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THE AMERICAN PUT OPTION CLOSE TO EXPIRY

R. MALLIER AND G. ALOBAIDI

ABSTRACT. We use an asymptotic expansion to study the behavior of the American put option close to expiry for the case where the dividend yield is less than or equal to the risk-free interest rate. Series solutions are obtained for the location of the free boundary and the price of the option in that limit.

1. INTRODUCTION

Options are derivative financial securities whose value is based on the value of some other underlying security. Equity options are options on an individual stock, or an index of such stocks such as the Dow Jones Industrial Average (DJIA) or S&P500, and such options are traded both on exchanges and over-the-counter. Although more exotic pay-offs are possible, exchange-traded options tend to be either vanilla calls or puts. If the price of the underlying stock is S and E is the exercise price of the option, which is specified in the option contract, a vanilla call pays an amount $\max(S-E,0)$ at expiry while a vanilla put pays $\max(E-S,0)$.

In addition to being classified by their pay-offs, options are also classified by when they can be exercised: European options can be exercised only at expiry while American options can be exercised at any time at or before expiry, while the less-common Bermudan options can be exercised on a finite number of pre-specified dates. Because they can be exercised only at expiry, it is straightforward to price European options using the Black-Scholes option pricing formula [**6**]: essentially, one calculates the probability distribution for the stock price at expiry and uses that to price the option. American options are considerably harder to price because the possibility of early exercise leads to a free boundary problem, with this free boundary separating the region where it is optimal to hold from the region where exercise is optimal. In theory, exercise should take place only on this free boundary, known as the optimal exercise boundary. This sort of free boundary problem is common in diffusion problems such as melting and solidification problems and is referred to as a Stefan problem. To date, no closed form solution is known for American options, except for one or two very special cases such as the American call without dividends when exercise is never optimal meaning that the option has the same value as a European.

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In recent years, several authors [**8, 5, 15, 11, 9, 10, 2, 3**] have considered the behavior of American options close to expiry, and in particular the behavior of the free boundary in this limit. Close to expiry, the remaining life of the option can be regarded as a small parameter which can be used to order an expansion. These studies have suggested that there are two distinct regimes in this limit, depending on the relative values of the risk-free rate r and the dividend yield D. For the call with $D < r$ and the put with $D > r$ [**8, 9, 2**], the free boundary is relatively well-behaved close to expiry. In this case, the free boundary starts from rE/D at expiry, and its location $S_f(t)$ can be written as a power series of the form

$$\ln \frac{S_f(t)D}{rE} \sim \sum_{n=1}^{\infty} x_n \tau^{n/2}, \tag{1}$$

where $\tau = \sigma^2(T-t)/2$ is the rescaled time remaining until expiry, σ being the volatility of the stock price. This $\tau^{1/2}$ behavior, referring to the leading term in the series, is considered normal for Stefan problems, and arises frequently in the classic works of Tao [**16**]–[**24**], who studied Stefan problems arising in solidification and melting with various boundary conditions.

For the call with $D > r$ and the put with $D < r$, the behavior of the free boundary is somewhat less well-behaved, and involves logarithms [**5, 15, 11, 9, 10, 3**], with the leading term in the series being $\sqrt{-\tau \ln \tau}$. This is unusual for Stefan problems, and indeed this kind of behavior was not encountered by Tao in any of the problems he considered [**16**]–[**24**]. In addition to the $\tau^{1/2}$ behavior mentioned above, Tao found several possibilities for the behavior of the free boundary in the various problems he considered. The richest behavior was found in [**24**], where he considered the Cauchy-Stefan problem with two different materials, one a solid and one a liquid, each occupying a semi-infinite region. For this problem, Tao found four different cases could occur, depending on the initial conditions: (i) the leading term in the series was $\tau^{1/2}$; (ii) the leading term was $\tau^{m/2}$ for integer $m > 1$, which was also found in [**18**]; (iii) no solidification ever occurred so there was no boundary; (iv) a pre-solidification period occurred during which the temperatures in both materials were redistributed, with solidification starting at some time $\tau_c > 0$ once the surface temperature had reached the freezing point. Despite the richness of the behavior found by Tao, he did not encounter the logarithmic behavior found in [**5, 15, 11, 9, 10, 3**], which appears to be caused by the boundary conditions at expiry.

In the next section, we will pose an expansion in τ to study the behavior of the American put close to expiry for $D \leq r$, and we will see that when $D = r$, the behavior of the free boundary is slightly different than when $D < r$: it will be be necessary to include logs in the expansion for $D < r$ while a special function known as the Lambert W function is necessary when $D = r$. The results of this expansion are discussed in section 3. The expansion we use is essentially along the lines of those used by Tao, although in our case it is necessary to make a change of variables to transform the Black-Scholes-Merton partial differential equation (PDE) which governs option prices into a more standard diffusion equation together with a forcing term so that we can use Tao's method. Finally in this section, we note that in previous work [**3**], we posed a similar sort of expansion for the problem considered here, but our analysis there was limited to demonstrating that it was necessary to include logs in the expansion and we did not explore the details, which are supplied in the current analysis, nor did we consider the case $r = D$, which again is included in this study.

2. Analysis

The starting point for our analysis is the Black-Scholes-Merton PDE [**6, 13**] for the price $V(S, t)$ of an equity derivative,

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0, \tag{2}$$

where S is the price of the underlying, r is the risk-free rate, D is the constant dividend yield, and σ is the volatility. For European options, this equation is valid for times $t < T$, with a pay-off at expiry $t = T$ of $V(S, T) = \max(E - S, 0)$ for a vanilla put, where E is the strike price of the option. The corresponding pay-off for a vanilla call is $V(S, T) = \max(S - E, 0)$.

For American-style options, the possibility of early exercise leads to the additional constraint that the value of the option cannot fall below the pay-off from immediate exercise, so that $V(S, t) \geq \max(E - S, 0)$ for an American put, and similarly $V(S, t) \geq \max(S - E, 0)$ for an American call. This constraint leads to conditions on the value of the option V and its delta or derivative with respect to S, $(\partial V/\partial S)$, at the free boundary, specifically that they must be continuous across the free boundary so that for a put $V = E - S$ and $(\partial V/\partial S) = -1$ on the free boundary. The condition on the delta $(\partial V/\partial S)$ is known as the smooth pasting or high contact condition of Samuelson [**14**]. The above constraint allows us to find the location of the optimal exercise boundary at expiry, which we will label S_0. At expiry, for a put we know $V(S, T) = \max(E - S, 0)$, and substituting this into the PDE (2) enables us to calculate $(\partial V/\partial t)$ at expiry; the sign of $(\partial V/\partial t)$ at expiry will then tell us whether the constraint is violated as we move away from expiry. If $(\partial V/\partial t) > 0$ at expiry, then as we move away from expiry, $V(S, t)$ will decrease and the constraint will be violated, so that the option should already have been exercised, while if $(\partial V/\partial t) \leq 0$, it should have been held to expiry, and if $(\partial V/\partial t) = 0$, the holder will have been ambivalent between holding and exercising. We can examine this behavior for a put by substituting the pay-off $V(S, T) = \max(E - S, 0) = (E - S)H(E - S)$ into (2), where

$$H(S) = \begin{cases} 1 & S < 0 \\ 0 & S < 0 \end{cases} \tag{3}$$

is the Heaviside step function. This tells us that for a put at expiry, we have

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= (rE - DS)H(E-S) + S\left[(E-S)(r-D) - \sigma^2 S\right]\delta(S-E) \\
&\quad + \frac{1}{2}\sigma^2 S^2 (E-S)\delta'(S-E) \\
&= \left\{ \begin{array}{cc} rE - DS & S < E \\ -\sigma^2 E^2 \delta(0) + (r-D)EH(0) & S = E \\ 0 & S > E \end{array} \right. , \qquad (4)
\end{aligned}
$$

where $\delta(S)$ is the delta function, which is defined to be zero for $S \neq 0$ and infinite for $S = 0$, subject to $\int_{-\infty}^{\infty} \delta(S)dS = 1$. From (4), we can see that there are two distinct cases. If $D > r$, so that the dividend yield is greater than the risk-free rate, we have

$$
\frac{\partial V}{\partial t} \left\{ \begin{array}{cc} > 0 & S < rE/D < E \\ = 0 & S = rE/D \\ < 0 & S > rE/D \end{array} \right. , \qquad (5)
$$

so that the free boundary will start at $S_0 = rE/D < E$. However, for $D \leq r$, we have

$$
\frac{\partial V}{\partial t} \left\{ \begin{array}{cc} > 0 & S < E \\ < 0 & S \geq E \end{array} \right. , \qquad (6)
$$

which means that the option should have been held to expiry if $S \geq E$ but exercised if $S < E$, so that the free boundary must start at $S_0 = E$.

The case $D > r$ can be recovered from [**8, 2**] by using put-call symmetry [**7, 12**]. Because of this, in the present study we will consider only the case $D \leq r$. Put-call symmetry means that this study also covers the call with $D \geq r$. Returning to (2), we will proceed as in [**8, 2, 3**] and make the change of variables $S = E\,\mathrm{e}^{x}$, $t = T - 2\tau/\sigma^2$ and $V(S,t) = E - S + Ev(x,\tau)$, which converts (2) into a more standard diffusion-like equation.

$$
\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k_2 - 1)\frac{\partial v}{\partial x} - k_1 v + f(x), \qquad (7)
$$

where the nonhomogeneous term is given by $f(x) = (k_1 - k_2)\,\mathrm{e}^{x} - k_1$ for the put, with $k_1 = 2r/\sigma^2$ and $k_2 = 2(r-D)/\sigma^2$. The restriction $D \leq r$ means that $k_2 \geq 0$. This equation (7) is valid for $\tau > 0$ and must be solved together with the payoff at expiry, $\tau = 0$,

$$
v(x,0) = \left\{ \begin{array}{cc} \mathrm{e}^{x} - 1 & x > 0 \\ 0 & x \leq 0 \end{array} \right. , \qquad (8)
$$

while on the free boundary, we have

$$
v = \frac{\partial v}{\partial x} = 0. \qquad (9)
$$

At expiry, we have

$$\frac{\partial v}{\partial \tau} = [(k_1 - k_2)\,\mathrm{e}^{\,x} - k_1]\,[1 + H(x)] + [k_2 - 1 - (k_2+1)\,\mathrm{e}^{\,x}]\,\delta(x) + [1 - \mathrm{e}^{\,x}]\,\delta^{'}(x)$$

$$= \left\{ \begin{array}{cc} 2(k_1 - k_2)\,\mathrm{e}^{\,x} - 2k_1 & x > 0 \\ -2\delta(0) - k_2\,[1 + H(0)] & x = 0 \\ (k_1 - k_2)\,\mathrm{e}^{\,x} - k_1 & x < 0 \end{array} \right. , \tag{10}$$

so that the free boundary must start at $x = 0$ at $\tau = 0$.

In the analysis that follows, strictly speaking the equation (7) is valid only where it is valid to hold the option, so that at expiry, we can only impose the initial condition on $x > 0$ that

$$v(x, 0) = \mathrm{e}^{\,x} - 1, \tag{11}$$

while for $x < 0$, it is assumed that the option has already been exercised so that we cannot impose the initial condition.

Following [**16**]–[**24**], we will seek series solutions to (7). We will consider first the case $D < r$, and consider the case $D = r$ later. If we try a series of the form

$$v(x, \tau) = \sum_{n=1}^{\infty} \tau^{n/2} F_n(\xi) \tag{12}$$

where $\xi = x/(2\sqrt{\tau})$ is a similarity variable, we run into the problems discussed in [**2**] and elsewhere, and it is necessary instead to assume a series of the form

$$v(x, \tau) = \tau^{1/2} F_1^{(0)}(\xi) + \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} \tau^{n/2}\,(-\ln\tau)^{-m}\,F_n^{(m)}(\xi). \tag{13}$$

The minus sign is included in $(-\ln\tau)$ because $\ln\tau$ is negative for $0 < \tau < 1$. It is worth noting that logarithms first enter in this series with the τ^1 terms rather than the leading $\tau^{1/2}$ term. In order to solve for the functions $F_n^{(m)}(\xi)$ in (13), we substitute this series (13) into (7) and group powers of τ and $(-\ln\tau)$. The resulting equations can be written in terms of the operator

$$L_n \quad \equiv \quad \frac{1}{4}\frac{d^2}{d\xi^2} + \frac{1}{2}\frac{d}{d\xi} - \frac{n}{2}. \tag{14}$$

For the terms independent of $(-\ln\tau)$, we find

$$\begin{aligned} L_1 F_1^{(0)} &= 0, \\ L_2 F_2^{(0)} &= \frac{1}{2}\,(1 - k_2)\,F_1^{(0)'} + k_2, \\ L_3 F_3^{(0)} &= \frac{1}{2}\,(1 - k_2)\,F_2^{(0)'} + k_1 F_1^{(0)} + 2\,(k_2 - k_1)\,\xi, \end{aligned} \tag{15}$$

with the general case for $n \geq 3$ given by

$$L_n F_n^{(0)} \quad = \quad \frac{1}{2}\,(1 - k_2)\,F_{n-2}^{(0)'} + k_1 F_{m-2}^{(0)} + \frac{2^{n-2}}{(n-2)!}\,(k_2 - k_1)\,\xi^n. \tag{16}$$

It is straightforward to find solutions to (15), (16) that satisfy the initial condition (11) by assuming that the solutions are of the form

$$F_n^{(0)}(\xi) = f_{n1}^{(0)}(\xi)\mathrm{erf}(\xi) + f_{n1}^{(0)}(\xi)\,\mathrm{e}^{-\xi^2} + f_{n3}^{(0)}(\xi), \tag{17}$$

where the f's are polynomials in ξ, and erf is the error function (and later on, erfc is the complementary error function). Using this approach, we find that

$$\begin{aligned}
F_1^{(0)} &= 2\xi + C_1^{(0)}\left[\pi^{-1/2}\,\mathrm{e}^{-\xi^2} + \xi\left(\mathrm{erfc}(-\xi) - 2\right)\right], \\
F_2^{(0)} &= 2\xi^2 + \left[\frac{1}{2}\left(k_1 - 1\right)C_1^{(0)} + \left(2\xi^2 + 1\right)C_2^{(0)}\right]\left(\mathrm{erfc}(-\xi) - 2\right) \\
&+ 2C_2^{(0)}\pi^{-1/2}\xi\,\mathrm{e}^{-\xi^2}, \\
F_3^{(0)} &= \frac{4}{3}\xi^3 + \left[-k_1C_1^{(0)} + 2\left(k_2 - 1\right)C_2^{(0)} + C_3^{(0)}\left(3 + 2\xi^2\right)\right]\xi\left(\mathrm{erfc}(-\xi) - 2\right) \\
&+ \pi^{-1/2}\,\mathrm{e}^{-\xi^2}\left[\frac{C_1^{(0)}}{4}\left((1 - k_2)^2 - 4k_1\right) + 2C_2^{(0)}\left(k_2 - 1\right) + 2C_3^{(0)}\left(1 + \xi^2\right)\right], \\
F_4^{(0)} &= \frac{2}{3}\xi^4 \\
&+ \left[\frac{1}{2}C_1^{(0)}k_1\left(1 - k_2\right) + \frac{1}{2}C_2^{(0)}\left(\left(1 - k_2\right)^2 - k_1\left(2 + \xi^2\right)\right)\right. \\
&\quad \left. + \frac{3}{2}C_3^{(0)}\left(k_2 - 1\right)\left(1 + 2\xi^2\right) + C_4^{(0)}\left(3 + 12\xi^2 + 4\xi^4\right)\right]\left(\mathrm{erfc}(-\xi) - 2\right) \\
&+ \pi^{-1/2}\xi\,\mathrm{e}^{-\xi^2}\left[\frac{1}{12}C_1^{(0)}\left(1 - k_2\right)^3 - 2k_1C_2^{(0)}\right. \\
&\quad \left. + 3\left(k_2 - 1\right)C_3^{(0)} + 2C_4^{(0)}\left(5 + 2\xi^2\right)\right].
\end{aligned} \tag{18}$$

In (18), the $C_n^{(0)}$ are constants which can be determined by applying the conditions on the free boundary.

For the moment, we will set aside the log terms and try to impose the conditions (9) on the free boundary, which we will assume that we can write as $x = x_f(\tau)$. If we assume that

$$x_f(\tau) \sim \sum_{n=1}^{\infty} x_n \tau^{n/2} \tag{19}$$

as $\tau \to 0$, we find that at leading order,

$$\begin{aligned}
C_1^{(0)}\left[\frac{x_1}{2}\mathrm{erfc}(-x_1/2) + \pi^{-1/2}\,\mathrm{e}^{-x_1^2/4}\right] + \left[1 - C_1^{(0)}\right]x_1 = 0, \\
C_1^{(0)}\mathrm{erfc}(-x_1/2) + 2\left[1 - C_1^{(0)}\right] = 0,
\end{aligned} \tag{20}$$

and in order to solve both of these equations simultaneously, we require that $\mathrm{e}^{-x_1^2/4} = 0$ so that $x_1 = \pm\infty$. Since we require the boundary to go down rather than up, we must choose $x_1 = -\infty$, so that $\mathrm{e}^{-x_1^2/4} = \mathrm{erfc}(-x_1/2) = 0$ and $C_1^{(0)} = 1$. In a moment, we will see that in our analysis, where we have grouped terms in powers of τ, the statement $\mathrm{e}^{-x_1^2/4} = \operatorname{erfc}(-x_1/2) = 0$ actually means that the terms $\mathrm{e}^{-x_1^2/4}$ and $\operatorname{erfc}(-x_1/2)$ are $O\left(\tau^{1/2}\right)$, so they vanish at this order but re-appear at a later order in the analysis.

At the next order, if we set $\mathrm{e}^{-x_1^2/4} = \operatorname{erfc}(-x_1/2) = 0$ and $C_1 = 1$, we find that we require

$$\begin{aligned} \frac{1}{2}\left(1 - 2C_2^{(0)}\right) x_1^2 + 1 - 2C_2^{(0)} - k_2 &= 0, \\ 2\left(1 - 2C_2^{(0)}\right) x_1 &= 0, \end{aligned} \tag{21}$$

and if we set $C_2^{(0)} = 1/2$, the second equation is satisfied, but the first reduces to $-k_2 = 0$, which is clearly wrong, except for the special case $D = r$ when we actually do have $k_2 = 0$. It is to deal with this inconsistency that we require $\mathrm{e}^{-x_1^2/4}$ and $\operatorname{erfc}(-x_1/2)$ to be $O\left(\tau^{1/2}\right)$, so that they enter into this equation and remove the inconsistency. To accomplish this, the expansion for $x_f(\tau)$ must be of the form

$$x_f(\tau) \sim \sum_{n=1}^{\infty} \tau^{n/2} f_n(-\ln\tau), \tag{22}$$

where

$$f_n(-\ln\tau) \sim (-\ln\tau)^{a_n} \sum_{m=0}^{\infty} x_n^{(m)}(-\ln\tau)^{-m}. \tag{23}$$

The presence of logs in the series (22), (23) for $x_f(\tau)$ and the functions f_n necessitate the presence of logs in the series (13) for $v(x,\tau)$

With this expression for x_f, on the free boundary we have

$$\begin{aligned} \mathrm{e}^{-\xi^2} &= \exp\left[-\frac{x_f^2}{4\tau}\right] \\ &\sim \mathrm{e}^{-f_1^2/4}\left[1 - \frac{1}{2}f_1 f_2 \tau^{1/2} + \left(\frac{1}{8}f_1^2 f_2^2 - \frac{1}{2}f_1 f_3 - \frac{1}{4}f_2^2\right)\tau + \cdots\right]. \end{aligned} \tag{24}$$

At leading order in this expression, we require that $\mathrm{e}^{-f_1^2/4} \sim O\left(\tau^{1/2}\right)$, so that $\exp\left[-\frac{x_1^{(0)2}}{4}(-\ln\tau)^{2a_1}\right] \sim \tau^{1/2}$ or $-\frac{x_1^{(0)2}}{4}(-\ln\tau)^{2a_1} \sim \frac{1}{2}\ln\tau$, which means that $a_1 = 1/2$ and $x_1^{(0)} = -\sqrt{2}$, and hence

$$\mathrm{e}^{-f_1^2/4} \quad\sim\quad \tau^{1/2}\,\mathrm{e}^{x_1^{(2)}/\sqrt{2}}\left[1 + \left(\frac{x_1^{(2)}}{\sqrt{2}} - \frac{x_1^{(1)2}}{4}\right)(-\ln\tau)^{-1} + \cdots\right]. \tag{25}$$

Similarly, we can show that

$$\begin{aligned} \operatorname{erfc}(-\xi) &= \operatorname{erfc}\left[-\frac{x_f}{2\sqrt{\tau}}\right] \\ &\sim \operatorname{erfc}\left[-\frac{f_1}{2}\right] + \frac{\mathrm{e}^{-f_1^2/4}}{\sqrt{\pi}}\left[f_2\tau^{1/2} + \left(f_3 - \frac{1}{4}f_1 f_2^2\right)\tau \cdots\right], \end{aligned} \tag{26}$$

and we can use the result that as $\zeta \to \infty$ [**1**],

$$\text{erfc}(\zeta) \sim \frac{\mathrm{e}^{-\zeta^2}}{\zeta\sqrt{\pi}}\left[1+\sum_{m=1}^{\infty}\frac{1\times 3\times\cdots\times(2m-1)}{(-2\zeta^2)^m}\right] \tag{27}$$

to give

$$\begin{aligned}\text{erfc}\left[-\frac{f_1}{2}\right] &\sim \tau^{1/2}\,(-\ln\tau)^{-1/2}\,\pi^{-1/2}\,\mathrm{e}^{\,x_1^{(2)}/\sqrt{2}} \\ &\times\left[\sqrt{2}+\left(x_1^{(1)}+x_1^{(2)}-\sqrt{2}-\frac{x_1^{(2)2}}{2\sqrt{2}}\right)(-\ln\tau)^{-1}+\cdots\right].\end{aligned} \tag{28}$$

Before we can compute the coefficients in the series (22), (23) for the location of the free boundary, it is necessary to solve for some of the terms involving logs in the series (13) for $v(x,\tau)$. Considering the terms at $O\left(\tau^{n/2}\,(-\ln\tau)^{-1}\right)$, at successive orders we find

$$\begin{aligned} L_2F_2^{(1)}(\xi) &= 0 \\ L_3F_3^{(1)}(\xi) &= \frac{1-k_2}{2}F_2^{(1)'}(\xi),\end{aligned} \tag{29}$$

with the general equation for $n \geq 3$ given by

$$L_nF_n^{(1)}(\xi) \quad = \quad \frac{1-k_2}{2}F_{n-2}^{(1)'}(\xi)+k_1F_{m-2}^{(1)}. \tag{30}$$

The solutions at the first few orders are given by

$$\begin{aligned} F_2^{(1)} &= C_2^{(1)}\left[2\pi^{-1/2}\xi\,\mathrm{e}^{-\xi^2}+\left(2\xi^2-1\right)(\text{erfc}(-\xi)-2)\right] \\ F_3^{(1)} &= 2C_2^{(1)}\,(k_2-1)\left[\xi\,(\text{erfc}(-\xi)-2)+\pi^{-1/2}\,\mathrm{e}^{-\xi^2}\right] \\ &+\ C_3^{(1)}\left[\left(3+2\xi^2\right)\xi\,(\text{erfc}(-\xi)-2)+2\left(1+\xi^2\right)\pi^{-1/2}\,\mathrm{e}^{-\xi^2}\right].\end{aligned} \tag{31}$$

It follows that the conditions (21) on the free boundary become

$$\begin{aligned}&\tau^{-1/2}\left[\frac{f_1}{2}\text{erfc}(-f_1/2)+\pi^{-1/2}\,\mathrm{e}^{-f_1^2/4}\right] \\ &\left[\frac{1}{2}\left(1-2C_2^{(0)}\right)f_1^2+1-2C_2^{(0)}-k_2\right]+(-\ln\tau)^{-1}\,C_2^{(1)}\left(2-f_1^2\right)+\cdots=0,\end{aligned} \tag{32}$$

and

$$\tau^{-1/2}\text{erfc}(-f_1/2)+2\left(1-2C_2^{(0)}\right)f_1-(-\ln\tau)^{-1}\,4C_2^{(1)}f_1+\cdots=0. \tag{33}$$

or

$$\begin{aligned}&(-\ln\tau)\left(1-2C_2^{(0)}\right)+\left(1-2C_2^{(0)}\right)\left(1-\frac{x_1^{(1)}}{\sqrt{2}}\right)-k_2-2C_2^{(1)} \\ &+(-\ln\tau)^{-1/2}\,\pi^{-1/2}\,\mathrm{e}^{\,x_1^{(2)}/\sqrt{2}}\left(\frac{x_1^{(1)2}}{4}-\frac{x_1^{(2)}}{\sqrt{2}}\right)+O\left((-\ln\tau)^{-1}\right)=0\end{aligned} \tag{34}$$

and

$$
\begin{aligned}
&(-\ln\tau)^{1/2}\,2^{3/2}\left(2C_2^{(0)}-1\right)\\
&+(-\ln\tau)^{-1/2}\left[2x_1^{(1)}\left(1-2C_2^{(0)}\right)+2^{5/2}C_2^{(1)}+2^{1/2}\pi^{-1/2}\,\mathrm{e}^{\,x_1^{(2)}/\sqrt{2}}\right]\\
&+(-\ln\tau)^{-1}\,\mathrm{e}^{\,x_1^{(2)}/\sqrt{2}}\pi^{-1/2}\left[x_1^{(2)}-2^{-3/2}x_1^{(1)2}\right]+O\left((-\ln\tau)^{-3/2}\right)=0
\end{aligned}
\tag{35}
$$

so that we find $C_2^{(0)}=1/2$, and that $C_2^{(1)}=-k_2/2$ and $\mathrm{e}^{\,x_1^{(1)}/\sqrt{2}}=2k_2\pi^{1/2}$, so $x_1^{(1)}=\sqrt{2}\ln\left(2k_2\pi^{1/2}\right)$, and $x_1^{(2)}=2^{-3/2}x_1^{(1)2}$. Similarly, at the next order in τ, we find $C_3^{(0)}=1/3$, and that $C_3^{(1)}=k_1-k_2$ and $a_1=0$ and $x_2^{(0)}=2k_1/k_2-1-k_2$.

It follows that the free boundary is given by

$$
\begin{aligned}
x_f(\tau)\quad\sim\quad&\sqrt{2\tau(-\ln\tau)}\left(-1+\left[\frac{\ln\left(2k_2\pi^{1/2}\right)}{-\ln\tau}\right]+\frac{1}{2}\left[\frac{\ln\left(2k_2\pi^{1/2}\right)}{-\ln\tau}\right]^2+\cdots\right)\\
&+\tau\left[2k_1/k_2-1-k_2+\cdots\right]+\cdots
\end{aligned}
\tag{36}
$$

with the value of the option given by

$$
\begin{aligned}
v(x,t)\sim\tau^{1/2}&\left[\xi\mathrm{erfc}(-\xi)+\frac{\mathrm{e}^{-\xi^2}}{\sqrt{\pi}}\right]\\
+\ \tau&\left[\left(\frac{k_2}{2}+\xi^2\right)\mathrm{erfc}(-\xi)+\frac{\xi\,\mathrm{e}^{-\xi^2}}{\sqrt{\pi}}-k_2\right]\\
+\ \tau&(-\ln\tau)^{-1}k_2\left[-\left(\frac{1}{2}+\xi^2\right)\mathrm{erfc}(-\xi)-\pi^{-1/2}\xi\,\mathrm{e}^{-\xi^2}+2\xi^2+1\right]+\cdots\\
+\ \tau^{3/2}&\left[\left(k_2-k_1+\frac{2}{3}\xi^2\right)\xi\mathrm{erfc}(-\xi)\right.\\
&\left.+\frac{\mathrm{e}^{-\xi^2}}{4\sqrt{\pi}}\left(k_2^2+2k_2-4k_1-\frac{1}{3}+\frac{8\xi^2}{3}\right)+2\left(k_1-k_2\right)\xi\right]\\
+\ \tau^{3/2}&(-\ln\tau)^{-1}\left[\left(2k_2+k_2^2-3k_1+2(k_2-k_1)\xi^2\right)\xi\mathrm{erfc}(\xi)\right.\\
&\left.+\frac{\mathrm{e}^{-\xi^2}}{\sqrt{\pi}}\left(2k_1-k_2-k_2^2+2(k_1-k_2)\xi^2\right)\mathrm{e}^{-\xi^2}\right]+\cdots\\
+\ \cdots&.
\end{aligned}
\tag{37}
$$

2.1. The case $D=r$

Looking at the solution (37) for $D<r$, we recall that we required $\mathrm{e}^{\,x_1^{(1)}/\sqrt{2}}=2k_2\pi^{1/2}$, so that if $k_2=0$, corresponding to $D=r$, we again have a problem. For this case, (7) becomes

$$
\frac{\partial v}{\partial\tau}=\frac{\partial^2 v}{\partial x^2}-\frac{\partial v}{\partial x}-k_1v+f(x),
\tag{38}
$$

with $f(x) = k_1\,(\mathrm{e}^{\,x} - 1)$. Proceeding as above, $F_1^{(0)}, F_2^{(0)}, F_3^{(0)}, \cdots$ are again given by (18), but with $k_2 = 0$. If we impose the conditions at the free boundary, at leading order we again find that $C_1^{(0)} = 1$ and at the next order we find $C_2^{(0)} = 1/2$, so that

$$\begin{aligned} F_1^{(0)} &= \pi^{-1/2}\,\mathrm{e}^{\,-\xi^2} + \xi\mathrm{erfc}(-\xi) \\ F_2^{(0)} &= \pi^{-1/2}\xi\,\mathrm{e}^{\,-\xi^2} + \xi^2\mathrm{erfc}(-\xi) \end{aligned} \tag{39}$$

while from the $O\left(\tau^{3/2}\right)$ terms we require

$$-4\left(C_3^{(0)} - \frac{1}{3}\right)\xi^3 + 2\left(k_1 + 1 - 3C_3^{(0)}\right)\xi = 0. \tag{40}$$

If we set $C_3^{(0)} = 1/3$, this equation reduces to $2k_1\xi = 0$, which again is clearly wrong, and to rectify this, the $\mathrm{erfc}(-\xi)$ and $\mathrm{e}^{\,-\xi^2}$ terms from $F_1^{(0)}$ must be added to (40) to balance the $2k_1\xi$ term. To do this, if we suppose that

$$x_f(\tau) \sim \sum_{n=1}^{\infty} \tau^{n/2} g_n(\tau), \tag{41}$$

then we require $\mathrm{e}^{\,-g_1^2/4} \sim \tau g_1$, as opposed to the relation $\mathrm{e}^{\,-f_1^2/4} \sim \tau^{1/2}$ for the case $D < r$, so that

$$g_1(\tau) \sim \left[2W_L\left(\frac{1}{2\tau^2}\right)\right]^{1/2} \tag{42}$$

where W_L is a special function, the Lambert W function, which is defined to be the solution to the equation $W_L(x)\,\mathrm{e}^{\,W_L(x)} = x$. It follows that

$$\begin{aligned} g_1(\tau) &\sim \left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{1/2} \sum_{m=0}^{\infty} x_1^{(m)} \left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-m} \\ g_n(\tau) &\sim \left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{a_n} \sum_{m=0}^{\infty} x_n^{(m)} \left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-m} \end{aligned} \tag{43}$$

This means that our series for $v(x,\tau)$ must be of the form

$$\begin{aligned} v(x,\tau) &= \tau^{1/2}F_1^{(0)}(\xi) + \tau F_2^{(0)}(\xi) \\ &+ \sum_{n=3}^{\infty}\sum_{m=0}^{\infty} \tau^{n/2}\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-m} F_n^{(m)}(\xi), \end{aligned} \tag{44}$$

with $F_1^{(0)}$ and $F_2^{(0)}$ given by (39) above while $F_3^{(1)}$ obeys

$$L_3F_3^{(1)} = 0 \tag{45}$$

with a solution

$$F_3^{(1)} = C_3^{(1)}\left[2\pi^{-1/2}\,\mathrm{e}^{\,-\xi^2}\left(1+\xi^2\right) - \xi\left(3+2\xi^2\right)\mathrm{erfc}(\xi)\right]. \tag{46}$$

It is straightforward to show that as $\tau \to 0$, the counterparts of (24), (25), (26) and (28) are

(47)
$$
\begin{aligned}
\mathrm{e}^{-g_1^2/4} \sim & \,\mathrm{e}^{-x_1^{(1)}/2}\tau \left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{1/2} \\
& \times \left(1 - \left[\frac{x_1^{(2)}}{2} + \frac{x_1^{(1)2}}{4}\right]\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-1} + \cdots\right) + O\left(\tau^{3/2}\right), \\
\operatorname{erfc}\left[-\frac{g_1}{2}\right] \sim & \,\pi^{-1/2}\,\mathrm{e}^{x_1^{(1)}/2}\tau \\
& \times \left(2 + \left[x_1^{(2)} + 2x_1^{(1)} - \frac{1}{2}x_1^{(1)2} - 4\right]\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-1} + \cdots\right) \\
& + O\left(\tau^{3/2}\right).
\end{aligned}
$$

The conditions on the free boundary then tell us that we require

$$
\begin{aligned}
2k_1 - 3C_3^{(1)} + 2\pi^{-1/2}\,\mathrm{e}^{x_1^{(1)}/2} + O\left(\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-1}\right) &= 0 \\
\left(\frac{1}{2}C_3^{(1)} - k_1\right)\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{1/2} + O\left(\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-1/2}\right) &= 0
\end{aligned}
\tag{48}
$$

so that $C_3^{(1)} = 2k_1$ and $\mathrm{e}^{x_1^{(1)}/2} = 2k_1\pi^{1/2}$, or equivalently $x_1^{(1)} = 2\ln\left(2k_1\pi^{1/2}\right)$. Hence for the case $r = D$, the value of the option is given by

$$
\begin{aligned}
v(x,t) \sim & \,\tau^{1/2}\left[\xi\operatorname{erfc}(-\xi) + \frac{\mathrm{e}^{-\xi^2}}{\sqrt{\pi}}\right] \\
& + \tau\left[\xi^2\operatorname{erfc}(-\xi) + \frac{\xi\,\mathrm{e}^{-\xi^2}}{\sqrt{\pi}}\right] \\
& + \tau^{3/2}\left[\left(\frac{2}{3}\xi^3 - \frac{1}{12} - k_1\right)\frac{\mathrm{e}^{-\xi^2}}{\sqrt{\pi}} + k_1\xi\operatorname{erfc}(\xi) + \frac{2\xi^3}{3}\operatorname{erfc}(-\xi)\right] \\
& + 2k_1\tau^{3/2}\left[\frac{2\,\mathrm{e}^{-\xi^2}}{\sqrt{\pi}}\left(1+\xi^2\right) - \xi\left(3+2\xi^2\right)\operatorname{erfc}(\xi)\right]\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-1} \\
& + \cdots
\end{aligned}
\tag{49}
$$

and the location of the free boundary is given by

$$
\begin{aligned}
x_f(\tau) \sim & \left[2\tau W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{1/2}\left(-1 + \ln\left(2k_1\pi^{1/2}\right)\left[W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-1} + \cdots\right) \\
& + O(\tau).
\end{aligned}
\tag{50}
$$

3. Discussion

In the previous section, we revisited the problem of the American put close to expiry and used an asymptotic expansion of the Black-Scholes-Merton PDE to find expressions for the location of the free boundary (36), (50) and the value of the option (37), (49) for the cases $D < r$ and $D = r$ in that limit. For the case $D < r$, we found that close to expiry, the location of the free boundary was given by $x_f(\tau) \sim -\sqrt{2\tau(-\ln\tau)}$,the location of the free boundary was given by $x_f(\tau) \sim -\sqrt{2\tau(-\ln\tau)}$, while for $D = r$ it was given by $x_f(\tau) \sim -\sqrt{2\tau W_L\left(\tau^{-2}/2\right)}$, where W_L was the Lambert W function. For the case $D > r$, put-call symmetry [**7, 12**] together with results from the call [**8, 2**] indicates that the behavior close to expiry in that case will be $x_f(\tau) \sim -x_0\sqrt{\tau}$ where x_0 is a constant that must be found numerically. The free boundary in terms of the stock price S is located at $S = S_f(\tau) = E\,\mathrm{e}^{x_f(\tau)}$, so that close to expiry we have

$$S_f \sim \begin{cases} -\sigma\sqrt{(T-t)(-\ln\frac{\sigma^2(T-t)}{2})} & D < r \\ -\sigma\sqrt{(T-t)W_L\left(\frac{2}{\sigma^4(T-t)^2}\right)} & D = r \\ -x_0\sigma\sqrt{(T-t)/2} & D > r. \end{cases} \tag{51}$$

These three behaviors are somewhat different. The $\tau^{1/2}$ behavior for $D > r$ is the standard behavior for Stefan problems found in the classic works of Tao [**16**]–[**24**]. The $\sqrt{-\tau\ln\tau}$ behavior for $r < D$, although previously found by a number of other authors working on the American put [**5, 15, 11, 9, 10, 3**] must be considered somewhat of an oddity for Stefan problems in that we are unaware of this behavior having been encountered in a physical problem, and it does not appear in Tao's work. We should mention that although [**5, 15, 11, 9, 10**] came across this behavior, they did not use our method: Barles *et al.* [**5**] found upper and lower bounds close to expiry and showed that these bounds approached each other, while [**15, 11, 9, 10**] all used integral equation approaches to reformulate the Black-Scholes-Merton PDE and associated boundary and initial conditions as integral equations which they solved asymptotically: Stamicar *et al.* [**15**] used Fourier transforms, Kuske and coworkers [**11, 9**] used Green's functions, and Knessl [**10**] used Laplace transforms to arrive at their respective integral equation, and inevitably various approximations such as Laplace's method were used to evaluate the integrals in those equations. Finally, the $\sqrt{\tau W_L\left(\tau^{-2}/2\right)}$ behavior for $r = D$ is even more unusual that the $\sqrt{-\tau\ln\tau}$ behavior discussed above: it does not appear to have been encountered in any physical problems, and it would appear that previous studies of American options have not encountered this behavior.

Although as we mentioned above, this problem has been studied previously by reformulating it as an integral equation, we believe that in some respects our approach has advantages over the integral equation approach. Firstly, in most if not all of the integral equation studies, it is necessary to use some sort of approximation to determine the asymptotic behavior of the integrals, and that sort of approximation is not required for the asymptotic expansion used here. Secondly, in a sense our analysis sheds more light on why logs are necessary when $D < r$, and the Lambert W function is necessary when $D = r$: for our expansion to work in those cases, we required the $\operatorname{erfc}(-\xi)$ and $\mathrm{e}^{-\xi^2}$ terms to appear at a later order in the expansion to balance certain other terms, so for $D < r$ we needed $\mathrm{e}^{-x_f^2/(4\tau)} \sim \sqrt{\tau}$ while for $D = r$ we needed $\mathrm{e}^{-x_f^2/(4\tau)} \sim \sqrt{\tau} x_f$.

In closing, we would note that although the present study was carried out for the American put with $D \le r$, it can trivially be modified to cover the American call with $D \ge r$ using put-call symmetry [**7, 12**], and also that we have recently presented a similar analysis for an American-style exotic, the lock-in option [**4**].

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R. Mallier, Department of Applied Mathematics, The University of Western Ontario, London ON N6A 5B7 Canada, *e-mail*: `rolandmallier@hotmail.com`

G. Alobaidi, Department of Mathematics, American University of Sharjah, Sharjah, United Arab Emirates, *e-mail*: `galobaidi@yahoo.ca`