)) \iff x \in \Gamma_f(y) \quad \text{and} \quad \Gamma_f(y) \subset \operatorname{lb}(x).$$

Moreover, by Theorem 5.3, we have $x \in \Gamma_f(y)$ if and only if $\operatorname{lb}(x) \subset \Gamma_f(y)$. □

Remark 5.5. Note that, in addition to the above assertions, by Theorem 2.8 we can also state that $g_f(y) = \Gamma_f(y) \cap \sup(\Gamma_f(y))$ for all $y \in Y$.

Now, as an extension of an observation of Pickert [**22**] and [**2**, Theorem 2.5], we can also prove the following

Theorem 5.6. *For any structures f on X to Y and g on Y to X, the following assertions are equivalent:*

(1) *f is increasingly g-normal;*

(2) *f is increasing and* $g \subset g_f$;

(3) $\Gamma_f(y) = \operatorname{lb}(g(y))$ *for all* $y \in Y$.

Proof. If (1) holds, then by Theorem 4.10 the structure f is increasing. Moreover, if $y \in Y$, then by the reflexivity of X we have $g(y) \leq g(y)$. Hence, by using (1), we can infer that $f(g(y)) \leq y$. Therefore, by the definition of Γ_f, we have $g(y) \in \Gamma_f(y)$.

Moreover, if $x \in \Gamma_f(y)$, then $f(x) \leq y$. Hence, by using (1), we can infer that $x \leq g(y)$. Therefore, $g(y) \in \operatorname{ub}(\Gamma_f(y))$ is also true. Hence, by Theorem 5.4, it is clear that

$$g(y) \in \Gamma_f(y) \cap \operatorname{ub}(\Gamma_f(y)) = g_f(y).$$

Therefore, (2) also holds.

While, if (2) holds, then $g(y) \in g_f(y)$ for all $y \in Y$. Hence, by using Theorem 5.4, we can infer that $\Gamma_f(y) = \operatorname{lb}(g(y))$ for all $y \in Y$. Therefore, (3) also holds. Finally, if (3) holds, then it is clear that, for any $x \in X$ and $y \in Y$, we have

$$f(x) \leq y \iff x \in \Gamma_f(y) \iff x \in \operatorname{lb}(g(y)) \iff x \leq g(y).$$

Therefore, (1) also holds. □

From the above theorem, by the second part of Remark 5.2, we can immediately get

Corollary 5.7. *If f is an increasingly g-normal structure on a poset X to Y, then $g = g_f$.*

Hence, it is clear that in particular we also have

Corollary 5.8. *If f is a structure on a poset X to Y, then there exists at most one structure g on Y to X such that f is increasingly g-normal.*

Moreover, by using Theorem 5.6, we can also easily establish the following

Theorem 5.9. *If f is an increasing structure on X to Y such that g_f is a structure on Y, then f is increasingly g_f–normal.*

Proof. In this case, g_f is already a selection of itself. Therefore, by Theorem 5.6, the required assertion is true. □

From the above theorem, by Corollary 4.11, we can also state

Corollary 5.10. *If f is an increasing structure on X to Y such that g_f is a structure on Y, then g_f is also increasing.*

Definition 5.11. For any structure f on X to Y, we define

$$\mathcal{Q}_f = \left\{ g \in X^Y : \ f \text{ is increasingly } g\text{-normal} \right\}.$$

Moreover, if in particular $\mathcal{Q}_f \neq \emptyset$, then we say that f is increasingly normal.

Now, by using Theorem 5.6, we can also easily prove the following two theorems.

Theorem 5.12. *If f is a structure on X to Y, then the following assertions are equivalent:*

(1) *f is increasingly normal;*

(2) *f is increasing and Y is the domain of g_f;*

(3) *for each $y \in Y$ there exists $x \in X$ such that $\Gamma_f(y) = \operatorname{lb}(x)$.*

Proof. If (1) holds, then by Definition 5.11 there exists a structure g on Y to X such that f is increasingly g-normal. Hence, by Theorem 5.6, it follows that f is increasing and $g(y) \in g_f(y)$ for all $y \in Y$. Therefore, $g_f(y) \neq \emptyset$ for all $y \in Y$, and thus (2) also holds.

While, if (2) holds, then by the Axiom of Choice, there exists a function g of Y into X such that $g \subset g_f$. Hence, by Theorem 5.6, it is clear that $\Gamma_f(y) = \operatorname{lb}(g(y))$ for all $y \in Y$. Therefore, (3) also holds.

Finally, if (3) holds, then by the Axiom of Choice there exists a function g of Y into X such that $\Gamma_f(y) = \operatorname{lb}(g(y))$ for all $y \in Y$. Hence, by Theorem 5.6, it is clear that f is g-normal. Therefore, (1) also holds. □

Theorem 5.13. *If f is an increasingly normal structure on X to Y, then $g_f = \bigcup \mathcal{Q}_f$.*

Proof. If $g \in \mathcal{Q}_f$, then by Theorem 5.6 we have $g \subset g_f$. Therefore, the inclusion $\bigcup \mathcal{Q}_f \subset g_f$ is true.

On the other hand, from Theorem 5.12 we can see that $g_f(y) \neq \emptyset$ for all $y \in Y$. Therefore, if $y \in Y$ and $x \in g_f(y)$, then by the Axiom of Choice there exists a function g of Y to X such that $x = g(y)$ and $g(t) \in g_f(t)$ for all $t \in Y \setminus \{y\}$. Moreover, from Theorem 5.6, we can see that $g \in \mathcal{Q}_f$. Hence, it is clear that

$$x \in \bigcup_{g \in \mathcal{Q}_f} \{g(y)\} = \Big(\bigcup_{g \in \mathcal{Q}_f} g\Big)(y) = \Big(\bigcup \mathcal{Q}_f\Big)(y).$$

Therefore, $g_f(y) \subset \left(\bigcup \mathcal{Q}_f\right)(y)$, and thus the inclusion $g_f \subset \bigcup \mathcal{Q}_f$ is also true. □

Moreover, in addition to Theorem 5.9 and Corollary 5.10, we can also prove

Theorem 5.14. *If f is an increasingly normal structure on poset X to Y, then g_f is an increasing structure on Y to X and $\mathcal{Q}_f = \{g_f\}$.*

Proof. In this case, by Definition 5.11, there exists $g \in \mathcal{Q}_f$. Hence, by Corollary 5.7, it follows that $g = g_f$, and thus $g_f \in \mathcal{Q}_f$. Hence, by Corollary 5.8, it is clear that $\mathcal{Q}_f = \{g_f\}$. Moreover, from Corollary 5.10, we can see that g_f is also increasing. □

6. Characterizations of increasingly regular structures

Definition 6.1. For any structure f on a proset X, we define two relations Λ_f and φ_f on X such that

$$\Lambda_f(x) = \{u \in X : f(u) \leq f(x)\} \qquad \text{and} \qquad \varphi_f(x) = \max\left(\Lambda_f(x)\right)$$

for all $x \in X$.

Remark 6.2. Note that thus we have $\Lambda_f(x) = f^{-1}[\mathrm{lb}\,(f(x))] = \left(f^{-1} \circ \mathrm{lb} \circ f\right)(x)$ for all $x \in X$.

Moreover, note that if in particular X is a poset, then by Theorem 2.9 the relation φ_f is already a function.

By Definition 3.3 and Theorem 4.5, the above definitions could only be naturally applied to increasing structures. Therefore, it is not surprising that most of our forthcoming theorems need increasing structures.

Theorem 6.3. *If f is a structure on X, then Λ_f is preorder on X.*

Proof. This is immediate from the reflexivity and transitivity of the range of f by the definition of Λ_f. □

Theorem 6.4. *If f is a structure on X, then*

$$\Lambda_f = \Gamma_f \circ f \qquad \textit{and} \qquad \varphi_f = g_f \circ f.$$

Proof. If $x \in X$, then for any $u \in X$ we have

$$u \in \Lambda_f(x) \iff f(u) \leq f(x) \iff u \in \Gamma_f\left(f(x)\right) \iff u \in \left(\Gamma_f \circ f\right)(x).$$

Therefore, $\Lambda_f(x) = (\Gamma_f \circ f)(x)$. Hence, it is clear

$$\begin{aligned}\varphi_f(x) = \max(\Lambda_f(x)) = \max((\Gamma_f \circ f)(x)) \\ = \max(\Gamma_f(f(x))) = g_f(f(x)) = (g_f \circ f)(x).\end{aligned}$$

Therefore, the required equalities are also true. □

Theorem 6.5. *If f is a structure on X, then the following assertions are equivalent:*

(1) *f is increasing;*

(2) *$\Lambda_f(x)$ is descending in X for all $x \in X$;*

(3) *$\Lambda_f(x_1) \subset \Lambda_f(x_2)$ for all $x_1, x_2 \in X$ with $x_1 \leq x_2$.*

Proof. Suppose that (1) holds and $x \in X$. Then, by Theorem 6.4, we have $\Lambda_f(x) = (\Gamma_f \circ f)(x) = \Gamma_f(f(x))$. Hence, by Theorem 5.3, we can see that $\Lambda_f(x)$ is descending in X, and thus (2) also holds.

Suppose now that (2) holds and $x_1, x_2 \in X$ such that $x_1 \leq x_2$. Then, by Theorem 6.3, we have $x_2 \in \Lambda_f(x_2)$. Hence, by (2) and the inequality $x_1 \leq x_2$, we can see that $x_1 \in \Lambda_f(x_2)$. Now, by Theorem 6.3, it is clear that $\Lambda_f(x_1) \subset \Lambda_f(\Lambda_f(x_2)) = (\Lambda_f \circ \Lambda_f)(x_2) \subset \Lambda_f(x_2)$. Therefore, (3) also holds.

Finally, suppose that (3) holds and $x_1, x_2 \in X$ such that $x_1 \leq x_2$. Then, by Theorem 6.3 and assertion (3), we have $x_1 \in \Lambda_f(x_1) \subset \Lambda_f(x_2)$. Hence, by the definition of Λ_f, it is clear that $f(x_1) \leq f(x_2)$. Therefore, (1) also holds. □

From Theorem 6.4, by Theorem 5.4, it is clear that we also have the following

Theorem 6.6. *If f is a structure on X, then*

(1) *$\varphi_f(x) = \Lambda_f(x) \cap \mathrm{ub}(\Lambda_f(x))$ for all $x \in X$;*

(2) *if f is increasing, then $\varphi_f(x) = \{u \in X : \mathrm{lb}(u) = \Lambda_f(x)\}$ for all $x \in X$.*

Remark 6.7. Now, by Remark 5.5, we can also state that $\varphi_f(x) = \Lambda_f(x) \cap \sup(\Lambda_f(x))$ for all $x \in X$.

Moreover, we can also easily prove the following

Theorem 6.8. *If f is a structure on X, then*

(1) *$\varphi_f(x) \subset \mathrm{ub}(x)$ for all $x \in X$;*

(2) *if f is increasing, then $\varphi_f(x_1) \subset \mathrm{lb}(\varphi_f(x_2))$ for all $x_1, x_2 \in X$ with $x_1 \leq x_2$.*

Proof. If $x \in X$, then by Theorem 6.3 we have $\{x\} \subset \Lambda_f(x)$. Hence, by Theorem 6.6, it is clear that

$$\varphi_f(x) \subset \mathrm{ub}(\Lambda_f(x)) \subset \mathrm{ub}(\{x\}) = \mathrm{ub}(x),$$

and thus (1) is true.

To prove (2), suppose that $x_1, x_2 \in X$ such that $x_1 \leq x_2$, and moreover $u \in \varphi_f(x_1)$ and $v \in \varphi_f(x_2)$. Then, by Theorems 6.6 and 6.5, we have

$$u \in \varphi_f(x_1) \subset \Lambda_f(x_1) \subset \Lambda_f(x_2) \qquad \text{and} \qquad v \in \varphi_f(x_2) \subset \mathrm{ub}(\Lambda_f(x_2)).$$

Hence, it is clear that $u \leq v$. Therefore, $u \in \mathrm{lb}\,(\varphi_f(x_2))$, and thus (2) is also true. $\square$

Remark 6.9. From the above theorem, we can at once see that if the structure f is increasing and φ_f is an operation on X, then φ_f is a preclosure operation.

However, this observation is of no particular importance for us since by using the following extension of [**20**, Theorem 1.12], we can prove a stronger statement.

Theorem 6.10. *If φ is an operation and f is a structure on X, then the following assertions are equivalent:*

(1) *f is increasingly φ-regular;*

(2) *f is increasing and $\varphi \subset \varphi_f$;*

(3) *$\Lambda_f(x) = \mathrm{lb}\,(\varphi(x))$ for all $x \in X$.*

Proof. If (1) holds, then by Theorem 4.5, the structure f is increasing. Moreover, if $x \in X$, then by Theorem 4.5 we have $f\,(\varphi(x)) \leq f(x)$. Hence, by the definition of Λ_f, it follows that $\varphi(x) \in \Lambda_f(x)$.

Moreover, if $u \in \Lambda_f(x)$, i.e., $f(u) \leq f(x)$, then by (1) we also have $u \leq \varphi(x)$. Therefore, $\varphi(x) \in \mathrm{ub}\,(\Lambda_f(x))$ is also true. Hence, by Theorem 6.6, it is clear that

$$\varphi(x) \in \Lambda_f(x) \cap \mathrm{ub}\,(\Lambda_f(x)) = \varphi_f(x).$$

Therefore, (2) also holds.

While, if (2) holds, then $\varphi(x) \in \varphi_f(x)$ for all $x \in X$. Hence, by using Theorem 6.6, we can infer that $\Lambda_f(x) = \mathrm{lb}\,(\varphi(x))$ for all $x \in X$. Therefore, (3) also holds. (Thus, by Remark 6.2, we can also state that $f^{-1} \circ \mathrm{lb} \circ f = \mathrm{lb} \circ \varphi$.)

Finally, if (3) holds, then it is clear that, for any $x_1, x_2 \in X$, we have

$$x_1 \leq \varphi(x_2) \iff x_1 \in \mathrm{lb}\,(\varphi(x_2)) \iff x_1 \in \Lambda_f(x_2) \iff f(x_1) \leq f(x_2).$$

Therefore, (1) also holds. $\square$

Now, as an immediate consequence of Theorem 6.10 and the second part of Remark 6.2, we can also state

Corollary 6.11. *If f is an increasingly φ-regular structure on a poset X, then $\varphi = \varphi_f$.*

Hence, it is clear that in particular we also have

Corollary 6.12. *If f is a structure on a poset X, then there exists at most one operation φ on X such that f is increasingly φ-regular.*

Moreover, by using Theorem 6.10, we can also easily establish the following

Theorem 6.13. *If f is an increasing structure on X such that φ_f is an operation on X, then f is φ_f-increasingly regular.*

Hence, by Theorem 4.7, it is clear that in particular we also have

Corollary 6.14. *If f is an increasing structure on X such that φ_f is an operation on X, then φ_f is a closure operation.*

Definition 6.15. For any structure f on a proset X, we define

$$\mathcal{O}_f = \big\{ \varphi \in X^X : f \text{ is increasingly } \varphi\text{-regular} \big\} .$$

Moreover, if in particular $\mathcal{O}_f \neq \emptyset$, then we say that f is increasingly regular.

By Corollary 3.8, we evidently have the following

Theorem 6.16. *If f is an increasingly normal structure on X to Y, then f is increasingly regular.*

Moreover, analogously to the corresponding results of Section 5, we can also easily establish the following three theorems.

Theorem 6.17. *If f is a structure on X, then the following assertions are equivalent:*

(1) *f is increasingly regular;*

(2) *f is increasing and X is the domain of φ_f;*

(3) *for each $x \in X$ there exists $u \in X$ such that $\Lambda_f(x) = \operatorname{lb}(u)$.*

Theorem 6.18. *If f is an increasingly regular structure on X, then $\varphi_f = \bigcup \mathcal{O}_f$.*

Theorem 6.19. *If f is an increasingly regular structure on a poset X, then φ_f is a closure operation on X and $\mathcal{O}_f = \{\varphi_f\}$.*

7. Injective and surjective increasingly regular structures

In addition to Theorem 6.3, we can also prove the following

Theorem 7.1. *If f is a structure on X onto Y, then the following assertions are equivalent:*

(1) *Λ_f is antisymmetric;*

(2) *f is injective and Y is antisymmetric.*

Proof. Suppose that (1) holds and $x_1, x_2 \in X$ such that $f(x_1) \leq f(x_2)$ and $f(x_2) \leq f(x_1)$. Then, by the definition of Λ_f, we have $x_1 \in \Lambda_f(x_2)$ and $x_2 \in \Lambda_f(x_1)$. Hence, by using (1), we can infer that $x_1 = x_2$, and thus $f(x_1) = f(x_2)$. Hence, since $Y = f[X]$, it is clear that the second part of (2) is true. Moreover, by the reflexivity of Y, it is clear that the first part of (2) is also true.

To prove the converse implication, suppose now that (2) holds and $x_1 \in \Lambda_f(x_2)$ and $x_2 \in \Lambda_f(x_1)$. Then, by the definition of Λ_f, we have $f(x_1) \leq f(x_2)$ and $f(x_2) \leq f(x_1)$. Hence, by the antisymmetry of Y, it follows that $f(x_1) = f(x_2)$. Thus, by the injectivity of f, we also have $x_1 = x_2$. Therefore, (1) also holds. □

However, Theorem 7.1 is of no particular importance for us since we also have the following

Theorem 7.2. *If f is an increasingly φ-regular structure on one poset X to another, then the following assertions are equivalent:*

(1) *f is injective;* (2) *φ is the identity function of X.*

Proof. Suppose that (1) holds and $x \in X$. Then, by Corollary 4.6, we have $f(\varphi(x)) = f(x)$. Hence, by (1), it follows that $\varphi(x) = x$. Thus, (2) also holds.

Suppose now that (2) holds and $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Then, by the reflexivity of the range of f, we also have $f(x_1) \leq f(x_2)$ and $f(x_2) \leq f(x_1)$. Hence, by using (2) and the assumed regularity of f, we can infer that $x_1 \leq \varphi(x_2) = x_2$ and $x_2 \leq \varphi(x_1) = x_1$. Now, by the antisymmetry of X, it is clear that $x_1 = x_2$. Therefore, (1) also holds. □

In this respect, it is also worth mentioning that, according to [**2**, Theorem 2.6], we also have the following

Theorem 7.3. *If f is an increasingly g-normal structure on one poset X to another Y and $\psi = f \circ g$, then the following assertions are equivalent:*

(1) *f is onto Y;*

(2) *g is an injection of Y;*

(3) *ψ is the identity function of Y.*

Proof. Define $\varphi = g \circ f$. Then, by Corollary 3.8, f is increasingly φ-regular. Moreover, by Corollary 4.6, we have

$$f(x) = f(\varphi(x)) = f(g(f(x)))$$

for all $x \in X$. Hence, if (1) holds, we can infer that

$$y = f(g(y)) = \psi(y)$$

for all $y \in Y$. Therefore, (2) and (3) also hold.

On the other hand, from Theorem 3.13 we know that now g is an increasingly f–normal structure on Y^* to X^*. Thus, by Corollary 3.8, the function g is an increasingly ψ-regular structure on Y^* to X^*. Hence, by Theorem 7.2, it is clear that (2) and (3) are equivalent. Now, to complete the proof, it remains to note only that if (3) holds, then we have $f(g(y)) = y$ for all $y \in Y$. Therefore, $f[R_g] = Y$, and thus in particular (1) also holds. □

Now, as an useful consequence of Theorems 7.2 and 7.3, we can also state

Corollary 7.4. *If f is an increasingly g-normal structure on one poset X to another Y such that f is injective and onto Y, then $g = f^{-1}$.*

Proof. Namely, by Theorem 7.3, we have $f \circ g = \Delta_Y$. Moreover, by Corollary 3.8 and Theorem 7.2, we also have $g \circ f = \Delta_X$. Therefore, $g = f^{-1}$ also holds. □

Moreover, by using Theorems 5.12, 6.16 and 6.17, we can also prove the following

Theorem 7.5. *If f is a structure on X onto Y, then the following assertions are equivalent:*

(1) *f is increasingly regular;* (2) *f is increasingly normal.*

Proof. If (1) holds, then by Theorems 6.17 and 6.4 we can see that f is increasing and

$$g_f\,(f(x)) = \varphi_f(x) \neq \emptyset$$

for all $x \in X$. Hence, by using that $Y = f[X]$, we can infer that $g_f(y) \neq \emptyset$ for all $y \in Y$. Therefore, by Theorem 5.12, assertion (1) also holds.

On the other hand, from from Theorem 6.16, we know that (2) implies (1) even if f is not onto Y. □

Remark 7.6. By Theorems 7.5 and 7.3, it is clear that, in the case of posets, the class of all increasingly regular structures coincides with the class of all increasingly injectively normal structures.

Therefore, the increasingly normal structures are more general objects, than the increasingly regular ones. However, despite this the latter ones are sometimes more natural and important means than the former ones.

8. A modification of the induced operation

By Theorem 4.5 and the results Section 6, we may also naturally introduce

Definition 8.1. For any structure f on a proset X, we define two relations Λ_f^* and φ_f^* on X such that

$$\Lambda_f^*(x) = \{u \in X : f(x) \leq f(u) \leq f(x)\} \qquad \text{and} \qquad \varphi_f^*(x) = \max\left(\Lambda_f^*(x)\right)$$

for all $x \in X$.

Remark 8.2. Note that if X is a poset, then by Theorem 2.9 the relation φ_f^* is already a function.

Moreover, note that if the range of f is a poset, then we simply have $\Lambda_f^*(x) = \{u \in X : f(u) = f(x)\}$ for all $x \in X$.

By using the corresponding definitions and Theorem 6.3, we can easily prove

Theorem 8.3. *If f is a structure on X, then*

(1) $\Lambda_f^* = \Lambda_f \cap \Lambda_f^{-1}$; (2) Λ_f^* *is an equivalence on* X.

Proof. To prove (1), note that if $x \in X$ and $u \in \Lambda_f^*(x)$, then by the definition of Λ_f^*, we have $f(x) \leq f(u) \leq f(x)$. Hence, by the definition of Λ_f, it follows that $u \in \Lambda_f(x)$ and $x \in \Lambda_f(u)$. Hence, we can already see that

$$u \in \Lambda_f(x) \cap \Lambda_f^{-1}(x) = \left(\Lambda_f \cap \Lambda_f^{-1}\right)(x).$$

Therefore, $\Lambda_f^*(x) \subset \left(\Lambda_f \cap \Lambda_f^{-1}\right)(x)$, and thus $\Lambda_f^* \subset \Lambda_f \cap \Lambda_f^{-1}$. Moreover, by reversing the above argument, we can easily see that the converse inclusion is also true.

Now, to prove (2), it remains only to note that, by Theorem 6.3, Λ_f is a preorder on X. Hence, it is clear that Λ_f^{-1}, and thus $\Lambda_f^* = \Lambda_f \cap \Lambda_f^{-1}$ is also a preorder on X. Now, since the latter relation is obviously symmetric the proof is complete. □

However, instead of an analogue of Theorem 6.6, we can only state the following

Theorem 8.4. *If f is a structure on X, then*

(1) $\varphi_f^*(x) = \Lambda_f^*(x) \cap \operatorname{ub}\Big(\Lambda_f^*(x)\Big)$ *for all $x \in X$;*

(2) *if f is increasing, then* $\varphi_f^*(x) = \Big\{u \in X : u \in \Lambda_f^*(x) \subset \operatorname{lb}(u)\Big\}$ *for all $x \in X$.*

Moreover, instead of an analogue of Theorem 6.10, we can only prove the following

Theorem 8.5. *If f is an increasingly φ-regular structure on X, then $\varphi \subset \varphi_f^*$.*

Proof. If $x \in X$, then by Theorem 4.5, we have $f(x) \leq f(\varphi(x)) \leq f(x)$. Hence, by the definition Λ_f^*, it follows that $\varphi(x) \in \Lambda_f^*(x)$. Moreover, by Theorems 6.10 and 6.6, we have $\varphi(x) \in \varphi_f(x) \subset \operatorname{ub}(\Lambda_f(x))$. Hence, by the inclusion $\Lambda_f^*(x) \subset \Lambda_f(x)$, it follows that $\varphi(x) \in \operatorname{ub}\Big(\Lambda_f^*(x)\Big)$ is also true. Therefore, by Theorem 8.4, we also have

$$\varphi(x) \in \Lambda_f^*(x) \cap \operatorname{ub}\big(\Lambda_f^*(x)\big) = \varphi_f^*(x).$$

Thus, the required inclusion is also true. $\square$

Remark 8.6. If φ is an operation and f is an increasing structure on X such that $\varphi \subset \varphi_f^*$, then in contrast to Theorem 6.10 we can only prove that $x_1 \leq \varphi(x_2)$ implies $f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in X$.

Now, as an immediate consequence of Theorems 8.5 and 6.18, we can also state the following

Theorem 8.7. *If f is an increasingly regular structure on X, then $\varphi_f \subset \varphi_f^*$.*

Proof. Namely, if $\varphi \in \mathcal{O}_f$, then by Theorem 8.5 we have $\varphi \subset \varphi_f^*$. Hence, by Theorem 6.18, it is clear that $\varphi_f = \bigcup \mathcal{O}_f \subset \varphi_f^*$. $\square$

From the above theorem, by Theorem 6.17 and Remark 8.2, it is clear that in particular we also have

Corollary 8.8. *If f is an increasingly regular structure on a poset X, then $\varphi_f = \varphi_f^*$.*

Now, as an immediate consequence of this corollary and Theorem 6.19, we can also state the following

Theorem 8.9. *If f is an increasingly regular structure on a poset X, then φ_f^* is a closure operation on X and $\mathcal{O}_f = \{\varphi_f^*\}$.*

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