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AN AMERICAN CONVERT CLOSE TO MATURITY

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Abstract. We use an asymptotic expansion to study the behavior of an American convertible bond close to maturity, under the assumptions that the underlying stock price obeys a lognormal random walk and the risk-free rate is given by either the Vasicek model or the Cox-Ingersoll-Ross model. Series solutions are obtained for the location of the free boundary and the price of the bond in that limit.

1. Introduction

A convertible bond, or *convert*, is debt which can be converted into the equity of the issuing corporation at certain times using a pre-determined exchange ratio [**17**], with the option to convert solely at the discretion of the bond holder, who will do so only if it is beneficial. If and when conversion occurs, new shares are issued by the corporation, with the existing shares diluted by the creation of the new ones. For arbitrage reasons, a convertible bond cannot be worth less than an otherwise identical non-convertible bond. To an issuer, convertible debt has the advantage of lower interest cost than straight debt, but with the drawback that the issuer faces capital structure uncertainty. In return for a reduced yield, an investor will receive a security with considerable upside potential along with downside protection. There is a large global market for convertible debt, with in excess of $400 billion in market value outstanding in 2000 [**14**], because of which the pricing of these securities is an important problem.

The behavior of a convertible bond can be classified into four regimes [**14**] according to the conversion premium, which is the excess an investor would pay to acquire the stock by buying the convertible and immediately converting rather than buying the stock itself. Most new issued converts tend to be *balanced converts*, which respond to changes in both the underlying stock price and the spot interest rate, with a correlation of about 55% to 80% with changes in the stock price, with around a 25% conversion premium of about 25%. Once the price of the underlying has risen, the convert tends to be an *equity substitute convert*, which responds much more to changes in the stock price than to interest rate changes, with a conversion premium of less than 15%. If the underlying stock

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price has declined so significantly that the conversion option is worth very little, the convert is a *busted convert* whose value approaches that of an otherwise identical non-convertible bond. *Distressed converts* are a sub-category of busted converts, where the stock price has fallen so much that there is a significant chance of bankruptcy.

As with other derivatives, contingent claims analysis has been used to value convertibles, and this approach dates back to [**8, 9, 15**]. Initially, the firm value was used as the underlying variable [**8**], with the analysis extended later to include stochastic interest rates [**9**] and the value of the stock rather than that of the firm [**24**]. Almost all of this earlier work led to a numerical rather than an analytical solution of the underlying equations for the value of a convertible bond, typically using binomial trees, while a later thread [**21, 23**] was analytical, using a Green's function approach to value securities whose value depended on both a stock price and interest rate. The present study is entirely analytical, using asymptotic analysis.

In our analysis, we consider a convertible bond, whose value depends on both the price S of the underlying stock, which is assumed to obey a lognormal random walk with constant volatility, as in the Black-Scholes-Merton option pricing model [**7, 26**], and on the interest rate r, which is assumed to follow a mean-reverting random walk given by either the Vasicek [**39**] or the Cox-Ingersoll-Ross (CIR) [**11, 12**] model. From these models, we have stochastic differential equations for both the stock price and the spot rate. By constructing a risk-free portfolio, it is possible to go from these stochastic differential equations to a partial differential equation (PDE) for the value of the convert [**40**], and this PDE is the starting point for our analysis in the next section.

American converts contain an embedded American-style option, and as with American equity options this embedded option can be exercised at any time at or prior to maturity. As a consequence, American converts are harder to price than their European counterparts, because the possibility of early exercise leads to a free boundary separating the region where it is optimal to hold from that where exercise is optimal. In theory, exercise should take place only on this free boundary, known as the optimal exercise boundary. This sort of free boundary problem is common in diffusion problems such as melting and solidification problems and is referred to as a Stefan problem, and a large number of studies have focused on the optimal exercise boundary for American equity options, and in particular on the behavior of this boundary close to expiry, including [**2, 3, 5, 10, 13, 16, 18, 20, 22, 25, 28, 38**]. In our analysis, we will consider an American zero coupon convert, which can be converted to one unit of stock at any time at or prior to maturity, and which pays an amount P, the principal, at maturity if the option to convert is not exercised, so that the pay-off on the free boundary is S and that at maturity is $\max(S, P)$. To prevent arbitrage, the value of the bond must be equal to the value of the stock on the free boundary. In addition, we have the smooth pasting or high contact conditions [**27**] that the option's delta (or derivative of its value with respect to asset price $\partial V/\partial S$) and rho (or derivative with respect to interest rate $\partial V/\partial r$) must both be continuous across the boundary. Since the stock is held on one side of the boundary, this means that $\partial V/\partial S = 1$ and $\partial V/\partial r = 0$ at the boundary. The location of the free boundary at maturity will be $S = P$, which motivates us to seek a free boundary of the form $S = S_f(r,t)$, where t is time.

In the present study, we will use a technique developed by Tao [**29**]–[**37**] for free boundary problems arising in melting and solidification. Tao used a series expansion in time to find the location of the moving surface of separation between two phases of a material, and in almost all of the cases he studied, he found that the location of the interface was proportional to $\tau^{1/2}$, τ being the time since the two phases were first put in contact. Tao's method has been applied to American equity options in the past [**2, 3, 13, 22**], and in those studies, a change of variables [**13, 40**] was use to transform the governing equations into the heat conduction equation studied by Tao, along with a nonhomogeneous term. Because we are seeking a free boundary of the form $S = S_f(r,t)$, we are able to use the same transformation in this study as was used for American equity options, with no transformation applied to r.

At this point, the condition on the delta merits further comment. At maturity, where it is optimal to hold the bond $\partial V/\partial S = 0$, yet on the free boundary we have $\partial V/\partial S = 1$ prior to maturity, so that there is a discontinuity in the delta. When similar discontinuities occur for American equity options [**3, 4, 5, 22**], they appear to lead to logarithmic behavior of the free boundary, which is therefore the behavior we expect here. Although this discontinuity is possible in the financial setting, it does not seem to occur in physical Stefan problems, which perhaps explains why Tao [**29**]–[**37**] never encountered logarithmic behavior.

The rest of our paper is as follows. We will present our analysis for American converts in Section 2, followed by a brief discussion of our results in Section 3.

2. Analysis

In this section we will discuss the value $V(S,r,t)$ of a convertible bond. We shall assume that the asset price S and spot interest rate r obey the stochastic differential equations,

$$\begin{aligned} \mathrm{d}S &= \mu S \mathrm{d}t + \sigma S \mathrm{d}X_1, \\ \mathrm{d}r &= u(r,t)\mathrm{d}t + w(r,t)\mathrm{d}X_2, \end{aligned} \tag{1}$$

where σ is the volatility of the stock price and μ is the drift, while $\mathrm{d}X_1$ and $\mathrm{d}X_2$ are both normally distributed with zero mean and variance $\mathrm{d}t$ and may be correlated, with $E[\mathrm{d}X_1\mathrm{d}X_2] = \rho\mathrm{d}t$ and $-1 \le \rho(r,S,t) \le 1$. S obeys a lognormal random walk, as in the Black-Scholes-Merton option pricing model, [**7, 26**]. Constructing a risk-free portfolio leads to the following PDE for V [**40**],

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \rho\sigma S w \frac{\partial^2 V}{\partial S \partial r} + \frac{w^2}{2}\frac{\partial^2 V}{\partial r^2} \\ &\qquad + (r-D)\, S\frac{\partial V}{\partial S} + (u - \lambda w)\,\frac{\partial V}{\partial r} - rV = 0, \end{aligned} \tag{2}$$

which for $t \leq T$, where T is the time at which the bond matures. In the above, D is the constant dividend yield of the stock, $\lambda(r, S, t)$ is the market price of interest rate risk and $u - \lambda w$ is the risk adjusted drift. Many of the popular one-factor interest rate models are special cases of the general affine model for which $u - \lambda w = a(t) - b(t)r$ and $w = (c(t)r - d(t))^{1/2}$ [**40**]. Two of these special cases are the Vasicek model [**39**] and the Cox-Ingersoll-Ross (CIR) model [**11, 12**], with $u - \lambda w = a - br$ for both models and $w = c$ for the Vasicek model and $w = cr^{1/2}$ for the CIR model, where a, b and c are constants rather than functions of t. The Vasicek model allows interest rates to become negative but is popular because it is extremely tractable.

If we specialize to either Vasicek or CIR, both of which are mean-reverting models, and also assume that the correlation ρ is constant, (2) becomes

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \rho\sigma c r^{\mu} S \frac{\partial^2 V}{\partial S \partial r} + \frac{c^2 r^{2\mu}}{2}\frac{\partial^2 V}{\partial r^2} + (r - D) S \frac{\partial V}{\partial S} + (a - br)\frac{\partial V}{\partial r} - rV = 0, \tag{3}$$

with $\mu = 0$ for Vasicek and $1/2$ for CIR. We will suppose the pay-off at maturity $t = T$ is $V(S, r, T) = \max(S, P)$, while at the free boundary $S = S_f(r, t)$ we have $V(S, r, t) = S$. We shall proceed along the same lines as [**2, 3, 13, 22**] and make the change of variables $V(S, r, t) = S + Pv(x, r, \tau)$, $S = P\,\mathrm{e}^x$ and $t = T - 2\tau/\sigma^2$, where Pv is the conversion premium, which transforms (3) into

$$\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2} + \rho\sigma c r^{\mu}\frac{\partial^2 v}{\partial x \partial r} + \frac{c^2 r^{2\mu}}{2}\frac{\partial^2 v}{\partial r^2} + \left(r - D - \frac{\sigma^2}{2}\right)\frac{\partial v}{\partial x} + (a - br)\frac{\partial v}{\partial r} - rV - De^x \tag{4}$$

with $v(x, r, 0) = \max(0, 1 - \mathrm{e}^x)$ at maturity while at the free boundary $x = x_f(r, \tau)$ we have $v = \partial v/\partial x = \partial v/\partial r = 0$. The bond should be held where $x < x_f(r, \tau)$ and converted where $x > x_f(r, \tau)$.

At maturity the free boundary starts at $S = P$ or equivalently $x = 0$. In the analysis that follows, strictly speaking the equation (4) is valid only for those parameter values where it is advantageous to hold the bond, so that at maturity, we can only impose the initial condition on $x < 0$, and the initial condition becomes $v \to 1 - \mathrm{e}^x$ as $\tau \to 0$.

To tackle the equation (4) and associated boundary and initial conditions, we shall follow Tao and seek a series solution. While Tao expanded in powers of $\tau^{1/2}$, in the current problem, the discontinuity in the delta mentioned above means that we must include logs as well as powers of $\tau^{1/2}$ in the expansion, and this seems to be the rule when there is a discontinuity in the delta at the free boundary [**3, 4, 5, 22**]. The series for $v(x, r, \tau)$ is therefore

$$v(x, r, \tau) = \tau^{1/2} V_1^{(0)}(\xi, r) + \sum_{n=2}^{\infty}\sum_{m=0}^{\infty} \tau^{n/2}(-\ln\tau)^{-m} V_n^{(m)}(\xi, r), \tag{5}$$

which is the same form as for the American put with $D < r$ considered in [**22**], with $\xi = x\tau^{-1/2}/2$ a similarity variable. The minus sign is included in $(-\ln\tau)$ because $\ln\tau$ is negative for $0 < \tau < 1$. It is worth noting that logarithms first enter in this series with the τ^1 terms rather than the leading $\tau^{1/2}$ term. We assume that the free boundary is located at $x = x_f(r,\tau)$ which we also write as a series,

$$x_f(r,\tau) \sim \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} x_n^{(m)}(r)\,\tau^{n/2}(-\ln\tau)^{1-n/2-m}, \tag{6}$$

with $x_1^{(0)}(r) = \sqrt{2}$. The leading order scaling of $x_f(\tau) \sim x_1^{(0)}(-\tau\ln\tau)^{1/2}$, which is the same as for the American options, is chosen because we need $|x_f(r,\tau)| \gg \mathcal{O}\left(\tau^{1/2}\right)$, and more specifically $\exp\left[-\frac{x_f^2}{4\tau}\right] \sim \mathcal{O}\left(\tau^{1/2}\right)$. The presence of logs in the series (6) for $x_f(r,\tau)$ necessitates the presence of logs in the series (5) for $v(x,r,\tau)$.

With this expansion, it follows that on the free boundary we have

$$\begin{aligned} \mathrm{e}^{-\xi^2} &= \exp\left[-\frac{x_f^2}{4\tau}\right] \sim \tau^{1/2}\,\mathrm{e}^{x_1^{(2)}/\sqrt{2}}\left[1+\mathcal{O}\left(\ln^{-1}\tau\right)\right], \\ \mathrm{erfc}(\xi) &= \mathrm{erfc}\left[\frac{x_f}{2\sqrt{\tau}}\right] \sim \left(\frac{2\tau}{-\pi\ln\tau}\right)^{1/2}\mathrm{e}^{x_1^{(2)}/\sqrt{2}}\left[1+\mathcal{O}\left(\ln^{-1}\tau\right)\right], \end{aligned} \tag{7}$$

where erfc is the complementary error function. and we have used the result [**1**] that as $\zeta \to \infty$, $\mathrm{erfc}(\zeta) \sim \frac{\mathrm{e}^{-\zeta^2}}{\zeta\sqrt{\pi}}\left[1+\sum_{m=1}^{\infty}\frac{(2m-1)!!}{(-2\zeta^2)^m}\right]$.

In our analysis, we substitute the assumed form for $v(x,\tau)$ (5) in the PDE (4) and group powers of $\tau^{1/2}$ and $-\ln\tau$. We find the following equations for the leading order terms at each power of $\tau^{1/2}$ in this expansion,

$$\mathcal{L}_n V_n^{(0)} = \begin{cases} 0 & n=0 \\ \dfrac{(2\xi)^{n-1}D}{(n-1)!\sigma^2} + \mathcal{M}_n V_{n-1}^{(0)} & n=1 \\ \dfrac{(2\xi)^{n-1}D}{(n-1)!\sigma^2} + \mathcal{M}_n V_{n-1}^{(0)} + \mathcal{N}_n V_{n-2}^{(0)} & n\geq 2 \end{cases}, \tag{8}$$

where

$$\begin{aligned} \mathcal{L}_n &\equiv \frac{1}{8}\frac{\partial^2}{\partial\xi^2} + \frac{\xi}{4}\frac{\partial}{\partial\xi} - \frac{n}{2}, \\ \mathcal{M}_n &= \left(\frac{D-r}{2\sigma^2}+\frac{1}{4}\right)\frac{\partial}{\partial\xi} - \frac{\rho c r^{\mu}}{2\sigma}\frac{\partial^2}{\partial\xi\partial r} \qquad \text{and} \\ \mathcal{N}_n &= \frac{r}{\sigma^2} + \frac{br-a}{\sigma^2}\frac{\partial}{\partial r} - \frac{c^2 r^{2\mu}}{2\sigma^2}\frac{\partial^2}{\partial r^2}. \end{aligned}$$

It is straightforward to write the solutions to the equations (8) which satisfy the initial condition that $v(x,0)=\max(1-\mathrm{e}^x,0)$ for $x\le 0$,

$$
\begin{aligned}
V_1^{(0)} &= -2\xi + \left[\frac{\mathrm{e}^{-\xi^2}}{\sqrt{\pi}} + \xi\,\mathrm{erfc}\,(-\xi)\right] C_1^{(0)}(r),\\
V_2^{(0)} &= -2\xi^2 - \frac{2r}{\sigma^2} + \left[\frac{2\xi\,\mathrm{e}^{-\xi^2}}{\sqrt{\pi}} + \left(1+2\xi^2\right)\,\mathrm{erfc}\,(-\xi)\right] C_2^{(0)}(r)\\
&+ \left[\left(1+\frac{2\,(D-r)}{\sigma^2}\right) C_1^{(0)}(r) - \frac{2\rho c r^\mu}{\sigma} C_1^{(0)\prime}(r)\right]\left[\frac{\xi\,\mathrm{e}^{-\xi^2}}{\sqrt{\pi}} + \xi^2\,\mathrm{erfc}\,(-\xi)\right]. \qquad (9)
\end{aligned}
$$

In (9), $V_1^{(0)}$ is the same for both the Vasicek and CIR models, but $V_2^{(0)}$ differs for the two models, because of the r^μ factor in the $C_1^{(0)\prime}(r)$ term. We would mention that since we can only impose the initial condition on $x<0$, the limit $\tau\to 0$ means that $\xi\to-\infty$. To impose the initial condition that $v\to 1-\mathrm{e}^x$ as $\tau\to 0$, we require that $\tau^{n/2}V_n^{(0)}\to -x^n/n!$, and we first set $\mathrm{e}^{-\xi^2}=\mathrm{erfc}(-\xi)=0$, and then replace ξ by $x\tau^{-1/2}/2$ and finally take the limit $\tau\to 0$.

Next, we impose the the conditions at the free boundary on (9). To do this, we replace x by (6), the series for $x_f\,(r,\tau)$, using the expressions (7) for $\mathrm{e}^{-\xi^2}$ and $\mathrm{erfc}(\xi)$ at the free boundary. This tells us that $C_1^{(0)}=1$ and $C_2^{(0)}=\frac{r-D}{\sigma^2}$, so that (9) becomes

$$
\begin{aligned}
V_1^{(0)} &= \frac{\mathrm{e}^{-\xi^2}}{\sqrt{\pi}} - \xi\,\mathrm{erfc}\,(\xi)\,,\\
V_2^{(0)} &= \frac{\xi\,\mathrm{e}^{-\xi^2}}{\sqrt{\pi}} + \left(\frac{D-r}{\sigma^2} - \xi^2\right)\mathrm{erfc}\,(\xi) - \frac{2D}{\sigma^2}.
\end{aligned} \qquad (10)
$$

Since $C_1^{(0)\prime}(r)=0$, $V_2^{(0)}$ is now the same for both models. However, if we perform the same procedure at the next order, we find that $V_3^{(0)}$ differs for the two models,

$$
\begin{aligned}
V_3^{(0)} = &\left[\frac{\rho c r^\mu}{\sigma^3} + \frac{(r-D)^2}{\sigma^4} - \frac{r+D}{\sigma^2} + \frac{2\xi^2}{3} - \frac{1}{12}\right]\frac{\mathrm{e}^{-\xi^2}}{\sqrt{\pi}}\\
&+ \left[\frac{2D\xi}{\sigma^2} - \frac{2\xi^3}{3}\right]\mathrm{erfc}\,(\xi) - \frac{4D\xi}{\sigma^2}.
\end{aligned} \qquad (11)
$$

In (10), we have the leading order terms at each power of $\tau^{1/2}$ and we can comment further on the discontinuity in $\partial v/\partial x$. From (10), at leading order, $\partial v/\partial x \sim -\frac{x+1}{2}\mathrm{erfc}\left(\frac{x}{2\tau^{1/2}}\right) + \mathcal{O}\left(\tau^{1/2}\right)$, which enables us to see the discontinuity: when $\tau=0$, $\mathrm{erfc}\left(\frac{x}{2\tau^{1/2}}\right)=2$ for $x<0$, while on the free boundary $x_f(\tau)$, $\mathrm{erfc}\left(\frac{x}{2\tau^{1/2}}\right)\sim\sqrt{\frac{2\tau}{-\pi\ln\tau}}\,\mathrm{e}^{x_1^{(2)}}/\sqrt{2}$. The complementary error function provides immediate smoothing of this discontinuity, as τ increases from zero.

For the next terms in the expansion, at $\tau^{n/2}/(-\ln\tau)$, we have the following equations,

$$\mathcal{L}_n V_n^{(1)} = \begin{cases} 0 & n=1 \\ \mathcal{M}_n V_{n-1}^{(1)} & n=2\,, \\ \mathcal{M}_n V_{n-1}^{(1)} + \mathcal{N}_n V_{n-2}^{(1)} & n\geq 3 \end{cases} \tag{12}$$

where $\mathcal{L}_n$, $\mathcal{M}_n$ and $\mathcal{N}_n$ are as above. It should be noted that these equations (12) do not involve the leading order terms $V_n^{(0)}$. The solution at the first order for both models is

$$V_2^{(1)} = C_2^{(1)}\left[\frac{2\xi\,\mathrm{e}^{-\xi^2}}{\sqrt{\pi}} + \left(1+2\xi^2\right)\,\mathrm{erfc}(-\xi)\right]. \tag{13}$$

With our expression for the free boundary (6), at leading order the conditions on the free boundary applied to (10,13) tell us that $C_2^{(1)}(r) = \frac{D}{\sigma^2}$ and $x_1^{(1)}(r) = -\sqrt{2}\ln\left[\frac{4\sqrt{\pi}D}{\sigma^2}\right]$ for both models. The analysis at the next order is rather involved, but tells us that $x_2^{(0)}(r) = -1-\frac{2(r-D)}{\sigma^2}$ for both models. We now know the behavior of the free boundary in the limit $\tau\to 0$,

$$\begin{aligned} x_f(\tau) &\sim \sqrt{-2\tau\ln\tau}\left(1+\frac{\ln\left(4\sqrt{\pi}D/\sigma^2\right)}{\ln\tau} + \mathcal{O}\left(\ln^{-2}\tau\right)\right) + \mathcal{O}\left(\tau\right), \\ S_f(t) &\sim P\exp\left[\sigma\sqrt{-(T-t)\ln\left[\frac{\sigma^2(T-t)}{2}\right]}\right. \\ &\qquad \times\left(1+\frac{\ln\left(4\sqrt{\pi}D/\sigma^2\right)}{\ln[\sigma^2(T-t)/2]} + \mathcal{O}\left(\ln^{-2}\left[\frac{\sigma^2(T-t)}{2}\right]\right)\right) \\ &\qquad \left. + \,\mathcal{O}\,\left(T-t\right)\right]. \end{aligned} \tag{14}$$

Our analysis of the convert is now complete. It should be noted that $x_{11}(r)\to\infty$ as $D\to 0+$, and also that the forcing term $-D\,\mathrm{e}^x$ in (4) vanishes in the same limit. Because of this, just as with American equity call options, an American zero coupon convert should never be exercised early if the underlying stock does not pay dividends.

3. Discussion

In the previous section, we used an asymptotic expansion of the governing PDE to study the behavior of a zero coupon American convert close to maturity, whose value depended on both the price S of the underlying stock, which was assumed assuming to obey a lognormal random walk with constant volatility, as in the Black-Scholes-Merton option pricing model [**7, 26**], and on the interest rate r, which was assumed to follow a mean-reverting random walk given by either the Vasicek [**39**] or the Cox-Ingersoll-Ross (CIR) [**11, 12**] model. The primary rationale for using these models are that they are popular models and also highly tractable. The Vasicek model has the undesirable property that interest rates can go negative, while the CIR model does not allow interest rates to change sign.

The principal results of this paper are two sets of expressions: an expression (14) for the location of the free boundary close to maturity, along with expressions (5,10,11,13) for the value of the bond in that limit. To the order shown, the free boundary was the same for both models considered, but this will not be true at subsequent orders. It is interesting to note that, provided D, the dividend yield of the stock, is positive, the location of the free boundary close to maturity is of the form $x_f(r,\tau) \sim \sqrt{\tau(-\ln\tau)}$, which is the same form as that for the American put with $D < r$ and the American call with $D > r$ [**3, 5, 22, 28**]. This differs from the $x_f(\tau) \sim x_1\sqrt{\tau}$ behavior for the American put with $D > r$ and the American call with $D < r$ which was also the behavior encountered most often by Tao [**29**]-[**37**], who pioneered the method used here, in his studies of Stefan problems arising in melting and solidification. Although Tao encountered several behaviors other than the $\sqrt{\tau}$ behavior, he did not come across the $\sqrt{\tau(-\ln\tau)}$ behavior found both here and with American options for the parameter ranges mentioned above. We suspect that this logarithmic behavior is caused by the discontinuity in $\partial V/\partial S$ which we discussed earlier, and since this discontinuity is unphysical, Tao did not encounter it. When $D = 0$, the American convert should never be converted prior to maturity.

In our analysis, we considered a fairly simple convert, an American zero coupon convert, which can be converted to one unit of stock at any time at or prior to maturity, and which pays an amount P, the principal, at maturity if the option to convert is not exercised, so that the pay-off on the free boundary is S and that at maturity is $\max(S,P)$. This convert has only one free boundary, on which the bond is exchanged for equity, and the addition of embedded call or put options which are found in some converts [**6, 17**] would lead to additional free boundaries.

As we mentioned above, our analysis used an asymptotic expansion of the governing PDE. An alternative approach might be to use the PDE and associated boundary and initial conditions to construct an integral equation formulation of the problem, thereby decoupling the location of the free boundary from the pricing of the convert. This approach has proven popular for American equity options [**10, 16, 18, 25, 20, 28, 38**]. One integral equation formulation which might be applied is of course the free boundary Green's function method [**19**], using the Green's function we presented in [**21, 23**] for a convert under the Vasicek model, although the corresponding Green's function for CIR model remains elusive. However, the resulting integral equations would involve a triple integral over x, r and τ rather than the double integrals found in [**10, 16, 18, 25, 38**] when this approach was applied to American equity options, and this might make asymptotic analysis of the integral equations problematic, although attempting this approach would likely still be a worthwhile endeavor.

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