

# APPROXIMATION OF FUNCTIONS OF TWO VARIABLES BY SOME LINEAR POSITIVE OPERATORS

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**ABSTRACT.** We introduce certain positive linear operators in weighted spaces of functions of two variables and we study approximation properties of these operators. We give theorems on the degree of approximation of functions from polynomial and exponential weighted spaces by introduced operators, using norms of these spaces.

Approximation properties of Szasz-Mirakyan operators

$$(1) \quad S_n(f; x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

$x \in R_0 = [0, +\infty)$ ,  $n \in N := \{1, 2, \dots\}$ , in polynomial weighted spaces  $C_p$  were examined in [1]. The space  $C_p$ ,  $p \in N_0 := \{0, 1, 2, \dots\}$ , is associated with the weighted function

$$(2) \quad w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1}, \quad \text{if } p \geq 1,$$

and consists of all real-valued functions  $f$ , continuous on  $R_0$  and such that  $w_p f$  is uniformly continuous and bounded on  $R_0$ . The norm on  $C_p$  is defined by the formula

$$(3) \quad \|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x \in R_0} w_p(x) |f(x)|.$$

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In [1] there were proved theorems on the degree of approximation of  $f \in C_p$  by the operators  $S_n$  defined by (1). From these theorems it was deduced that

$$(4) \quad \lim_{n \rightarrow \infty} S_n(f; x) = f(x),$$

for every  $f \in C_p$ ,  $p \in N_0$  and  $x \in R_0$ . Moreover the convergence (4) is uniform on every interval  $[x_1, x_2]$ ,  $x_2 > x_1 \geq 0$ .

The Szasz-Mirakyan operators are important in approximation theory. They have been studied intensively, in connection with different branches of analysis, such as numerical analysis. Recently in many papers various modifications of  $S_n$  were introduced [4]–[8], [12]–[15], [19], [20]. Approximation properties of modified Szasz-Mirakyan operators

$$(5) \quad L_n(f; r; x) := \frac{1}{g((nx+1)^2; r)} \sum_{k=0}^{\infty} \frac{(nx+1)^{2k}}{(k+r)!} f\left(\frac{k+r}{n(nx+1)}\right),$$

$$x \in R_0, \quad n \in N,$$

where

$$(6) \quad g(t; r) := \sum_{k=0}^{\infty} \frac{t^k}{(k+r)!}, \quad t \in R_0,$$

i.e.

$$g(0; r) = \frac{1}{r!}, \quad g(t, r) = \frac{1}{t^r} \left( e^t - \sum_{j=0}^{r-1} \frac{t^j}{j!} \right) \quad \text{if } t > 0,$$

in polynomial weighted spaces were examined in [13].

In [13] it was proved that if  $f \in C_p$ ,  $p \in N_0$ , then

$$(7) \quad \|L_n(f; r; \cdot) - f(\cdot)\|_p \leq M_1 \omega_1 \left( f; C_p; \frac{1}{n} \right), \quad n, r \in N,$$

where

$$(8) \quad \omega_1(f; C_p; t) := \sup_{0 \leq h \leq t} \|\Delta_h f(\cdot)\|_p, \quad t \in R_0,$$

where  $\Delta_h f(x) := f(x+h) - f(x)$  for  $x, h \in R_0$  and  $M_1 = \text{const} > 0$ .

In particular, if  $f \in C_p^1$ ,  $p \in N_0$ , then

$$(9) \quad \|L_n(f; r; \cdot) - f(\cdot)\|_p \leq \frac{M_2}{n}, \quad n, r \in N,$$

where  $M_2 = \text{const} > 0$ . The above inequalities estimate the rate of uniform convergence of  $\{L_n(f; r; \cdot)\}$

In [14] there were proved theorems on the degree of approximation of  $f \in C_p$  by operators  $A_n$  defined by

$$(10) \quad A_n(f; r; \alpha; x) := \frac{1}{g((n^\alpha x + 1)^2; r)} \sum_{k=0}^{\infty} \frac{(n^\alpha x + 1)^{2k}}{(k+r)!} f\left(\frac{k+r}{n^\alpha(n^\alpha x + 1)}\right).$$

The degree of approximation is similar and in some cases better than for approximation by  $L_n$ .

Similar results in exponential weighted spaces can be found in [15], [17].

Thus the question arises, whether the operators introduced in [18] for function of two variables can be similarly modified. In connection with this question we introduce the operators (15).

## 1. Preliminaries

**1.1.** For given  $p, q \in N_0$ , we define the weighted function

$$(11) \quad w_{p,q}(x, y) := w_p(x)w_q(y), \quad (x, y) \in R_0^2 := R_0 \times R_0,$$

and the weighted space  $C_{p,q}$  of all real-valued functions  $f$  continuous on  $R_0^2$  for which  $w_{p,q}f$  is uniformly continuous and bounded on  $R_0^2$ . The norm on  $C_{p,q}$  is defined by the formula

$$(12) \quad \|f\|_{p,q} \equiv \|f(\cdot, \cdot)\|_{p,q} := \sup_{(x,y) \in R_0^2} w_{p,q}(x, y) |f(x, y)|.$$

The modulus of continuity of  $f \in C_{p,q}$  we define as usual by the formula

$$(13) \quad \omega(f, C_{p,q}; t, s) := \sup_{0 \leq h \leq t, 0 \leq \delta \leq s} \|\Delta_{h,\delta} f(\cdot, \cdot)\|_{p,q}, \quad t, s \geq 0,$$

where  $\Delta_{h,\delta} f(x, y) := f(x+h, y+\delta) - f(x, y)$  and  $(x+h, y+\delta) \in R_0^2$ . Moreover let  $C_{p,q}^1$  be the set of all functions  $f \in C_{p,q}$  which first partial derivatives belong also to  $C_{p,q}$ .

From (13) it follows that

$$(14) \quad \lim_{t,s \rightarrow 0^+} \omega(f, C_{p,q}; t, s) = 0$$

for every  $f \in C_{p,q}$ ,  $p, q \in N_0$ .

**1.2.** In this paper we introduce the following class of operators in  $C_{p,q}$ .

**Definition 1.** Fix  $r, s \in N := \{1, 2, \dots\}$  and  $\alpha > 0$ . Define a class of operators  $A_{m,n}(f; r, s, \alpha)$  by the formula

$$(15) \quad A_{m,n}(f; r, s, \alpha; x, y) \equiv A_{m,n}(f; x, y) := \frac{1}{g((m^\alpha x + 1)^2; r) g((n^\alpha y + 1)^2; s)} \\ \cdot \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(m^\alpha x + 1)^{2j}}{(j+r)!} \frac{(n^\alpha y + 1)^{2k}}{(k+s)!} f\left(\frac{j+r}{m^\alpha(m^\alpha x + 1)}, \frac{k+s}{n^\alpha(n^\alpha y + 1)}\right)$$

for  $(x, y) \in R_0^2$ ,  $m, n \in N$ .

The methods used to prove the Lemmas and the Theorems are similar to those used in construction of modified Szasz-Mirakyan operators [16], [18].

From (15), (10), (6) we deduce that  $A_{m,n}(f; r, s)$  are well defined in every space  $C_{p,q}$ ,  $p, q \in N_0$ . Moreover for fixed  $r, s \in N$  and  $\alpha > 0$  we have

$$(16) \quad A_{m,n}(1; r, s, \alpha; x, y) = 1 \quad \text{for } (x, y) \in R_0^2, \quad m, n \in N,$$

and if  $f \in C_{p,q}$  and  $f(x, y) = f_1(x)f_2(y)$  for all  $(x, y) \in R_0^2$ , then

$$(17) \quad A_{m,n}(f; r, s, \alpha; x, y) = A_m(f_1; r, \alpha; x)A_n(f_2; s, \alpha; y)$$

for all  $(x, y) \in R_0^2$  and  $m, n \in N$ .

In this paper by  $M_k(\beta_1, \beta_2)$  we shall denote suitable positive constants depending only on indicated parameters  $\beta_1, \beta_2$ .

## 2. Lemmas and theorems

**2.1.** In this section we shall give some properties of the above operators, which we shall apply to the proofs of the main theorems.

From (10) and (6) we get for  $x \in R_0$  and  $n \in N$

$$(18) \quad \begin{aligned} A_n(1; r, \alpha; x) &= 1, \\ A_n(t-x; r, \alpha; x) &= \frac{1}{n^\alpha} + \frac{1}{n^\alpha(n^\alpha x + 1)(r-1)!g((n^\alpha x + 1)^2; r)} \end{aligned}$$

$$(19) \quad A_n((t-x)^2; r, \alpha; x) = \frac{2}{n^{2\alpha}} + \frac{r + (n^\alpha x + 1)^2 - 2n^\alpha x(n^\alpha x + 1)}{n^{2\alpha}(n^\alpha x + 1)^2(r-1)!g((n^\alpha x + 1)^2; r)}.$$

In the paper [14] was proved the following lemma for  $A_n(f; r, \alpha)$  defined by (10).

**Lemma 1.** For every fixed  $p \in N_0$ ,  $r \in N$  and  $\alpha > 0$  there exist positive constants  $M_i \equiv M_i(p, r)$ ,  $i = 3, 4$ , such that for all  $x \in R_0$ ,  $n \in N$

$$(20) \quad w_p(x) A_n(1/w_p(t); r, \alpha; x) \leq M_1,$$

$$(21) \quad w_p(x) A_n((t-x)^2/w_p(t); r, \alpha; x) \leq \frac{M_2}{n^{2\alpha}}.$$

Applying Lemma 1 we shall prove the main lemma on  $A_{m,n}$  defined by (15).

**Lemma 2.** Fix  $p, q \in N_0$ ,  $r, s \in N$  and  $\alpha > 0$ . Then there exists a positive constant  $M_5 \equiv M_5(p, q, r, s)$  such that

$$(22) \quad \|A_{m,n}(1/w_{p,q}(t, z); r, s, \alpha; \cdot, \cdot)\|_{p,q} \leq M_5 \quad \text{for } m, n \in N.$$

Moreover for every  $f \in C_{p,q}$  we have

$$(23) \quad \|A_{m,n}(f; r, s, \alpha; \cdot, \cdot)\|_{p,q} \leq M_5 \|f\|_{p,q} \quad \text{for } m, n \in N, r, s \in N.$$

The formulas (15), (5) and the inequality (23) show that  $A_{m,n}$ ,  $m, n \in N$ , defined by (15) are linear positive operators from the space  $C_{p,q}$  into  $C_{p,q}$ .

*Proof.* The inequality (22) follows immediately from (11), (17) and (20).

From (15) and (12) we get for  $f \in C_{p,q}$  and  $r, s \in N$

$$\|A_{m,n}(f; r, s, \alpha)\|_{p,q} \leq \|f\|_{p,q} \|A_{m,n}(1/w_{p,q}; r, s, \alpha)\|_{p,q}, \quad m, n \in N,$$

which by (22) implies (23). This completes the proof of Lemma 2. □

**2.2.** Now we shall give two theorems on the degree of approximation of functions by  $A_{m,n}$ .

**Theorem 1.** Suppose that  $f \in C_{p,q}^1$  with fixed  $p, q \in N_0$ . Then there exists a positive constant  $M_6 = M_6(p, q, r, s)$  such that for all  $m, n \in N$  and  $r, s \in N$

$$(24) \quad \|A_{m,n}(f; r, s, \alpha; \cdot, \cdot) - f(\cdot, \cdot)\|_{p,q} \leq M_4 \left\{ \frac{1}{m^\alpha} \|f'_x\|_{p,q} + \frac{1}{n^\alpha} \|f'_y\|_{p,q} \right\}.$$

*Proof.* Let  $(x, y) \in R_0^2$  be a fixed point. Then for  $f \in C_{p,q}^1$  we have

$$f(t, z) - f(x, y) = \int_x^t f'_u(u, z) du + \int_y^z f'_v(x, v) dv, \quad (t, z) \in R_0^2.$$

From this and by (16) we get

$$(25) \quad A_{m,n}(f(t, z); r, s, \alpha; x, y) - f(x, y) = A_{m,n} \left( \int_x^t f'_u(u, z) du; r, s, \alpha; x, y \right) + A_{m,n} \left( \int_y^z f'_v(x, v) dv; r, s, \alpha; x, y \right).$$

By (2), (11), (12) we have

$$\left| \int_x^t f'_u(u, z) du \right| \leq \|f'_x\|_{p,q} \left| \int_x^t \frac{du}{w_{p,q}(u, z)} \right| \leq \|f'_x\|_{p,q} \left( \frac{1}{w_{p,q}(t, z)} + \frac{1}{w_{p,q}(x, z)} \right) |t - x|,$$

which by (2), (10) (11), (15) and (16)–(18) implies that

$$\begin{aligned} w_{p,q}(x, y) \left| A_{m,n} \left( \int_x^t f'_u(u, z) du; r, s, \alpha; x, y \right) \right| &\leq w_{p,q}(x, y) A_{m,n} \left( \left| \int_x^t f'_u(u, z) du \right|; r, s, \alpha; x, y \right) \\ &\leq \|f'_x\|_{p,q} w_{p,q}(x, y) \left\{ A_{m,n} \left( \frac{|t - x|}{w_{p,q}(t, z)}; r, s, \alpha; x, y \right) \right. \\ &\quad \left. + A_{m,n} \left( \frac{|t - x|}{w_{p,q}(x, z)}; r, s, \alpha; x, y \right) \right\} \\ &\leq \|f'_x\|_{p,q} w_q(y) A_n \left( \frac{1}{w_q(z)}; s; \alpha y \right) \\ &\quad \cdot \left\{ w_p(x) A_m \left( \frac{|t - x|}{w_p(t)}; r, \alpha; x \right) + A_m(|t - x|; r; x) \right\}. \end{aligned}$$



Applying the Hölder inequality and (18)–(21), we get

$$\begin{aligned} A_m(|t-x|; r, \alpha; x) &\leq \left\{ A_m((t-x)^2; r, \alpha; x) A_m(1; r, \alpha; x) \right\}^{\frac{1}{2}} \\ &\leq \frac{M_7(p, r)}{m^\alpha}, \end{aligned}$$

$$\begin{aligned} w_p(x) A_m\left(\frac{|t-x|}{w_p(t)}; r, \alpha; x\right) \\ &\leq \left\{ w_p(x) A_m\left(\frac{(t-x)^2}{w_p(t)}; r, \alpha; x\right) \right\}^{\frac{1}{2}} \left\{ w_p(x) A_m\left(\frac{1}{w_p(t)}; r, \alpha; x\right) \right\}^{\frac{1}{2}} \\ &\leq \frac{M_8(p, r)}{m^\alpha} \end{aligned}$$

for  $x \in R_0$  and  $m \in N$ . This implies that

$$w_{p,q}(x, y) \left| A_{m,n} \left( \int_x^t f'_u(u, z) du; r, s, \alpha; x, y \right) \right| \leq \frac{M_9(p, q, r, s)}{m^\alpha} \|f'_x\|_{p,q}, \quad m \in N.$$

Analogously we obtain

$$w_{p,q}(x, y) \left| A_{m,n} \left( \int_y^z f'_v(x, v) dv; r, s, \alpha; x, y \right) \right| \leq \frac{M_{10}(p, q, r, s)}{n^\alpha} \|f'_y\|_{p,q}, \quad n \in N.$$

Combining these estimations, we derive from (25)

$$w_{p,q}(x, y) |A_{m,n}(f; r, s; x, y) - f(x, y)| \leq M_{11} \left\{ \frac{1}{m^\alpha} \|f'_x\|_{p,q} + \frac{1}{n^\alpha} \|f'_y\|_{p,q} \right\},$$

for all  $m, n \in N$ , where  $M_{11} = M_{11}(p, q, r, s) = \text{const} > 0$ . This ends the proof of (24). □

**Theorem 2.** Suppose that  $f \in C_{p,q}$ ,  $p, q \in N_0$ . Then there exists a positive constant  $M_{11} \equiv M_{11}(p, q, r, s)$  such that

$$(26) \quad \|A_{m,n}(f; r, s, \alpha; \cdot, \cdot) - f(\cdot, \cdot)\|_{p,q} \leq M_{11} \omega \left( f, C_{p,q}; \frac{1}{m^\alpha}, \frac{1}{n^\alpha} \right)$$

for all  $m, n \in N$ ,  $r, s \in N$  and  $\alpha > 0$ .

*Proof.* We apply the Steklov function  $f_{h,\delta}$  for  $f \in C_{p,q}$

$$(27) \quad f_{h,\delta}(x, y) := \frac{1}{h\delta} \int_0^h du \int_0^\delta f(x+u, y+v) dv, \quad (x, y) \in R_0^2, h, \delta > 0.$$

From (27) it follows that

$$\begin{aligned} f_{h,\delta}(x, y) - f(x, y) &= \frac{1}{h\delta} \int_0^h du \int_0^\delta \Delta_{u,v} f(x, y) dv, \\ (f_{h,\delta})'_x(x, y) &= \frac{1}{h\delta} \int_0^\delta (\Delta_{h,v} f(x, y) - \Delta_{0,v} f(x, y)) dv, \\ (f_{h,\delta})'_y(x, y) &= \frac{1}{h\delta} \int_0^h (\Delta_{u,\delta} f(x, y) - \Delta_{u,0} f(x, y)) du. \end{aligned}$$

This implies that  $f_{h,\delta} \in C_{p,q}^1$  for  $f \in C_{p,q}$  and  $h, \delta > 0$ . Moreover

$$(28) \quad \|f_{h,\delta} - f\|_{p,q} \leq \omega(f, C_{p,q}; h, \delta),$$

$$(29) \quad \|(f_{h,\delta})'_x\|_{p,q} \leq 2h^{-1} \omega(f, C_{p,q}; h, \delta),$$

$$(30) \quad \|(f_{h,\delta})'_y\|_{p,q} \leq 2\delta^{-1} \omega(f, C_{p,q}; h, \delta),$$

for all  $h, \delta > 0$ . Observe that

$$\begin{aligned} w_{p,q}(x, y) |A_{m,n}(f; r, s, \alpha; x, y) - f(x, y)| \\ \leq w_{p,q}(x, y) \{ |A_{m,n}(f(t, z)f_{h,\delta}(t, z); r, s, \alpha; x, y)| \\ + |A_{m,n}(f_{h,\delta}(t, z); r, s, \alpha; x, y) - f_{h,\delta}(x, y)| \\ + |f_{h,\delta}(x, y) - f(x, y)| \} := T_1 + T_2 + T_3. \end{aligned}$$

By (12), (23) and (28) we obtain

$$\begin{aligned} T_1 \leq \|A_{m,n}(f - f_{h,\delta}; r, s, \alpha; \cdot, \cdot)\|_{p,q} \leq M_5 \|f - f_{h,\delta}\|_{p,q} \leq M_5 \omega(f, C_{p,q}; h, \delta), \\ T_3 \leq \omega(f, C_{p,q}; h, \delta). \end{aligned}$$

Applying Theorem 1 and (29) and (30), we get

$$\begin{aligned} T_2 \leq M_6 \left\{ \frac{1}{m^\alpha} \|(f_{h,\delta})'_x\|_{p,q} + \frac{1}{n^\alpha} \|(f_{h,\delta})'_y\|_{p,q} \right\} \\ \leq 2M_6 \omega(f, C_{p,q}; h, \delta) \left\{ h^{-1} \frac{1}{m^\alpha} + \delta^{-1} \frac{1}{n^\alpha} \right\}. \end{aligned}$$

From the above we deduce that there exists a positive constant  $M_{13} \equiv M_{13}(p, q, r, s)$  such that

$$(31) \quad \|A_{m,n}(f; r, s, \alpha; \cdot, \cdot) - f(\cdot, \cdot)\|_{p,q} \leq M_{13} \omega(f, C_{p,q}; h, \delta) \left\{ 1 + h^{-1} \frac{1}{m^\alpha} + \delta^{-1} \frac{1}{n^\alpha} \right\},$$

for  $m, n \in N$  and  $h, \delta > 0$ . Now, for  $m, n \in N$  setting  $h = \frac{1}{m^\alpha}$  and  $\delta = \frac{1}{n^\alpha}$  to (31), we obtain (26).  $\square$

From Theorem 2 and the property (14) it follows that

**Corollary.** Let  $f \in C_{p,q}$ ,  $p, q \in N_0$ . Then for  $r, s \in N$  and  $\alpha > 0$  we have

$$(32) \quad \lim_{m,n \rightarrow \infty} \|A_{m,n}(f; r, s, \alpha; \cdot, \cdot) - f(\cdot, \cdot)\|_{p,q} = 0.$$

### 3. Preliminaries

**3.1.** Let as in [15], for a fixed  $p, q > 0$ ,

$$(33) \quad v_{2p}(x) := \exp(-2px), \quad x \in R_0,$$

and

$$(34) \quad v_{2p,2q}(x, y) := v_{2p}(x)v_{2q}(y), \quad (x, y) \in R_0^2.$$

Denote by  $C_{2p,2q}$  the set of all real-valued functions  $f$  continuous on  $R_0^2$  for which  $v_{2p,2q}f$  is uniformly continuous and bounded on  $R_0^2$ . The norm on  $C_{2p,2q}$  is defined by

$$(35) \quad \|f\|_{2p,2q} \equiv \|f(\cdot, \cdot)\|_{2p,2q} := \sup_{(x,y) \in R_0^2} v_{2p,2q}(x, y) |f(x, y)|.$$

The modulus of continuity of function  $f \in C_{2p,2q}$  we define as in section 1.1. by formula

$$\omega(f, C_{2p,2q}; t, z) := \sup_{0 \leq h \leq t, 0 \leq \delta \leq z} \|\Delta_{h,\delta} f(\cdot, \cdot)\|_{2p,2q}, \quad t, z \geq 0,$$

and we have

$$(36) \quad \lim_{t,z \rightarrow 0+} \omega(f, C_{2p,2q}; t, z) = 0 \quad \text{for } f \in C_{2p,2q}.$$

Analogously as in section 1.1, for fixed  $p, q > 0$ , we denote by  $C_{2p,2q}^1$  the set of all functions  $f \in C_{2p,2q}$  which first partial derivatives belong also to  $C_{2p,2q}$ .

**3.2.** Similarly as in Section II we introduce

**Definition 2.** Fix  $r, s \in N$  and  $\alpha > 0$ . For functions  $f \in C_{2p,2q}$ ,  $p, q > 0$ , we define the operators

$$(37) \quad B_{m,n}(f; p, q, r, s, \alpha; x, y) \equiv B_{m,n}(f; x, y) := \frac{1}{g((m^\alpha x + 1)^2; r) g((n^\alpha y + 1)^2; s)} \\ \cdot \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(m^\alpha x + 1)^{2j}}{(j+r)!} \frac{(n^\alpha y + 1)^{2k}}{(k+s)!} f\left(\frac{j+r}{m^\alpha(m^\alpha x + 1) + 2p}, \frac{k+s}{n^\alpha(n^\alpha y + 1) + 2q}\right)$$

for  $(x, y) \in R_0^2$ ,  $m, n \in N$ .

In [15] there were examined the operators

$$(38) \quad B_n(f; x) \equiv B_n(f; q, r, \alpha; x) \\ := \frac{1}{g((n^\alpha x + 1)^2; r)} \sum_{k=0}^{\infty} \frac{(n^\alpha x + 1)^{2k}}{(k+r)!} f\left(\frac{k+r}{n^\alpha(n^\alpha x + 1) + 2q}\right)$$

for functions  $f$  of one variable, belonging to exponential weighted spaces.

In this paper we shall give similar results for operators  $B_{m,n}(f)$ .

## 4. Lemmas and theorems

**4.1.** In this section we shall give some properties of the above operators, which we shall apply to the proofs of the main theorems. From (37) and (6) we deduce that  $B_{m,n}(f)$  is well-defined in every space  $C_{2p,2q}$ ,  $p, q > 0, r, s \in N$ . In particular

$$(39) \quad B_{m,n}(1; x, y) = 1, \quad (x, y) \in R_0^2, \quad m, n \in N,$$

and if  $f \in C_{2p,2q}$  and  $f(x, y) = f_1(x)f_2(y)$  for all  $(x, y) \in R_0^2$ , then

$$(40) \quad B_{m,n}(f; p, q, r, s, \alpha; x, y) = B_m(f_1; p, r, \alpha; x)B_n(f_2; q, s, \alpha; y)$$

for all  $(x, y) \in R_0^2$  and  $m, n \in N$ . Moreover from (38) and (6) we get

$$(41) \quad B_n(1; q, r; x) = 1 \quad x \in R_0, \quad n \in N.$$

In the paper [15] the following two lemmas for  $B_n(f; q, r; \cdot)$  defined by (38) were proved.

**Lemma 3.** *Let  $q, \alpha > 0$ ,  $r \in N$  be fixed numbers. Then for all  $n \in N$  and  $x \in R_0$ , we have*

$$\begin{aligned} B_n(t - x; q, r, \alpha; x) &= \frac{(n^\alpha x + 1)^2}{n^\alpha(n^\alpha x + 1) + 2q} - x + \\ &\quad + \frac{1}{(n(nx + 1) + 2q)(r - 1)!g((nx + 1)^2; r)}, \\ B_n((t - x)^2; q, r, \alpha; x) &= \left( \frac{(n^\alpha x + 1)^2}{n^\alpha(n^\alpha x + 1) + 2q} - x \right)^2 + \left( \frac{n^\alpha x + 1}{n^\alpha(n^\alpha x + 1) + 2q} \right)^2 \\ &\quad + \frac{r + (n^\alpha x + 1)^2 - 2x(n^\alpha(n^\alpha x + 1) + 2q)}{(n^\alpha(n^\alpha x + 1) + 2q)^2(r - 1)!g((n^\alpha x + 1)^2; r)}, \\ B_n(e^{2qt}; q, r, \alpha; x) &= \frac{g((n^\alpha x + 1)^2 e^{2q/(n^\alpha(n^\alpha x + 1) + 2q)}; r)}{g((n^\alpha x + 1)^2; r)} e^{2qr/(n^\alpha(n^\alpha x + 1) + 2q)}, \end{aligned}$$

$$\begin{aligned}
B_n((t-x)^2 e^{2qt}; q, r, \alpha; x) &= \left[ \left( \frac{(n^\alpha x + 1)^2}{n^\alpha(n^\alpha x + 1) + 2q} e^{2q/(n^\alpha(n^\alpha x + 1) + 2q)} - x \right)^2 \right. \\
&+ \left. \left( \frac{n^\alpha x + 1}{n^\alpha(n^\alpha x + 1) + 2q} \right)^2 e^{2q/(n^\alpha(n^\alpha x + 1) + 2q)} \right] B_n(e^{2qt}; q, r, \alpha; x) \\
&+ \frac{r + (n^\alpha x + 1)^2 e^{2q/(n^\alpha(n^\alpha x + 1) + 2q)} - 2x(n^\alpha(n^\alpha x + 1) + 2q)}{(n^\alpha(n^\alpha x + 1) + 2q)^2 (r-1)! g((n^\alpha x + 1)^2; r)} e^{2qr/(n^\alpha(n^\alpha x + 1) + 2q)}.
\end{aligned}$$

**Lemma 4.** For every fixed  $q, \alpha > 0$  and  $r \in N$  there exist positive constants  $M_i \equiv M_i(p, r)$ ,  $i = 14, 15$ , such that for all  $x \in R_0$ ,  $n \in N$

$$\begin{aligned}
v_{2q}(x) B_n(1/v_{2q}(t); q, r, \alpha; x) &\leq M_{14}, \\
v_{2q}(x) B_n((t-x)^2/v_{2q}(t); q, r, \alpha; x) &\leq \frac{M_{15}}{n^{2\alpha}}.
\end{aligned}$$

Applying (33) – (35) and (39) – (41) and Lemma 4 and arguing as in the proof of Lemma 2, we can prove the basic property of  $B_{m,n}(f)$ .

**Lemma 5.** For fixed  $p, q, \alpha > 0$  and  $r, s \in N$  there exists a positive constant  $M_{16} \equiv M_{16}(p, q, r, s)$  such that

$$(42) \quad \|B_{m,n}(1/v_{2p,2q}(t, z); p, q, r, s, \alpha; \cdot, \cdot)\|_{2p,2q} \leq M_{16} \quad \text{for } m, n \in N.$$

Moreover for every  $f \in C_{2p,2q}$  we have

$$(43) \quad \|B_{m,n}(f; p, q, r, s; \cdot, \cdot)\|_{2p,2q} \leq M_{16} \|f\|_{2p,2q} \quad \text{for } m, n \in N, r, s \in N.$$

The formula (37) and the inequality (43) show that  $B_{m,n}$ ,  $m, n \in N$ , are linear positive operators from the space  $C_{2p,2q}$  into  $C_{2p,2q}$ .

**4.2.** Applying Lemma 3–Lemma 5 and (33)–(35) and (39)–(41) and reasoning as in the proof of Theorem 1, we can prove the following

**Theorem 3.** Suppose that  $f \in C_{2p,2q}^1$  with given  $p, q > 0$  and  $r, s \in N$ . Then there exists a positive constant  $M_{17} = M_{17}(p, q, r, s)$  such that for all  $m, n \in N$  and  $\alpha > 0$

$$\|B_{m,n}(f; p, q, r, s, \alpha; \cdot, \cdot) - f(\cdot, \cdot)\|_{2p,2q} \leq M_{17} \left\{ \frac{1}{m^\alpha} \|f'_x\|_{2p,2q} + \frac{1}{n^\alpha} \|f'_y\|_{2p,2q} \right\}.$$

**Theorem 4.** Suppose that  $f \in C_{2p,2q}$ ,  $p, q, \alpha > 0$ ,  $r, s \in N$ . Then there exists a positive constant  $M_{18} \equiv M_{18}(p, q, r, s)$  such that

$$(44) \quad \|B_{m,n}(f; p, q, r, s; \cdot, \cdot) - f(\cdot, \cdot)\|_{2p,2q} \leq M_{18} \omega \left( f, C_{2p,2q}; \frac{1}{m^\alpha}, \frac{1}{n^\alpha} \right),$$

for all  $m, n \in N$ .

*Proof.* Similarly as in the proof of Theorem 2 we shall apply the Steklov function  $f_{h,\delta}$  for  $f \in C_{2p,2q}$ , defined by (27). Analogously as in (28)–(30) we get

$$(45) \quad \|f_{h,\delta} - f\|_{2p,2q} \leq \omega(f, C_{2p,2q}; h, \delta),$$

$$(46) \quad \|(f_{h,\delta})'_x\|_{2p,2q} \leq 2h^{-1} \omega(f, C_{2p,2q}; h, \delta),$$

$$(47) \quad \|(f_{h,\delta})'_y\|_{2p,2q} \leq 2\delta^{-1} \omega(f, C_{2p,2q}; h, \delta)$$

for all  $h, \delta > 0$ , which show that  $f_{h,\delta} \in C_{2p,2q}^1$  if  $f \in C_{2p,2q}$  and  $h, \delta > 0$ .

Now, for  $B_{m,n}$ , we can write

$$\begin{aligned} & v_{2p,2q}(x, y) |B_{m,n}(f; p, q, r, s, \alpha; x, y) - f(x, y)| \\ & \leq v_{2p,2q}(x, y) \{ |B_{m,n}(f(t, z) - f_{h,\delta}(t, z); p, q, r, s, \alpha; x, y)| \\ & \quad + |B_{m,n}(f_{h,\delta}(t, z); p, q, r, s, \alpha; x, y) - f_{h,\delta}(x, y)| \\ & \quad + |f_{h,\delta}(x, y) - f(x, y)| \} := T_1 + T_2 + T_3. \end{aligned}$$



By (35), (43) and (45), we get

$$\begin{aligned} T_1 &\leq \|B_{m,n}(f - f_{h,\delta}; p, q, r, s, \alpha; \cdot, \cdot)\|_{2p,2q} \\ &\leq M_{16} \|f - f_{h,\delta}\|_{2p,2q} \leq M_{14} \omega(f, C_{2p,2q}; h, \delta), \\ T_3 &\leq \omega(f, C_{2p,2q}; h, \delta). \end{aligned}$$

Applying Theorem 3 and (46) and (47), we get

$$\begin{aligned} T_2 &\leq M_{17} \left\{ \frac{1}{m^\alpha} \|(f_{h,\delta})'_x\|_{2p,2q} + \frac{1}{n^\alpha} \|(f_{h,\delta})'_y\|_{2p,2q} \right\} \\ &\leq 2M_{17} \omega(f, C_{2p,2q}; h, \delta) \left\{ h^{-1} \frac{1}{m^\alpha} + \delta^{-1} \frac{1}{n^\alpha} \right\}. \end{aligned}$$

From the above we deduce that there exists a positive constant  $M_{19} \equiv M_{19}(p, q, r, s)$  such that

$$(48) \quad \begin{aligned} &\|B_{m,n}(f; p, q, r, s, \alpha; \cdot, \cdot) - f(\cdot, \cdot)\|_{2p,2q} \\ &\leq M_{19} \omega(f, C_{2p,2q}; h, \delta) \left\{ 1 + h^{-1} \frac{1}{m} + \delta^{-1} \frac{1}{n} \right\}, \end{aligned}$$

for  $m, n \in N$  and  $h, \delta > 0$ . Now, for  $m, n \in N$  setting  $h = \frac{1}{m^\alpha}$  and  $\delta = \frac{1}{n^\alpha}$  to (48), we obtain (44). □

Theorem 4 and (36) imply

**Corollary.** *Let  $f \in C_{2p,2q}$ ,  $p, q, \alpha > 0$ ,  $r, s \in N$ . Then*

$$\lim_{m,n \rightarrow \infty} \|B_{m,n}(f; p, q, r, s, \alpha; \cdot, \cdot) - f(\cdot, \cdot)\|_{p,q} = 0.$$

**Remark.** Theorems and Corollaries in our paper show that  $A_{m,n}$  and  $B_{m,n}$ ,  $m, n \in N$ , give for  $\alpha > 1/2$  a better degree of approximation of functions belonging to weighted spaces of functions of two variables than classical Szasz-Mirakyan operator  $S_{m,n}$ , examined for continuous and bounded functions in [11].

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