NOTE ON THE Ψ-BOUNDEDNESS OF THE SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS

A. DIAMANDESCU

Abstract. It is proved a necessary and sufficient condition for the existence of Ψ-bounded solutions of a linear nonhomogeneous system of ordinary differential equations.

1. Introduction

The purpose of this note is to give a necessary and sufficient condition so that the nonhomogeneous system

$$x' = A(t)x + f(t) \tag{1}$$

have at least one Ψ-bounded solution for every continuous and Ψ-bounded function f, in supplementary hypothesis that $A(t)$ is a Ψ-bounded matrix on $\mathbb{R}_+$.

Here, Ψ is a continuous matrix function. The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the components of the solutions.

The problem of Ψ-boundedness of the solutions for systems of ordinary differential equations has been studied by many authors, as e.q. O. Akinyele [1], A. Constantin [3], C. Avramescu [2], T. Hallam [8], J. Morchalo [10]. In these papers, the function Ψ is a scalar continuous function (and increasing, differentiable and bounded in [1], nondecreasing and such that $\Psi(t) \geq 1$ on $\mathbb{R}_+$ in [3]).

Received March 17, 2004.
2000 *Mathematics Subject Classification.* Primary 34D05, 34C11.
Key words and phrases. Ψ-bounded solution, matrix function Ψ.

Let $\mathbb{R}^d$ be the Euclidean d-space. For $x = (x_1, x_2, \ldots, x_d)^T \in \mathbb{R}^d$, let $\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_d|\}$ be the norm of x. For a $d \times d$ real matrix A, we define the norm $|A|$ by $|A| = \sup_{\|x\| \leq 1} \|Ax\|$. Let $\Psi_i : \mathbb{R}_+ \to (0, \infty)$, $i = 1, 2, \ldots, d$, be continuous functions and

$$\Psi = \mathrm{diag}[\Psi_1, \Psi_2, \ldots, \Psi_d].$$

Definition 1.1. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}^d$ is said to be Ψ-bounded on $\mathbb{R}_+$ if $\Psi(t)\varphi(t)$ is bounded on $\mathbb{R}_+$.

Let A be a continuous $d \times d$ real matrix and the associated linear differential system

$$y' = A(t)y. \tag{2}$$

Let Y be the fundamental matrix of (2) for which $Y(0) = \mathrm{I}_d$ (identity $d \times d$ matrix).

Let X_1 denote the subspace of $\mathbb{R}^d$ consisting of all vectors which are values of Ψ-bounded solutions of (2) for $t = 0$ and let X_2 an arbitrary fixed subspace of $\mathbb{R}^d$, supplementary to X_1.

We suppose that X_2 is a closed subspace of $\mathbb{R}^d$. We denote by P_1 the projection of $\mathbb{R}^d$ onto X_1 (that is P_1 is a bounded linear operator $P_1 : \mathbb{R}^d \to \mathbb{R}^d$, $P_1^2 = P_1$, $\mathrm{Ker}\, P_1 = X_2$) and $P_2 = \mathrm{I} - P_1$ the projection onto X_2.

In our papers [5] and [6] we have proved the following results (Lemma 1, Lemma 2 and respectively Theorem 2.1.):

Lemma 1. *Let $Y(t)$ be an invertible matrix which is a continuous function of t on $\mathbb{R}_+$ and let P be a projection. If there exist a continuous function $\varphi : \mathbb{R}_+ \to (0, \infty)$ and a positive constant M such that*

$$\int_0^t \varphi(s)|\Psi(t)Y(t)PY^{-1}(s)\Psi^{-1}(s)|\, ds \leq M, \qquad \textit{for all } t \geq 0,$$

and

$$\int_0^\infty \varphi(s)\, ds = +\infty,$$

then, there exists a constant $N > 0$ *such that*

$$|\Psi(t)Y(t)P| \leq N e^{-M^{-1}\int_0^t \varphi(s)\,ds}, \qquad \textit{for all } t \geq 0.$$

Consequently,

$$\lim_{t\to\infty} |\Psi(t)Y(t)P| = 0.$$

Lemma 2. *Let* $Y(t)$ *be an invertible matrix which is a continuous function of* t *on* $\mathbb{R}_+$ *and let* P *be a projection. If there exists a constant* $M > 0$ *such that*

$$\int_t^\infty |\Psi(t)Y(t)PY^{-1}(s)\Psi^{-1}(s)|\,ds \leq M, \qquad \textit{for all } t \geq 0,$$

then, for any vector $x_0 \in \mathbb{R}^d$ *such that* $Px_0 \neq 0$,

$$\limsup_{t\to\infty} \|\Psi(t)Y(t)Px_0\| = +\infty.$$

Theorem 2.1. *If* A *is a continuous* $d \times d$ *matrix, then the system* (1) *has at least one* Ψ*-bounded solution on* $\mathbb{R}_+$ *for every continuous and* Ψ*-bounded function* f *on* $\mathbb{R}_+$ *if and only if there is a positive constant* K *such that*

$$\int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)|\,ds + \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)|\,ds \leq K, \tag{3}$$

for all $t \geq 0$.

2. The main results

In this section we give the main results of this note.

Theorem 2.1. *Let* A *be a continuous* $d \times d$ *real matrix such that*

$$|\Psi(t)A(t)\Psi^{-1}(t)| \leq L, \qquad for\ all\ t \geq 0.$$

Let $\Psi(t)$ *such that*

$$|\Psi(t)\Psi^{-1}(s)| \leq M, \qquad for\ t \geq s \geq 0.$$

Then, the system (1) *has at least one* Ψ*-bounded solution on* $\mathbb{R}_+$ *for every continuous and* Ψ*-bounded function* f *on* $\mathbb{R}_+$ *if and only if there are two positive constants* K_1 *and* α *such that*

$$\begin{aligned} |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| &\leq K_1 \mathrm{e}^{-\alpha(t-s)}, \qquad 0 \leq s \leq t, \\ |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| &\leq K_1 \mathrm{e}^{-\alpha(s-t)}, \qquad 0 \leq t \leq s, \end{aligned} \tag{4}$$

Proof. First, we prove the "only if" part.

From the hypotheses and Theorem 2.1, [6], it follows that there is a positive constant K such that

$$\int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)|\, ds + \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)|\, ds \leq K,$$

for all $t \geq 0$.

From $Y'(t) = A(t)Y(t), \ \ t \geq 0$, it follows that

$$Y(t) = Y(s) + \int_s^t A(u)Y(u)\,du, \qquad \text{for } t \geq s \geq 0.$$

Therefore,

$$\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) = \Psi(t)\Psi^{-1}(s) + \int_s^t \Psi(t)A(u)Y(u)Y^{-1}(s)\Psi^{-1}(s)du.$$

Thereafter, for $t \geq s \geq 0$,

$$\begin{aligned} &|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)| \\ &\leq |\Psi(t)\Psi^{-1}(s)| + \int_s^t |\Psi(t)\Psi^{-1}(u)| \|\Psi(u)A(u)\Psi^{-1}(u)\| |\Psi(u)Y(u)Y^{-1}(s)\Psi^{-1}(s)|\,du. \end{aligned}$$

From the hypotheses and Gronwall's inequality it follows that

$$|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)| \leq M\mathrm{e}^{LM(t-s)}, \qquad t \geq s \geq 0. \tag{5}$$

Now, we show that (3) and (5) imply (4).
For $v \in \mathbb{R}^d$ and $0 \leq s \leq t \leq s+1$, we have

$$\begin{aligned} \|\Psi(t)Y(t)P_1 v\| &= \|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)\Psi(s)Y(s)P_1 v\| \\ &\leq |\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)| \cdot \|\Psi(s)Y(s)P_1 v\| \\ &\leq M\mathrm{e}^{LM} \|\Psi(s)Y(s)P_1 v\| \end{aligned} \tag{6}$$

For $P_1 v \neq 0$, let

$$q(t) = \|\Psi(t)Y(t)P_1 v\|^{-1} \quad \text{and} \quad Q(t) = \int_0^t q(s)\,ds.$$

We have

$$q(t) \geq M^{-1}\mathrm{e}^{-LM} q(s), \qquad \text{for } 0 \leq s \leq t \leq s+1.$$

Thus,

$$Q(s+1) = \int_0^{s+1} q(u)\,du \geq \int_s^{s+1} q(u)\,du \geq M^{-1}\mathrm{e}^{-LM} q(s).$$

From Lemma 1, [5], it follows that

$$\|\Psi(t)Y(t)P_1 v\| \leq K Q^{-1}(s+1)\mathrm{e}^{-K^{-1}(t-s-1)}, \qquad \text{for } t \geq s+1$$

and hence

$$\begin{aligned} \|\Psi(t)Y(t)P_1 v\| &\leq KM\mathrm{e}^{LM} q^{-1}(s)\mathrm{e}^{-K^{-1}(t-s-1)} \\ &= KM\mathrm{e}^{LM}\mathrm{e}^{-K^{-1}(t-s-1)}\|\Psi(s)Y(s)P_1 v\|, \qquad \text{for } t \geq s+1. \end{aligned} \tag{7}$$

From (6) and (7) it results that

$$\|\Psi(t)Y(t)P_1 v\| \leq N_1 \mathrm{e}^{-K^{-1}(t-s)}\|\Psi(s)Y(s)P_1 v\|, \tag{8}$$

for $t \geq s$ and $v \in \mathbb{R}^d$, where $N_1 = M\mathrm{e}^{LM+K^{-1}}\max\{1, K\}$.

Similarly, for $P_2 v \neq 0$, let

$$r(t) = \|\Psi(t)Y(t)P_2 v\|^{-1}.$$

From (3) and Lemma 2, [5], it follows that the function $R(t) = \int_t^\infty r(u)\,du$ exists for $t \geq 0$ and

$$r^{-1}(\mathrm{t}) \int_t^T r(u)\,du \leq K, \qquad \text{for } T \geq t \geq 0. \tag{9}$$

Hence,

$$R'(t) = -r(t) \leq -K^{-1}R(t)$$

and then,

$$R(t) \leq R(t_0)\mathrm{e}^{-K^{-1}(t-t_0)}, \qquad t \geq t_0 \geq 0. \tag{10}$$

On the other hand, for t $\geq$ s $\geq$ 0, we have

$$\begin{aligned} r^{-1}(t) = \|\Psi(t)Y(t)P_2v\| &= \|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)\Psi(s)Y(s)P_2v\| \\ &\leq |\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)| \cdot \|\Psi(s)Y(s)P_2v\| \\ &\leq M\mathrm{e}^{LM(t-s)}r^{-1}(s). \end{aligned}$$

Consequently,

$$r(s) \geq M^{-1}\mathrm{e}^{-LM(s-t)}r(t), \qquad s \geq t \geq 0.$$

Hence,

$$R(t) \geq M^{-1}r(t)\int\limits_t^\infty \mathrm{e}^{-LM(s-t)}\,ds = L^{-1}M^{-2}r(t).$$

Combining this with (9) and (10), we obtain, for $t \geq t_0 \geq 0$:

$$\begin{aligned} \|\Psi(t)Y(t)P_2v\| &= r^{-1}(t) \geq L^{-1}M^{-2}R^{-1}(t) \\ &\geq L^{-1}M^{-2}R^{-1}(t_0)\mathrm{e}^{K^{-1}(t-t_0)} \\ &= (LM^2)^{-1}\mathrm{e}^{K^{-1}(t-t_0)}\|\Psi(t_0)Y(t_0)P_2v\|. \end{aligned}$$

It results that

$$\|\Psi(t)Y(t)P_2v\| \leq N_2\mathrm{e}^{-K^{-1}(s-t)}\|\Psi(s)Y(s)P_2v\|, \tag{11}$$

for $s \geq t \geq 0$, $v \in \mathbb{R}^d$, where $N_2 = LM^2$.

Now, we show that

$$p_i(t) = |\Psi(t)Y(t)P_iY^{-1}(t)\Psi^{-1}(t)|, \qquad i = 1, 2,$$

are bounded for $t \geq 0$. Let $\sigma > 0$ be such that

$$p = N_2^{-1}\mathrm{e}^{K^{-1}\sigma} - N_1\mathrm{e}^{-K^{-1}\sigma} > 0.$$

From (8) and (11) we deduce that

$$\begin{aligned} |\Psi(t+\sigma)Y(t+\sigma)P_1Y^{-1}(t)\Psi^{-1}(t)| &\leq N_1\mathrm{e}^{-K^{-1}\sigma}p_1(t), \\ |\Psi(t+\sigma)Y(t+\sigma)P_2Y^{-1}(t)\Psi^{-1}(t)| &\geq N_2^{-1}\mathrm{e}^{K^{-1}\sigma}p_2(t). \end{aligned}$$

Hence,

$$\begin{aligned} &|p_1^{-1}(t)\Psi(t+\sigma)Y(t+\sigma)P_1Y^{-1}(t)\Psi^{-1}(t) \\ &+ p_2^{-1}(t)\Psi(t+\sigma)Y(t+\sigma)P_2Y^{-1}(t)\Psi^{-1}(t)| \geq p. \end{aligned}$$

It follows that

$$|\Psi(t+\sigma)Y(t+\sigma)Y^{-1}(t)\Psi^{-1}(t)(p_1^{-1}(t)\Psi(t)Y(t)P_1Y^{-1}(t)\Psi^{-1}(t) \\ + p_2^{-1}(t)\Psi(t)Y(t)P_2Y^{-1}(t)\Psi^{-1}(t))| \geq p,$$

or

$$\begin{aligned} p \ &\leq |p_1^{-1}(t)\Psi(t)Y(t)P_1Y^{-1}(t)\Psi^{-1}(t) \\ &\quad + p_2^{-1}(t)\Psi(t)Y(t)P_2Y^{-1}(t)\Psi^{-1}(t)|M\mathrm{e}^{LM\sigma}. \end{aligned}$$

Therefore,

$$\begin{aligned} &pM^{-1}\mathrm{e}^{-LM\sigma} \\ &\quad \leq \ |p_1^{-1}(t)I_d + (p_2^{-1}(t) - p_1^{-1}(t))\Psi(t)Y(t)P_2Y^{-1}(t)\Psi^{-1}(t)| \\ &\quad \leq \ p_1^{-1}(t) + |p_2^{-1}(t) - p_1^{-1}(t)|p_2(t) = p_1^{-1}(t)(1 + |p_1(t) - p_2(t)|) \\ &\quad = \ p_1^{-1}(t)(1 + \big||\Psi(t)Y(t)P_1Y^{-1}(t)\Psi^{-1}(t)| - |\Psi(t)Y(t)P_2Y^{-1}(t)\Psi^{-1}(t)|\big|) \\ &\quad \leq \ p_1^{-1}(t)(1 + |\Psi(t)Y(t)P_1Y^{-1}(t)\Psi^{-1}(t) + \Psi(t)Y(t)P_2Y^{-1}(t)\Psi^{-1}(t)|) \\ &\quad = \ 2p_1^{-1}(t) \end{aligned}$$

It follows that

(12) $$p_1(t) \leq 2Mp^{-1}\mathrm{e}^{LM\sigma} = \overline{M}, \qquad t \geq 0.$$

Similarly,

(13) $$p_2(t) \leq \overline{M}, \qquad t \geq 0.$$

Finally, by (8), (11), (12) and (13) we deduce that

$$\begin{aligned} |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| &\le K_1 e^{-K^{-1}(t-s)}, \qquad 0 \le s \le t \\ |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| &\le K_1 e^{-K^{-1}(s-t)}, \qquad 0 \le t \le s, \end{aligned}$$

where $K_1 = \overline{M}\max\{N_1, N_2\}$.

Now, we prove the "if" part.
From (4), for $t \ge 0$ we have

$$\int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)|\,ds + \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)|\,ds$$

$$\le \quad K_1 \int_0^t e^{-\alpha(t-s)}\,ds + K_1 \int_t^\infty e^{-\alpha(s-t)}\,ds < \frac{2K_1}{\alpha}.$$

From this and Theorem 2.1, [6], it follows the conclusion of theorem. The proof is now complete. □

Remark 2.1. If $\Psi(t)$ and fundamental matrix $Y(t)$ do not fulfil the condition (5), then the conditions (4) may not be true.

This is shown by the

Example 2.1. Consider the linear system (2) with $A(t) = \begin{pmatrix} -2 & e^t \\ 0 & 2 \end{pmatrix}$.
A fundamental matrix for the system (2) is

$$Y(t) = \begin{pmatrix} e^{-2t} & \frac{1}{5}(e^{3t} - e^{-2t}) \\ 0 & e^{2t} \end{pmatrix}.$$

Consider

$$\Psi(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}.$$

We have

$$\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) = \begin{pmatrix} e^{-3(t-s)} & \frac{1}{5}e^{2t}(1-e^{-5(t-s)}) \\ 0 & 1 \end{pmatrix}.$$

This shows that (5) is not satisfied.

Instead,

$$\Psi(t)\Psi^{-1}(s) = \begin{pmatrix} e^{-(t-s)} & 0 \\ 0 & e^{-2(t-s)} \end{pmatrix},$$

is bounded for $0 \le s \le t$.

But then, in this case, we have

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thereafter,

$$\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s) = \begin{pmatrix} e^{-3(t-s)} & \frac{1}{5}e^{-3t}(1-e^{5s}) \\ 0 & 0 \end{pmatrix},$$

which is unbounded for $0 \le s \le t$.

Thus, the conditions (4) is not true.

Remark 2.2. If in Theorem 2.1 we put $\Psi(\mathrm{t}) = \mathrm{I}_d$, then the conclusion of the Theorem 3, Chapter V, [4], follows.

We prove finally a theorem in which we will see that the asymptotic behavior of solutions of (1) is determined completely by the asymptotic behavior of $f(t)$ as $t \to \infty$.

Theorem 2.2. *Suppose that:*

1. *the fundamental matrix $Y(t)$ of (2) satisfies the conditions*

$$|\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \le Ke^{-\alpha(t-s)}, \qquad 0 \le s \le t,$$
$$|\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \le Ke^{-\alpha(s-t)}, \qquad 0 \le t \le s,$$

   where K and α are positive constants and P_1, P_2 are supplementary projections, $P_i \neq 0$;

2. *the continuous and Ψ-bounded function $f : \mathbb{R}_+ \to \mathbb{R}^d$ satisfies one of the following conditions:*
   a) $\lim_{t\to\infty} \|\Psi(t)f(t)\| = 0$,
   b) $\int_0^\infty \|\Psi(t)f(t)\|\, dt$ *is convergent,*
   c) $\lim_{t\to\infty} \int_t^{t+1} \|\Psi(s)f(s)\|\, ds = 0.$

Then, every Ψ-bounded solution $x(t)$ of (1) is such that

$$\lim_{t\to\infty} \|\Psi(t)x(t)\| = 0.$$

Proof. a) It follows from the Theorem 2.1, [6].
b) It is similar to the proof of Theorem 2.1, [6].
c) By the hypothesis 2, it follows that there exists a positive constant C such that

$$\int_t^{t+1} \|\Psi(s)f(s)\|\, ds \le C, \qquad \text{for all } t \ge 0.$$

Let $x(t)$ be a Ψ-bounded solution of (1). There is a positive constant M such that $\|\Psi(t)x(t)\| \leq M$, for all $t \geq 0$. Consider the function

$$y(t) = x(t) - Y(t)P_1x(0) - \int_0^t Y(t)P_1Y^{-1}(s)f(s)\,ds + \int_t^\infty Y(t)P_2Y^{-1}(s)f(s)\,ds,$$

for all $t \geq 0$.
For $v \geq t \geq 0$ we have

$$\begin{aligned} \|\int_t^v P_2Y^{-1}(s)f(s)\,ds\| &\leq \int_t^v \|P_2Y^{-1}(s)f(s)\|\,ds \\ &\leq |Y^{-1}(t)\Psi^{-1}(t)| \int_t^v \left|\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)\right| \cdot \|\Psi(s)f(s)\|\,ds \\ &\leq K|Y^{-1}(t)\Psi^{-1}(t)| \int_t^v \mathrm{e}^{-\alpha(s-t)}\|\Psi(\mathrm{s})\mathrm{f}(\mathrm{s})\|\,ds \\ &\leq KC(1-\mathrm{e}^{-\alpha})^{-1}|Y^{-1}(t)\Psi^{-1}(t)|, \end{aligned}$$

by using a Lemma of J. L. Massera and J. J. Schäffer, [9].

It follows that the integral

$$\int_t^\infty Y(t)P_2Y^{-1}(s)f(s)\,ds$$

is convergent.

Clearly, the function $y(t)$ is continuously differentiable on $\mathbb{R}_+$.
For $t \geq 0$, we have

$$\begin{aligned} y'(t) &= x'(t) - Y'(t)P_1x(0) - Y'(t)\int_0^t P_1Y^{-1}(s)f(s)\,ds - Y(t)P_1Y^{-1}(t)f(t) \\ &+ Y'(t)\int_t^\infty P_2Y^{-1}(s)f(s)\,ds - Y(t)P_2Y^{-1}(t)f(t) \\ &= A(t)x(t) + f(t) - A(t)Y(t)P_1x(0) - A(t)Y(t)\int_0^t P_1Y^{-1}(s)f(s)ds \\ &+ A(t)Y(t)\int_t^\infty P_2Y^{-1}(s)f(s)ds - Y(t)(P_1+P_2)Y^{-1}(t)f(t) \\ &= A(t)y(t). \end{aligned}$$

Thus, the function $y(t)$ is a solution of the linear system (2).
Since the hypothesis 1. implies that $\lim_{t\to\infty} \Psi(t)Y(t)P_1 = 0$ (see Lemma 1, [5]), there exists a positive constant N such that $|\Psi(t)Y(t)P_1| \leq N$ for all $t \geq 0$.

It follows that

$$
\begin{aligned}
\|\Psi(t)y(t)\| &\leq \|\Psi(t)x(t)\| + |\Psi(t)Y(t)P_1| \cdot \|x(0)\| \\
&\quad + \int_0^t \left|\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)\right| \cdot \|\Psi(s)f(s)\|\,ds \\
&\quad + \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)|\|\Psi(s)f(s)\|\,ds \\
&\leq M + N\|x(0)\| + K\int_0^t \mathrm{e}^{-\alpha(t-s)}\|\Psi(s)f(s)\|\,ds \\
&\quad + K\int_t^\infty \mathrm{e}^{-\alpha(s-t)}\|\Psi(s)f(s)\|\,ds \\
&\leq \mathrm{M} + \mathrm{N}\|x(0)\| + 2KC(1-\mathrm{e}^{-\alpha})^{-1}, \qquad \text{for all } t \geq 0,
\end{aligned}
$$

by using of above Lemma of Massera and Schäffer.

Thus, the function $y(t)$ is a Ψ-bounded solution of the linear system (2).

On the other hand, $P_1y(0) = 0$. Therefore, $y(t) = Y(t)y(0) = Y(t)P_2y(0)$. If $P_2y(0) \neq 0$, from the Lemma 2, [5], it follows that $\limsup_{t\to\infty}\|\Psi(t)y(t)\| = +\infty$, which is contradictory. Thus, $P_2y(0) = 0$ and then $y(t) = 0$ for $t \geq 0$.

Thus, for $t \geq 0$ we have

$$
x(t) = Y(t)P_1x(0) + \int_0^t Y(t)P_1Y^{-1}(s)f(s)\,ds - \int_t^\infty Y(t)P_2Y^{-1}(s)f(s)\,ds.
$$

Now, for a given $\varepsilon > 0$, there exists $t_1 \geq 0$ such that

$$\int_t^{t+1} \|\Psi(s)f(s)\|\, ds < \varepsilon(4K)^{-1}(1 - \mathrm{e}^{-\alpha}), \qquad \text{for all } t \geq t_1.$$

Moreover, there exists $t_2 > t_1$ such that, for $t \geq t_2$,

$$|\Psi(t)Y(t)P_1| \leq \frac{\varepsilon}{2}\left(\|x(0)\| + \int_0^{t_1} \|Y^{-1}(s)f(s)\|\, ds\right)^{-1}.$$

Then, for $t \geq t_2$ we have, by using of above Lemma of Massera and Schäffer,

$$\begin{aligned}
\|\Psi(t)x(t)\| &\leq |\Psi(t)Y(t)P_1|\|x(0)\| + \int\limits_0^{t_1} |\Psi(t)Y(t)P_1|\|Y^{-1}(s)f(s)\|\,ds \\
&+ \int\limits_{t_1}^{t} |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)|\|\Psi(s)f(s)\|\,ds \\
&+ \int_t^{\infty} |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)|\|\Psi(s)f(s)\|\,ds \\
&\leq |\Psi(t)Y(t)P_1|(\|x(0)\| + \int_0^{t_1} \|Y^{-1}(s)f(s)\|\,ds) \\
&+ K\int\limits_{t_1}^{t} \mathrm{e}^{-\alpha(t-s)}\|\Psi(s)f(s)\|\,ds + K\int\limits_{t}^{\infty} \mathrm{e}^{-\alpha(s-t)}\|\Psi(s)f(s)\|\,ds \\
&< \varepsilon.
\end{aligned}$$

This shows that $\lim\limits_{t\to\infty} \|\Psi(t)x(t)\| = 0$.
The proof is now complete. □

Remark 2.3. If in Theorem we put $A(t) = A$, $\Psi(t) = \varphi^k(t)I_d$, then the conclusion of the Theorem 3.1, [3], follows.

Remark 2.4. If the function f does not fulfill the condition 2 of the theorem, then $\Psi(t)x(t)$ may be such that

$$\lim_{t\to\infty} \|\Psi(t)x(t)\| \neq 0.$$

This can be seen from

Example 2.2. Consider the linear system (1) with

$$A(t) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad \text{and} \qquad f(t) = \begin{pmatrix} \mathrm{e}^{(a+1)t} \\ \mathrm{e}^{(b-2)t} \end{pmatrix},$$

where $a, b \in \mathbb{R}$.

A fundamental matrix for the homogeneous system (2) is

$$Y(t) = \begin{pmatrix} \mathrm{e}^{at} & 0 \\ 0 & \mathrm{e}^{bt} \end{pmatrix}.$$

Consider

$$\Psi(t) = \begin{pmatrix} \mathrm{e}^{-(a+1)t} & 0 \\ 0 & \mathrm{e}^{(1-b)t} \end{pmatrix}.$$

The first condition of the Theorem 2.2. is satisfied with

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \alpha = 1, \qquad K = 1.$$

Then, we have $\|\Psi(t)f(t)\| = 1$ for all $t \geq 0$ and

$$\Psi(t)x(t) = \begin{pmatrix} c_1\mathrm{e}^{-t} + 1 \\ c_2\mathrm{e}^{t} - \frac{1}{2}\mathrm{e}^{-t} \end{pmatrix} \not\to \quad 0 \quad \text{as} \quad t \to \infty.$$

Remark 2.5. This Example shows that the components of the solution $x(t)$ have a mixed asymptotic behavior.

Acknowledgment. The author would like to thank very much the referee of this paper for valuable comments and suggestions.

1. Akinyele O. *On partial stability and boundedness of degree k*, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **8** 65 (1978), 259–264.

2. Avramescu C., *Asupra comportării asimptotice a soluţiilor unor ecuaţii funcţionale*, Analele Universităţii din Timişoara, Seria Ştiinţe Matematice – Fizice **VI** (1968), 41–55.

3. Constantin A., *Asymptotic properties of solutions of differential equations*, Analele Universităţii din Timişoara, Seria Ştiinţe Matematice **XXX** fasc. 2–3 (1992), 183–225.

4. Coppel W. A., *Stability and Asymptotic Behavior of Differential Equations*, Health, Boston, 1965.

5. Diamandescu A. *On the Ψ-instability of a nonlinear Volterra integro-differential system*, Bulletin Mathématique (Bucarest), (to appear).

6. ______, *A note on the Ψ-boundedness for differential systems*, submitted for publication.

7. Gaşpar D., *Analiză funcţională*, Ed. Facla, Timişoara, 1981.

8. Hallam T. G., *On asymptotic equivalence of the bounded solutions of two systems of differential equations*, Mich. Math. Journal **16** (1969), 353–363.

9. Massera J. L. and Schäffer J. J., *Linear differential equations and functional analysis*, Ann. of Math. **67** (1958), 517–573.

10. Morchalo J., *On (Ψ-L_p)Analele Ştiinţifice ale Universităţii "Al. I. Cuza" Iaşi* **XXXVI***, s I – a, Matematică (1990) f. 4, 353–360.*

A. Diamandescu, Department of Applied Mathematics, University of Craiova, 13, "Al. I. Cuza" st., 200585 Craiova, Romania., *e-mail*: adiamandescu@central.ucv.ro