A SPECIAL CONGRUENCE LATTICE OF A REGULAR SEMIGROUP

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Abstract. Let S be a regular semigroup and $\mathcal{C}$ its lattice of congruences. We consider the sublattice Λ of $\mathcal{C}$ generated by σ-the least group, τ-the greatest idempotent pure, μ-the greatest idempotent separating and β-the least band congruence on S. To this end, we study the following special cases: (1) any three of these congruences generate a distributive lattice, (2) Λ is distributive, (3) the restriction of the K-relation to Λ is a congruence and (4) a further special case. In each of these instances, we provide several characterizations. Our basic concept is that of a c-triple which represents an abstraction of $(\Lambda; K|_\Lambda, T|_\Lambda)$.

1. Introduction and summary

The most effective approach to congruences on regular semigroups is the kernel-trace approach. As a natural derivative, we have the kernel (respectively, trace) relation K (respectively, T), which relates two congruences having the same kernel (respectively, trace). Both these relations have their classes intervals, so one may speak of the least and the greatest elements of K- and T-classes. For the equality and the universal relations, these extremal congruences are just about omnipresent in any study of congruences on regular semigroups. The lattice they generate influences the structure of the semigroup in an essential way.

Let S be a regular semigroup and $\mathcal{C}$ its congruence lattice. We are concerned here with the sublattice Λ of $\mathcal{C}$ generated by the set $\Gamma = \{\sigma, \tau, \mu, \beta\}$, where σ is the least group congruence, τ the greatest idempotent pure congruence, μ the greatest idempotent separating congruence and β the least band congruence on S. There are

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four special cases of particular interest to us: (1) any three elements of Γ generate a distributive lattice, (2) Λ is distributive, (3) the restriction of the K-relation to Λ is a congruence, (4) a further special case.

In order to see that the four cited congruences are not arbitrarily chosen but belong to a system, we outline the kernel-trace approach to congruences on a regular semigroup. Continuing the above notation, let $\rho \in \mathcal{C}$. Then $\ker \rho$, the *kernel* of ρ, is the union of all idempotent ρ-classes; $\operatorname{tr} \rho$, the *trace* of ρ, equals $\rho|_E$ where E is the set of all idempotents of S. The relations K and T defined on $\mathcal{C}$ by

$$\lambda \, K \, \rho \Longleftrightarrow \ker \lambda = \ker \rho, \qquad \lambda \, T \, \rho \Longleftrightarrow \operatorname{tr} \lambda = \operatorname{tr} \rho,$$

are the *kernel* and *trace relations* on $\mathcal{C}$ (or of S), respectively. These are equivalence relations whose classes are intervals and for $\rho \in \mathcal{C}$, we can write the classes to which it belongs as

$$\rho K = [\rho_K, \rho^K], \qquad \rho T = [\rho_T, \rho^T].$$

Now denoting by ϵ and ω the equality and the universal relations on S, respectively, we get

$$\sigma = \omega_T, \quad \tau = \epsilon^K, \quad \mu = \epsilon^T, \quad \beta = \omega_K.$$

For S/σ has only one idempotent so that σ has the greatest possible trace, and one shows that σ is the least with this property. Idempotent pure congruences coincide with congruences whose kernel consists of idempotents only, and τ is the greatest such by definition. Similarly, idempotent separating congruences coincide with congruences whose trace is the equality relation, and μ is the greatest by definition. Finally, S/β is an idempotent semigroup so that its kernel is the greatest possible, and β is evidently the least such.

In the case that K is a congruence on $\mathcal{C}$, we have proved in [2] that Λ is a homomorphic image of the free distributive lattice on Γ subject to relations $\tau \leq \sigma$ and $\mu \leq \beta$. In [3] we gave a classification of regular semigroups based on the properties of the sublattice of $\mathcal{C}$ generated by the set $\{\sigma, \tau, \beta\}$. Here our basic concept is an abstraction of $(\Lambda; K|_\Lambda, T|_\Lambda)$ which we call a *c*-triple.

After the needed concepts and symbolism in Section 2, we give in Section 3 a list of relations that play a central role in our deliberations and establish some of their elementary properties. The case when the lattice Λ described above has the property that any three elements of Γ generate a distributive lattice is characterized in Section 4 in several ways. A similar analysis can be found in Section 5 for the case that Λ is distributive, in Section 6 for the case that $K|_\Lambda$ is a congruence and in Section 7 for a further special case.

2. Terminology and Notation

For concepts and symbolism we generally follow the book [1]. We now list some most frequent or special notation and terminology. The equality and the universal relations on a set X are denoted by ϵ_X and ω_X, respectively, with or without a subscript. For a lattice L and $\lambda, \rho \in L$, $\lambda \leq \rho$, let

$$[\lambda, \rho] = \{\theta \in L \mid \lambda \leq \theta \leq \rho\}.$$

Let S be a semigroup. Then $E(S)$ denotes its set of idempotents and $\mathcal{C}(S)$ its congruence lattice. For $\rho \in \mathcal{C}(S)$, its kernel and trace, as well as relations K and T, and also σ, τ, μ, β for S were defined in Section 1. We now write the last four symbols with subscript S and let

$$\Gamma_S = \{\sigma_S, \tau_S, \mu_S, \beta_S\},$$

Λ_S be the sublattice of $\mathcal{C}(S)$ generated by $\Gamma_S, K_S = K|_{\Lambda_S}$ and $T_S = T|_{\Lambda_S}$.

Next we abstract certain properties of these symbols. The following represents our principal concept.

Definition 2.1. We call $(\Lambda; K, T)$ a *c-triple* if

(i) Λ is a lattice generated by $\Gamma_\Lambda = \{\sigma, \tau, \mu, \beta\}$,
(ii) K is a $\wedge$-congruence on Λ,
(iii) T is a congruence on Λ,
(iv) $K \cap T = \epsilon_\Lambda$,
(v) Λ has a least element ϵ and a greatest element ω,
(vi) $[\epsilon, \tau]$ and $[\beta, \omega]$ are K-classes,
(vii) $[\epsilon, \mu]$ and $[\sigma, \omega]$ are T-classes,

(viii) if $\beta \wedge (\sigma \vee \mu) = \mu \vee (\sigma \wedge \beta)$, then $(\sigma \vee \mu) \wedge (\tau \vee \beta) = \tau \vee (\sigma \wedge \beta) \vee \mu$.

We can think of c-triples (c for congruence) as objects of a category. For them we shall also need morphisms.

Definition 2.2. Let $(\Lambda; K, T)$ and $(\Lambda'; K', T')$ be c-triples with $\Gamma_\Lambda = \{\sigma, \tau, \mu, \beta\}$ and $\Gamma_{\Lambda'} = \{\sigma', \tau', \mu', \beta'\}$. Define a mapping $\gamma_{\Lambda,\Lambda'}$ by

$$\gamma_{\Lambda,\Lambda'} : \alpha \longrightarrow \alpha' \qquad (\alpha \in \Gamma_\Lambda).$$

A mapping $\varphi : \Lambda \longrightarrow \Lambda'$ is *K-preserving* if

$$\lambda\, K\, \rho \Longrightarrow \lambda\varphi\, K'\, \rho\varphi \qquad (\lambda, \rho \in \Lambda).$$

Then $\varphi : (\Lambda; K, T) \longrightarrow (\Lambda'; K', T')$ is a *morphism* if φ is a K-preservinghomomorphism of Λ onto Λ' which extends $\gamma_{\Lambda,\Lambda'}$; if also φ is injective and $\varphi^{-1} : (\Lambda'; K', T') \longrightarrow (\Lambda; K, T)$ is a morphism, then φ is an *isomorphism*.

We shall also need a generalization of the concept of modularity.

Definition 2.3. Let $(\Lambda; K, T)$ be a c-triple. A sublattice Σ of Λ is *T-modular* if

$$\lambda \le \rho,\ \lambda \wedge \theta = \rho \wedge \theta,\ \lambda \vee \theta = \rho \vee \theta,\ \lambda\, T\, \rho \implies \lambda = \rho \qquad (\lambda, \rho, \theta \in \Sigma).$$

We now show that a c-triple is indeed an abstraction of $(\Lambda_S; K_S, T_S)$ for any regular semigroup S.

Lemma 2.4. *Let S be a regular semigroup. Then $(\Lambda_S; K_S, T_S)$ is a c-triple.*

Proof. In Definition 2.1, items (i)–(vii) are well known. For item (viii), we assume that $\beta \wedge (\sigma \vee \mu) = \mu \vee (\sigma \wedge \beta)$ and let $\lambda = (\sigma \vee \mu) \wedge (\tau \wedge \beta)$ and $\rho = \tau \vee (\sigma \wedge \beta) \vee \mu$. One shows easily that $\tau \le \sigma$, $\mu \le \beta$ so that $\lambda \ge \rho$ and also that $\lambda\, T\, \rho$. Further,

$$\begin{aligned} \ker\ \lambda &= \ker\,(\sigma \vee \mu) \cap \mathit{ker}\,(\tau \vee \beta) = \ker\,(\sigma \vee \mu) \cap \ker\ \beta \\ &= \ker\,[(\sigma \vee \mu) \wedge \beta] = \ker\,[\mu \vee (\sigma \wedge \beta)] \\ &\le \ker\,[\tau \vee \mu \vee (\sigma \wedge \beta)] = \ker\ \rho. \end{aligned}$$

Since $\lambda \ge \rho$, we get $\mathit{ker}\ \lambda \supseteq \ker\ \rho$ and thus $\lambda\, K\, \rho$. This together with $\lambda\, T\, \rho$ yields $\lambda = \rho$, as required. $\square$

Throughout the paper, we let

$$\Gamma = \{\sigma, \tau, \mu, \beta\}.$$

In all diagrams, the K-relation is depicted by full lines, the T-relation by dash-dot lines, the inclusion by broken lines.

3. Relations

We list first the relevant relations and then establish some of their properties and mutual relationships. We conclude the section with a special case.

We shall consider the following relations on the lattice Λ of a c-triple $(\Lambda; K, T)$ where $\Gamma_\Lambda = \{\sigma, \tau, \mu, \beta\}$.

(A) $\tau \leq \sigma$.
(B) $\mu \leq \beta$.
(C) $\sigma \wedge (\tau \vee \beta) = \tau \vee (\sigma \wedge \beta)$.
(D) $\beta \wedge (\sigma \vee \mu) = \mu \vee (\sigma \wedge \beta)$.
(E) $\sigma \wedge (\tau \vee \mu) = \tau \vee (\sigma \wedge \mu)$.
(F) $\beta \wedge (\tau \vee \mu) = \mu \vee (\tau \wedge \beta)$.
(G) $\sigma \wedge (\tau \vee \mu) \wedge \beta = (\sigma \wedge \mu) \vee (\tau \wedge \beta)$.
(H) $(\sigma \vee \mu) \wedge (\tau \vee \beta) = \tau \vee (\sigma \wedge \beta) \vee \mu$.
(I) $\sigma \vee \mu \; K \; \mu$.
(J) $\sigma \wedge \mu \; K \; \sigma$.
(K) $\tau \vee \mu \; K \; \mu$.
(L) $\tau \vee \beta \; T \; \tau$.
(M) $\tau \wedge \beta \; T \; \beta$.

Note the symmetries in the above relations. By the interchanges $\sigma \leftrightarrow \beta$ and $\tau \leftrightarrow \mu$, we get the interchanges

$$(C) \longleftrightarrow (D), \qquad (E) \longleftrightarrow (F).$$

By the interchanges $\sigma \leftrightarrow \mu$, $\tau \leftrightarrow \beta$, $\wedge \leftrightarrow \vee$, we obtain the interchanges

$$(C) \longleftrightarrow (F), \qquad (D) \longleftrightarrow (E), \qquad (G) \longleftrightarrow (H).$$

In the first lemma, we establish some simple but fundamental properties of c-triples.

Lemma 3.1. *Let $(\Lambda; K, T)$ be a c-triple. Then relations $(A) - (C)$ hold. Letting $\mathcal{X}$ and $\mathcal{Y}$ stand for the left and the right side, respectively, of the relations $(D) - (G)$, we have $\mathcal{X}\ T\ \mathcal{Y} \leq \mathcal{X}$.*

Proof. That conditions (A) and (C) hold is the content of [3, Lemma 3.2]. Since $\mu \wedge \beta\ K\ \mu \wedge \omega = \mu$ and $\mu \wedge \beta\ T\ \epsilon \wedge \beta = \epsilon\ T\ \mu$, we get $\mu \wedge \beta = \mu$ and (B) holds. Taking into account that T is a congruence and $\sigma\ T\ \omega$, $\mu\ T\,\epsilon$, the assertions concerning T follow in a straightforward manner. The inclusions follow easily from conditions (A) and (B). □

The next lemma exhibits some useful implications among our relations.

Lemma 3.2. *The following implications hold for c-triples.*

$$\begin{array}{ccccccccccc} (H) & \Longleftarrow & (D) & \Longleftarrow & (I) & \Longrightarrow & (K) & \Longrightarrow & (G) & \Longrightarrow & (E) \\ & & & & \Downarrow & & \Downarrow & & & & \\ & & & & (J) & & (F) & & & & \end{array}$$

Proof. By Definition 2.1(viii), (D) implies (H).
Assume that (I) holds. Then

$$\beta \wedge (\sigma \vee \mu) K\ \omega \wedge \mu = \mu = \mu \wedge [\mu \vee (\sigma \wedge \beta)]$$

$$K\,(\sigma \vee \mu) \wedge [\mu \vee (\sigma \wedge \beta)] = \mu \vee (\sigma \wedge \beta)$$

which together with Lemma 3.1 implies (D). Also using (A), we obtain

$$\tau \vee \mu = (\tau \vee \mu) \wedge (\sigma \vee \mu)\, K\, (\tau \vee \mu) \wedge \mu = \mu$$

and (K) holds. Clearly (J) holds as well.

Assume that (K) holds. First let $\lambda = \beta \wedge (\tau \vee \mu)$ and $\rho = \mu \vee (\tau \wedge \beta)$. Hence by (K),

$$\lambda \wedge \rho\, K\, \omega \wedge \mu \wedge [\mu \vee (\tau \wedge \beta)] = \omega \wedge \mu\, K\, \beta \wedge (\tau \vee \mu) = \lambda$$

and by Lemma 3.1, we get $\lambda \wedge \rho\, T\, \lambda$ so that $\lambda \wedge \rho = \lambda$. But then $\lambda \leq \rho$ which together with Lemma 3.1 yields that $\lambda = \rho$. Therefore (F) holds. Next let $\lambda = \sigma \wedge (\tau \vee \mu) \wedge \beta$ and $\rho = (\sigma \wedge \mu) \vee (\tau \wedge \beta)$. Then

$$\begin{aligned}
\lambda \wedge \rho = {} & \sigma \wedge (\tau \vee \mu) \wedge \beta \wedge [(\sigma \wedge \mu) \vee (\tau \wedge \beta)] \\
& K\, \sigma \wedge \mu \wedge \omega \wedge [(\sigma \wedge \mu) \vee (\tau \wedge \beta)] = \sigma \wedge \mu \wedge \omega && \text{by } (K) \\
& K\, \sigma \wedge (\tau \vee \mu) \wedge \beta = \rho && \text{by } (K)
\end{aligned}$$

and by Lemma 3.1, we get $\lambda \wedge \rho\, T\, \lambda$ so that $\lambda \wedge \rho = \lambda$. But then $\lambda \leq \rho$ which together with Lemma 3.1 yields that $\lambda = \rho$. Therefore also (G) holds.

Assume that (G) holds. Let $\lambda = \sigma \wedge (\tau \vee \mu)$ and $\rho = \tau \vee (\sigma \wedge \mu)$. Then

$$\begin{aligned}
\lambda \wedge \rho = {} & [\sigma \wedge (\tau \vee \mu) \wedge \omega] \wedge \rho\, K\, [\sigma \wedge (\tau \vee \mu) \wedge \beta] \wedge \rho \\
& = [(\sigma \wedge \mu) \vee (\tau \wedge \beta)] \wedge [\tau \vee (\sigma \wedge \mu)] && \text{by } (G) \\
& = (\sigma \wedge \mu) \vee (\tau \wedge \beta) \\
& = \sigma \wedge (\tau \vee \mu) \wedge \beta\, K\, \sigma \wedge (\tau \vee \mu) \wedge \omega && \text{by } (G) \\
& = \sigma \wedge (\tau \vee \mu) = \lambda.
\end{aligned}$$

By Lemma 3.1, we also have $\lambda \wedge \rho \, T \, \lambda$ and thus $\lambda \wedge \rho = \lambda$. But then $\lambda \leq \rho$ which together with Lemma 3.1 yields that $\lambda = \rho$ and (E) holds. ☐

4. Any three elements of Γ generate a distributive lattice

We start with a lemma that provides for the lattice Λ_1 a representation by generators and relations. The main result is supplemented by several diagrams eliciting a property of relation (C) and independence of relations (E), (D) and (F).

Lemma 4.1. *The lattice Λ_1 in Diagram 1 is a free lattice on Γ subject to relations (C), (D), (E), (F).*

Proof. This can be established by applying an unpublished result of Barry Wolk of the University of Manitoba. Its application requires a lengthy verification which is omitted. ☐

We are now ready for the main result of this section.

Theorem 4.2. *The following conditions on a c-triple $(\Lambda; K, T)$ are equivalent.*

(i) *Any three elements of Γ_Λ generate a distributive lattice.*
(ii) *Any three elements of Γ_Λ generate a T-modular lattice.*
(iii) *Λ satisfies relations $(D), (E)$ and (F).*
(iv) *The map $\gamma_{\Lambda_1, \Lambda}$ extends uniquely to a morphism $\delta_1 : (\Lambda_1; K_1, T_1) \longrightarrow (\Lambda; K, T)$ (see Diagram 1).*

Proof. (i) *implies* (ii). It suffices to observe that T-modularity is a weakening of modularity which is in turn a weakening of distributivity.

(ii) *implies* (iii). For each of the relations $(D), (E), (F)$, we must set up the hypothesis of T-modularity.

(D) Let $\lambda = \mu \vee (\sigma \wedge \beta)$ and $\rho = \beta \wedge (\sigma \vee \mu)$. By Lemma 3.1, we have $\lambda \leq \rho$ and $\lambda \, T \, \rho$. Further,

$$\lambda \wedge \sigma = [\mu \vee (\sigma \wedge \beta)] \wedge \sigma \leq (\mu \vee \beta) \wedge \sigma = \beta \wedge \sigma$$

$$= [\mu \vee (\sigma \wedge \beta)] \wedge (\sigma \wedge \beta) \leq [\mu \vee (\sigma \wedge \beta)] \wedge \sigma = \lambda \wedge \sigma$$

and equality prevails;

$$\rho \wedge \sigma = [\beta \wedge (\sigma \vee \mu)] \wedge \sigma = \beta \wedge \sigma$$

and thus $\lambda \wedge \sigma = \beta \wedge \sigma = \rho \wedge \sigma$. Also

$$\lambda \vee \sigma = [\mu \vee (\sigma \wedge \beta)] \vee \sigma = \mu \vee \sigma,$$

$$\rho \vee \sigma = [\beta \wedge (\sigma \vee \mu)] \vee \sigma \leq [\beta \wedge (\sigma \vee \mu)] \vee (\sigma \vee \mu) = \mu \vee \sigma$$

$$= (\beta \wedge \mu) \vee \sigma \leq [\beta \wedge (\sigma \vee \mu)] \vee \sigma = \rho \vee \sigma$$

and equality prevails; thus $\lambda \vee \sigma = \mu \vee \sigma = \rho \vee \sigma$. The hypothesis implies that $\lambda = \rho$ and thus (D) holds.

(E) Let $\lambda = \tau \vee (\sigma \wedge \mu)$ and $\rho = \sigma \wedge (\tau \vee \mu)$. An argument closely similar to the one above, using Lemma 3.1, shows that

$$\lambda \leq \rho, \qquad \lambda \wedge \mu = \rho \wedge \mu, \qquad \lambda \vee \mu = \rho \vee \mu, \qquad \lambda T \rho$$

and the hypothesis implies that $\lambda = \rho$ so that (E) holds.

(F) Let $\lambda = \mu \vee (\tau \wedge \beta)$ and $\rho = \beta \wedge (\tau \vee \mu)$. Again an argument analogous to the one above, using Lemma 3.1, shows that

$$\lambda \leq \rho, \qquad \lambda \wedge \tau = \rho \wedge \tau, \qquad \lambda \vee \tau = \rho \vee \tau, \qquad \lambda T \rho$$

and the hypothesis implies that $\lambda = \rho$, that is (F) holds.

(iii) *implies* (iv). According to Lemma 3.1, relation (C) holds. By Lemma 4.1, Λ_1 is the free lattice on Γ_{Λ_1} subject to relations (C), (D), (E) and (F). Hence the mapping $\gamma_{\Lambda_1,\Lambda}$ extends uniquely to a homomorphism δ_1 of Λ_1 onto Λ. It is easy to check that the K-relation on the lattice Λ_1 is the least $\wedge$-congruence on Λ_1 having $[\epsilon, \tau]$ and $[\beta, \omega]$ as its classes. On the lattice Λ, the K-relation is a $\wedge$-congruence with $[\epsilon, \tau]$ and $[\beta, \omega]$ as its classes. Hence δ_1 preserves the K-relation.

(iv) *implies* (i). Easy inspection shows that any three elements of Γ_{Λ_1} generate a distributive lattice. This is carried over to Λ by δ_1 so that any three elements of Γ_Λ generate a distributive lattice. $\square$

Proposition 4.3. *Let $(\Lambda; K, T)$ be a c-triple satisfying relations $(D), (E), (F)$. Then*

$$[\epsilon, \mu],\ [\tau \wedge \beta, \mu \vee (\tau \wedge \beta)],\ [\tau, \tau \vee \mu],\ [\sigma \wedge \beta, \beta],\ [\sigma \wedge (\tau \vee \beta), \tau \vee \beta],\ [\sigma, \omega]$$

constitutes the complete set of T-classes (with possible collapsing).

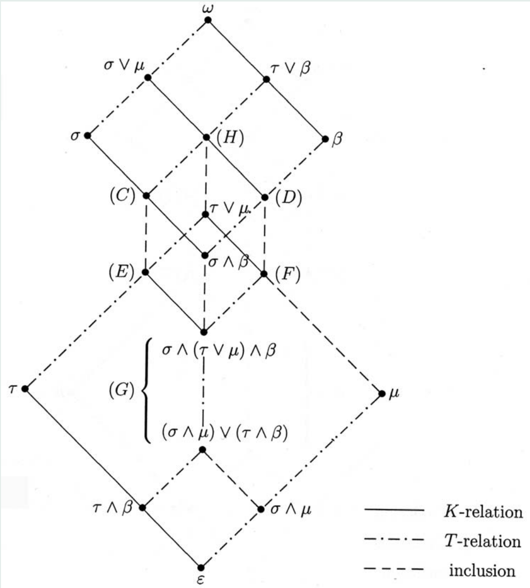

Diagram 1. $(\Lambda_1; K_1, T_1)$.

Proof. By the definition of a c-triple, $[\epsilon, \mu]$ and $[\sigma, \omega]$ are T-classes. The remaining classes are obtained by meets and/or joins of ends of these intervals with suitable elements, see Diagram 1. $\square$

In Diagram 1, the K-relation, is the least possible, for it is obtained from the definition of a c-triple by forming the meet of $\{\beta, \omega\}$ with suitable elements. This is not the only possible K-relation. The T-relation follows the assertion of Proposition 4.3.

5. The distributive case

We start with a representation of the lattice Λ_2 by generators and relations. The main result consists of several equivalent conditions one of which is the requirement that in the c-triple $(\Lambda; K, T)$, Λ be distributive.

Lemma 5.1. *The lattice Λ_2 in Diagram 2 is a free lattice on Γ subject to relations (C), (D), (F), (G).*

Proof. Let θ denote the congruence on the lattice Λ_1 in Diagram 1 induced by (G). Since the elements $\lambda = \sigma \wedge (\tau \vee \mu) \wedge \beta$ and $\rho = (\tau \wedge \beta) \vee (\sigma \wedge \mu)$ act the same way on all other elements of Λ_1, the θ-classes are $\{\lambda, \rho\}$ and singletons. Now noting that by Lemma 3.2, (G) implies (E), the assertion is a direct consequence of Lemma 4.1. $\square$

It was proved in [2, Theorem 4.2] that Λ_2 is the free distributive lattice on Γ_{Λ_2} subject to relations (A) and (B) and that neither of these relations may be omitted.

We can now establish the desired result.

Theorem 5.2. *The following conditions on a c-triple $(\Lambda; K, T)$ are equivalent.*

(i) *The lattice Λ is distributive.*
(ii) *The lattice Λ is T-modular.*
(iii) *Λ satisfies relations (D), (F), (G).*
(iv) *The map $\gamma_{\Lambda_2,\Lambda}$ extends uniquely to a morphism $\delta_2 : (\Lambda_2; K_2, T_2) \longrightarrow (\Lambda; K, T)$ (see Diagram 2).*

Proof. (i) *implies* (ii). Observe that T-modularity is a weakening of modularity which is in turn a weakening of distributivity.

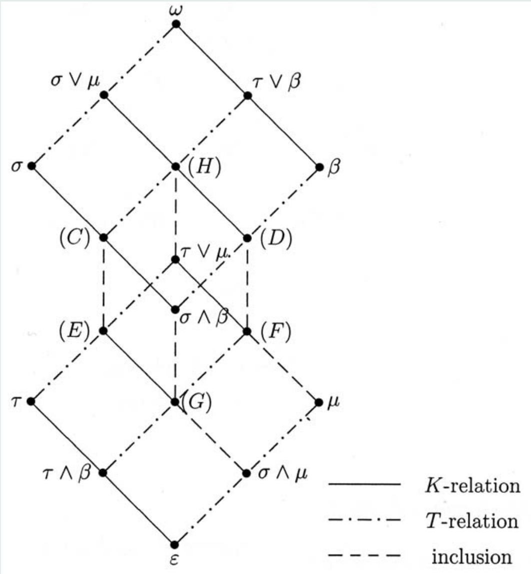

Diagram 2. $(\Lambda_2; K_2, T_2)$.

(ii) *implies* (iii). By Theorem 4.2, Λ satisfies relations (D) and (F). Let $\lambda = (\sigma \wedge \mu) \vee (\tau \wedge \beta)$ and $\rho = \sigma \wedge (\tau \vee \mu) \wedge \beta$. By Lemma 3.1, we have $\lambda \leq \rho$ and $\lambda T \rho$. Further (A) implies

$$\begin{aligned} \lambda \wedge \mu &= [(\sigma \wedge \mu) \vee (\tau \wedge \beta)] \wedge \mu \leq \sigma \wedge \mu \\ &\leq [(\sigma \wedge \mu) \vee (\tau \wedge \beta)] \wedge \mu = \lambda \wedge \mu \end{aligned}$$

and equality prevails;

$$\rho \wedge \mu = [\sigma \wedge (\tau \vee \mu) \wedge \beta] \wedge \mu = \sigma \wedge \mu$$

and thus $\lambda \wedge \mu = \sigma \wedge \mu = \rho \wedge \mu$. Also

$$\lambda \vee \mu = [(\sigma \wedge \mu) \vee (\tau \wedge \beta)] \vee \mu = (\tau \wedge \beta) \vee \mu,$$

and using condition (F), we get

$$\begin{aligned} \rho \vee \mu = [\sigma \wedge (\tau \vee \mu) \wedge \beta] \vee \mu &\leq \beta \wedge (\tau \vee \mu) = \mu \vee (\tau \wedge \beta) \\ &\leq [\sigma \wedge (\tau \vee \mu) \wedge \beta] \vee \mu = \rho \vee \mu \end{aligned}$$

and equality prevails so that $\lambda \vee \mu = (\tau \wedge \beta) \vee \mu = \rho \vee \mu$. The hypothesis implies that $\lambda = \rho$ and thus (G) holds.

(iii) *implies* (iv). By Lemma 3.2, (G) implies (E). Now applying Theorem 4.2, part (iv) of that theorem guarantees the existence of a unique K-preserving homomorphism δ_1 of Λ_1 onto Λ extending $\gamma_{\Lambda_1,\Lambda}$. In view of Lemma 5.1, since S satisfies (G), δ_1 factors into a homomorphism of Λ_1 onto Λ_2 and a homomorphism δ_2 of Λ_2 onto Λ. It is clear that δ_2 is the unique extension of $\gamma_{\Lambda_2,\Lambda}$ to a homomorphism of Λ_2 onto Λ. Since the K-relation on Λ_2 in Diagram 2 is the least one possible, it follows that δ_2 is K-preserving.

(iv) *implies* (i). Since Λ_2 is distributive, so is every of its homomorphic images. $\square$

In Diagram 2, the K-relation is the least K-relation but it is not the only possible one. The T-relation has the classes listed in Proposition 4.3.

6. The case when K is a congruence

We again establish equivalent conditions on a c-triple $(\Lambda; K, T)$ now with K being a congruence on Λ. We start with a description of all c-triples $(\Lambda_2; K, T)$ with Λ_2 as in the preceding section.

Lemma 6.1. *Let $(\Lambda_2; K, T)$ be a c-triple with the lattice Λ_2 depicted in Diagram 2. Then $T = T_2$ as in Diagram 2 and*

$$(1) \qquad [\beta, \omega], \quad [\mu \vee (\sigma \wedge \beta), \sigma \vee \mu], \quad [\sigma \wedge \beta, \sigma], \quad [\epsilon, \tau]$$

are K-classes and for the remaining K-classes, one of the following cases occurs:

Case 1: $[\mu \vee (\tau \wedge \beta), \tau \vee \mu], \{\mu\}, [\sigma \wedge (\tau \vee \mu) \wedge \beta, \tau \vee (\sigma \wedge \mu)], \{\sigma \wedge \mu\}$,

Case 2: $[\mu \vee (\tau \wedge \beta), \tau \vee \mu], \{\mu\}, [\sigma \wedge \mu, \tau \vee (\sigma \wedge \mu)]$,

Case 3: $[\mu, \tau \vee \mu], [\sigma \wedge \mu, \tau \vee (\sigma \wedge \mu)]$.

Only in Case 3 is K a congruence.

Proof. The assertion that $T = T_2$ follows directly from the restrictions on T namely that $[\epsilon, \mu]$ and $[\sigma, \omega]$ are T-classes and T is a congruence. From the requirement that $[\epsilon, \tau]$ and $[\beta, \omega]$ are K-classes and K is a $\wedge$-congruence, we obtain that the intervals listed in (1) are K-classes and $\beta \wedge (\tau \vee \mu) \, K \, \tau \vee \mu$ and $\sigma \wedge (\tau \vee \mu) \wedge \beta \; K \; \sigma \wedge (\tau \vee \mu)$. It is checked readily that in all three cases listed above, the corresponding equivalence relations satisfy all the requisite conditions. In Cases 1 and 2, we have $\epsilon \, K \, \tau$ but $\mu \, K \, \tau \vee \mu$ fails so K is not a congruence. In Case 3 it is easy to see that K is a congruence (the same way as for T). $\square$

Notation 6.2. Let $(\Lambda_2; K_2', T_2)$ be the triple with K_2' the equivalence relation on Λ_2 with classes as in Case 3 of Lemma 6.1. By Lemma 6.1, $(\Lambda_2; K_2', T_2)$ is a c-triple with K_2' a congruence.

We can now establish the desired result.

Theorem 6.3. *The following conditions on a c-triple $(\Lambda; K, T)$ are equivalent.*

(i) *K is a congruence.*
(ii) *Λ is distributive and satisfies relation (K).*
(iii) *Λ satisfies relations (D) and (K).*
(iv) *The map $\gamma_{\Lambda_2,\Lambda}$ extends uniquely to a morphism $\delta_2' : (\Lambda_2; K_2', T_2) \longrightarrow (\Lambda; K, T)$ (see Notation 6.2).*

Proof. (i) *implies* (ii). Indeed,

$$\beta \wedge (\sigma \vee \mu)\, K\, \omega \wedge (\sigma \vee \mu) = \sigma \vee \mu = (\sigma \wedge \omega) \vee \mu\, K\, (\sigma \wedge \beta) \vee \mu$$

which together with Lemma 3.1(d) yields (D). Also $\tau \vee \mu\, K\, \epsilon \vee \mu = \mu$ so (H) holds. Now Lemma 3.2 implies that (F) and (G) are valid which by Theorem 5.2 gives that Λ is distributive.

(ii) *implies* (iii). This follows directly from Theorem 5.2.

(iii) *implies* (iv). By Lemma 3.2, (K) implies (F) and (G). Now Theorem 5.2 implies that $\gamma_{\Lambda_2,\Lambda}$ can be extended uniquely to a morphism $\delta_2 : (\Lambda_2; K_2, T_2) \longrightarrow (\Lambda; K, T)$. Hence (H) implies that $\delta_2 : (\Lambda_2; K_2', T_2) \longrightarrow (\Lambda; K, T)$ is a morphism.

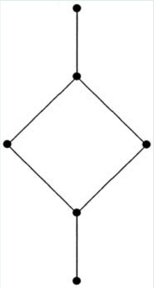

Diagram 3. .

(iv) *implies* (i). Define a relation $\overline{K}$ by

$$\lambda \, \overline{K} \, \rho \Longleftrightarrow \lambda\delta_2 \; K \; \rho\delta_2 \qquad (\lambda, \rho \in \Lambda_2).$$

By straightforward checking we see that $\overline{K}$ is a $\wedge$-congruence on Λ. If $\lambda, \rho \in \Lambda_2$ are such that $\lambda \, K_2' \, \rho$, then by hypothesis $\lambda\delta_2 \; K \; \rho\delta_2$ and thus $\lambda \, \overline{K} \, \rho$. It follows that $K_2' \subseteq \overline{K}$. Also $\overline{K}$-classes are convex and are unions of K_2'-classes. The lattice Λ_2/K_2' is depicted in Diagram 3.

It now follows easily that by forming convex unions ofK_2'-classes in Λ_2 we obtain a congruence on Λ_2. Therefore $\overline{K}$ is a congruence.

Let $\lambda, \rho, \theta \in \Lambda_2$ be such that $\lambda\delta_2 \, K \, \rho\delta_2$. Then $\lambda \, \overline{K} \, \rho$ and thus $\lambda \vee \theta \; \overline{K} \; \rho \vee \theta$ which implies that

$$\lambda\delta_2 \vee \theta\delta_2 = (\lambda \vee \theta)\delta_2 \; K \, (\rho \vee \theta)\delta_2 = \rho\delta_2 \vee \theta\delta_2.$$

Therefore K is a congruence on Λ. □

We have the following uniqueness statement.

Corollary 6.4. *Let* $(\Lambda; K, T)$ *and* $(\Lambda; K', T')$ *be c-triples satisfying relations* (D) *and* (K)*. Then* $K = K'$ *and* $T = T'$*.*

Proof. In view of Theorem 6.3 and Lemma 6.1, the set listed in Lemma 6.1 under Case 3 constitutes the complete collection of K-classes. It follows that $K = K'$. The assertion $T = T'$ is a consequence of Proposition 4.3. □

7. A special case

We consider here the case when K is a congruence and $\sigma \wedge \beta \leq \tau \vee \mu$. The principal interest of this case is that for any $\rho \in \Gamma$, the values of ρ_K, ρ_T, ρ^K, ρ^T can be given a simple explicit formula. We treat this case in the same manner as we did the other cases; in addition we start and finish with some ramifications of the restrictions involved.

Proposition 7.1. *The lattice Λ_3 in Diagram 3 is a free lattice on Γ subject to relations (C), (D), (F) and*

$$(\sigma \wedge \mu) \vee (\tau \wedge \beta) = \sigma \wedge \beta. \tag{2}$$

Proof. Let ρ denote the congruence on the lattice Λ_1 in Diagram 1 induced by the relation (2). It is easy to see that the ρ-classes are the four vertical intervals in the upper part of Diagram 1. By Lemma 3.2, (G) implies (E). The assertion now follows from Lemma 4.1. □

For a c-triple $(\Lambda; K, T)$ and $\rho \in \Lambda$, we denote by ρK and ρT the K-and T-classes of ρ, respectively. If these classes are intervals, we write

$$\rho K = [\rho_K, \rho^K], \qquad \rho T = [\rho_T, \rho^T].$$

We start with a condition on the K-relation.

Lemma 7.2. *The following conditions on a c-triple $(\Lambda; K, T)$ are equivalent.*

(i) *Relations $(D), (K)$ and (J) hold.*
(ii) *Relation (I) holds.*
(iii) $\mu^K = \sigma \vee \mu$.
(iv) $\sigma_K = \sigma \wedge \mu$.

Proof. (i) *implies* (ii). By Theorem 6.3, K is a congruence which together with (J) implies that $(\sigma \vee \mu)\, K\, (\sigma \wedge \mu) \vee \mu = \mu$ and (I) holds.

(ii) *implies* (iii). Let $\theta \in \Lambda$ be such that $\theta\ K\ \mu$. Then

$$(\sigma \vee \mu) \wedge \theta\, K\, (\sigma \vee \mu) \wedge \mu = \mu\ K\ \theta,$$
$$(\sigma \vee \mu) \wedge \theta\ T\ (\omega \vee \mu) \wedge \theta = \theta$$

and thus $(\sigma \vee \mu) \wedge \theta = \theta$ so that $\theta \leq \sigma \vee \mu$. Therefore $\mu^K = \sigma \vee \mu$.

(iii) *implies* (iv). The hypothesis implies that $\mu\,K\,\sigma \vee \mu$ which yields

$$\sigma \wedge \mu\,K\,\sigma \wedge (\sigma \vee \mu) = \sigma.$$

Now $\operatorname{tr}(\sigma \wedge \mu) = \varepsilon$ and $\sigma \wedge \mu$ has the least possible T-value. But then $\sigma_K = \sigma \wedge \mu$.

(iv) *implies* (i). Since $\mu \leq \mu \vee (\sigma \wedge \beta) \leq \mu \vee \sigma$ and $\mu\,K\,\mu \vee \sigma$, we get

$$\mu \vee (\sigma \wedge \beta)\,K\,\mu \vee \sigma = \omega \wedge (\mu \vee \sigma)\,K\,\beta \wedge (\mu \vee \sigma)$$

and (D) holds. Similarly, $\mu \leq \tau \vee \mu \leq \sigma \vee \mu$ implies that $\mu\,K\,\tau \vee \mu$ and (K) holds. Also $\mu\,K\,\sigma \vee \mu$ implies that $\sigma \wedge \mu\,K\,\sigma \wedge (\sigma \vee \mu) = \sigma$ whence (J) holds. □

We consider next a condition on the T-relation.

Lemma 7.3. *The following conditions on a c-triple $(\Lambda; K, T)$ are equivalent.*

(i) *Relation (M) holds.*
(ii) *Relation (L) holds.*
(iii) $\tau^T = \tau \vee \beta$.
(iv) $\beta_T = \tau \wedge \beta$.

Proof. (i) *implies* (ii). Indeed, $\tau \vee (\tau \wedge \beta)\,T\,\tau \vee \beta$ whence $\tau\,T\,\tau \vee \beta$ and (L) holds.

(ii) *implies* (iii). Let $\theta \in \Lambda$ be such that $\theta\,T\,\tau$. Then

$$(\tau \vee \beta) \wedge \theta\;K\;\omega \wedge \theta = \theta, \qquad (\tau \vee \beta) \wedge \theta\;T\;(\tau \vee \beta) \wedge \tau = \tau\;T\;\theta$$

and thus $(\tau \vee \beta) \wedge \theta = \theta$ so that $\theta \leq \tau \vee \beta$. Therefore $\tau^T = \tau \vee \beta$.

(iii) *implies* (iv). The hypothesis implies that $\tau\;T\;\tau \vee \beta$ which yields

$$\tau \wedge \beta\,T\,(\tau \vee \beta) \wedge \beta = \beta$$

Now $\tau \wedge \beta\;K\;\epsilon \wedge \beta = \epsilon$ so $\tau \wedge \beta$ has the least possible K-value. But then $\beta_T = \tau \wedge \beta$.

(iv) *implies* (i). This is trivial. □

Lemma 7.4. *The identity mapping on* Γ *extends uniquely to a* K*-preserving homomorphism* ψ *of* Λ_2 *onto* Λ_3 *inducing the congruence generated by the relation* $\sigma \wedge \beta \leq \tau \vee \mu$*.*

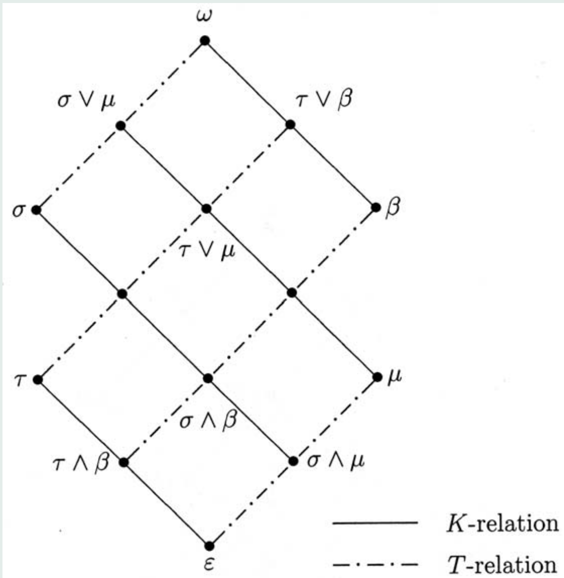

Diagram 4. $(\Lambda_3; K_3, T_3)$.

Proof. Let ρ be the congruence on Λ_2 induced by the relation $\sigma \wedge \beta \leq \tau \vee \mu$. Then ρ is generated by the pair

$$(\sigma \wedge \beta, \sigma \wedge (\tau \vee \mu) \wedge \beta).$$

From Diagram 2, we immediately see that ρ has classes the vertical intervals in the diagram. It is now clear that $\Lambda_2/\rho \cong \Lambda_3$. The induced homomorphism ψ of Λ_2 onto Λ_3 is obviously a unique extension of $\gamma_{\Lambda_2,\Lambda_3}$ and is K-preserving. □

We are now ready for the main result of this section.

Theorem 7.5. *The following conditions on a c-triple* $(\Lambda; K, T)$ *are equivalent.*

(i) K *is a congruence and* $\sigma \wedge \beta \leq \tau \vee \mu$.
(ii) Λ *satisfies relations* (I) *and* (L).
(iii) $\mu^K = \sigma \vee \mu$, $\tau^T = \tau \vee \beta$.
(iv) *The map* $\gamma_{\Lambda_3,\Lambda}$ *extends uniquely to a morphism* $\delta_3 : (\Lambda_3; K_3, T_3) \longrightarrow (\Lambda; K, T)$ *(see Diagram 4).*

Proof. (i) *implies* (ii). By Theorem 6.3, we have that Λ satisfies (D) and (K). The hypothesis implies that

$$\sigma = \sigma \wedge \omega \; K \; \sigma \wedge \beta = \sigma \wedge \beta \wedge (\tau \vee \mu) \; K \; \sigma \wedge \omega \wedge (\epsilon \vee \mu) = \sigma \wedge \mu.$$

Now Lemma 7.2 yields that (I) holds. In addition,

$$\beta = \omega \wedge \beta \; T \; \sigma \wedge \beta = \sigma \wedge \beta \wedge (\tau \vee \mu) \; T \; \omega \wedge \beta \wedge (\tau \vee \epsilon) = \beta \wedge \tau$$

which by Lemma 7.3 implies that (L) holds.

(ii) *implies* (iii). This follows directly from Lemmas 7.2 and 7.3.

(iii) *implies* (i). By Lemmas 7.3 and 7.4, we have the validity of (D), (K), (J) and (M). Hence Theorem 6.3 implies that K is a congruence and thus

$$\sigma \wedge \beta \wedge (\tau \vee \mu) \; K \; \sigma \wedge \omega \wedge (\epsilon \wedge \mu) = \sigma \wedge \mu \; K \; \sigma = \sigma \wedge \omega \; K \; \sigma \wedge \beta,$$

$$\sigma \wedge \beta \wedge (\tau \vee \mu) \; T \; \omega \wedge \beta \wedge (\tau \vee \epsilon) = \beta \wedge \tau \; T \; \beta = \omega \wedge \beta \; T \; \sigma \wedge \beta$$

whence $\sigma \wedge \beta \wedge (\tau \vee \mu) = \sigma \wedge \beta$, that is $\sigma \wedge \beta \leq \tau \vee \mu$.

(i) *implies* (iv). By Theorem 6.3, the map $\gamma_{\Lambda_2,\Lambda}$ extends uniquely to a K-preserving homomorphism δ_2 of Λ_2 onto Λ. Since Λ satisfies $\sigma \wedge \beta \leq \tau \vee \mu$, Lemma 7.4 implies that δ_2 factors through the mapping ψ in that lemma which implies that δ_3 has the enunciated property.

(iv) *implies* (i). With the mapping ψ in Lemma 7.4, and the extension δ_3 in the hypothesis, we obtain that the composition $\psi\delta_3$ is a unique extension of $\gamma_{\Lambda_2,\Lambda}$ to a K-preserving homomorphism of Λ_2 onto Λ. Now Theorem 6.3 implies that K is a congruence. Since the lattice Λ_3 obviously satisfies $\sigma \wedge \beta \leq \tau \vee \mu$, so does its homomorphic image Λ. $\square$

For a regular semigroup S, we may interpret $\rho_K, \rho_T, \rho^K, \rho^T$ not only within Λ_S, as above, but also within the congruence lattice $\mathcal{C}(S)$ of S by an essentially identical argument.

Any cryptogroup S (completely regular semigroup with $\mathcal{H}$ a congruence) satisfies the conditions of Theorem 7.5. Indeed, in S we have $\ker \mu = S$ so $\sigma \vee \mu \; K \; \mu$ and (I) holds, $\operatorname{tr} \beta = \epsilon$ so $\tau \vee \beta \; T \; \tau$ and (J) holds. We also have $\beta = \mu$ which implies that $\tau \wedge \beta = \epsilon$ and $\sigma \vee \mu = \omega$; see Diagram 4.

Independence of the relations and realization of c-triples by regular semigroups will be the subject of a subsequent communication.

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