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EFFECTIVE ASYMPTOTICS FOR SOME NONLINEAR RECURRENCES AND ALMOST DOUBLY-EXPONENTIAL SEQUENCES

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Abstract. We develop a technique to compute asymptotic expansions for recurrent sequences of the form $a_{n+1} = f(a_n)$, where $f(x) = x - ax^\alpha + bx^\beta + o(x^\beta)$ as $x \to 0$, for some real numbers α, β, a, and b satisfying $a > 0$, $1 < \alpha < \beta$. We prove a result which summarizes the present stage of our investigation, generalizing the expansions in [Amer. Math Monthly, Problem E 3034[1984, 58], Solution [1986, 739]]. One can apply our technique, for instance, to obtain the formula: $a_n = \frac{\sqrt{3}}{\sqrt{n}} - \frac{3\sqrt{3}}{10}\frac{\ln n}{n\sqrt{n}} + \frac{9\sqrt{3}}{50}\frac{\ln n}{n^2\sqrt{n}} + o\left(\frac{\ln n}{n^{5/2}}\right)$, where $a_{n+1} = \sin(a_n)$, $a_1 \in \mathbb{R}$. Moreover, we consider the recurrences $a_{n+1} = a_n^2 + g_n$, and we prove that under some technical assumptions, a_n is almost doubly-exponential, namely $a_n = \lfloor k^{2^n} \rfloor$, $a_n = \lfloor k^{2^n} \rfloor + 1$, $a_n = \lfloor k^{2^n} - \frac{1}{2} \rfloor$, or $a_n = \lfloor k^{2^n} + \frac{5}{2} \rfloor$ for some real number k, generalizing a result of Aho and Sloane [Fibonacci Quart. 11 (1973), 429–437].

1. Introduction

Obtaining an *exact* formula for the terms of a sequence given by a recurrence may not, in general, be possible. It is the intent of this paper to investigate and give asymptotics for sequences given by recurrences of the form $a_{n+1} = f(a_n)$, where $f(x) = x - ax^\alpha + bx^\beta + o(x^\beta)$ as $x \to 0$, for some real numbers α, β, a, and b satisfying $a > 0$, $1 < \alpha < \beta$. We also consider the same recurrence where $f(x) = x - x^2$ and give more detailed asymptotics. Moreover, we prove a few results concerning almost doubly-exponential sequences $a_{n+1} = a_n^2 + g_n$, where $-a_n + 1 < g_n < 2a_n$, generalizing a result of Aho and Sloane [**1**]. For standard notations consult [**3**], or any other book on differential and integral calculus.

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2. Asymptotics of Nonlinear Recurrences

The first part of the next lemma is known as Cesàro's lemma, and the second part is just a small variation of the first. For completeness, we include a proof of the second part of this lemma.

Lemma 1 (Cesàro). *Let $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$ two sequences of real numbers satisfying one of the following conditions:*

(i) *$\{v_n\}_{n\in\mathbb{N}}$ is eventually a strictly increasing sequence converging to infinity, or*

(ii) *$\{v_n\}_{n\in\mathbb{N}}$ is eventually a strictly decreasing sequence converging to zero, and u_n converges to zero.*

If the limit of the sequence $\frac{u_{n+1}-u_n}{v_{n+1}-v_n}$ exists, then the limit of the sequence $\frac{u_n}{v_n}$ exists, and we have the equality

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{u_{n+1}-u_n}{v_{n+1}-v_n}. \tag{1}$$

Proof. Suppose we are given an $\epsilon>0$, and by our hypothesis, for some integer n_0 and some real number l we have

$$\left|\frac{u_{n+1}-u_n}{v_{n+1}-v_n}-l\right|<\epsilon,\quad n\ge n_0.$$

Using (ii), the above inequality can be equivalently written in the form

$$-\epsilon(v_n-v_{n+1})<u_n-u_{n+1}-l(v_n-v_{n+1})<\epsilon(v_n-v_{n+1}),\quad n\ge n_0.$$

Adding up these inequalities from $n\ge n_0$ to some larger integer $m>n\ge n_0$, we get

$$-\epsilon(v_n-v_{m+1})<u_n-u_{m+1}-l(v_n-v_{m+1})<\epsilon(v_n-v_{m+1}),\quad m>n\ge n_0.$$

Letting m go to infinity in the above inequality and taking into account that $u_m\to 0$ and $v_m\to 0$, we obtain

$$-\epsilon v_n\le u_n-lv_n\le \epsilon v_n,\quad n\ge n_0,$$

which gives finally, after dividing by v_n, the conclusion of our lemma. □

Theorem 2. *Suppose f is a real-valued continuous function defined on the interval $I=(0,\delta)$ (for some δ), which has the form $f(x)=x-ax^\alpha+bx^\beta+o(x^\beta)$ as $x\to 0$, for some real numbers α,β, a, and b satisfying $a>0$, $1<\alpha<\beta$. Then, for a_0 sufficiently small, the orbit sequence $a_n=f(a_{n-1})$, satisfies one of the following:*

(i) *if $\beta=2\alpha-1$, then*

$$a_n=\frac{1}{[a(\alpha-1)]^{\frac{1}{\alpha-1}}}\left(\frac{1}{n}\right)^{1/(\alpha-1)}+\frac{b-\frac{a^2\alpha}{2}}{[a(\alpha-1)]^{\frac{2\alpha-1}{\alpha-1}}}\frac{\ln n}{n^{\alpha/(\alpha-1)}}+o\left(\frac{\ln n}{n^{\alpha/(\alpha-1)}}\right),$$

(ii) *if* $\beta > 2\alpha - 1$, *then*

$$a_n = \frac{1}{[a(\alpha-1)]^{\frac{1}{\alpha-1}}}\left(\frac{1}{n}\right)^{1/(\alpha-1)} - \frac{\frac{a^2\alpha}{2}}{[a(\alpha-1)]^{\frac{2\alpha-1}{\alpha-1}}}\frac{\ln n}{n^{\alpha/(\alpha-1)}} + o\left(\frac{\ln n}{n^{\alpha/(\alpha-1)}}\right).$$

(iii) *if* $\beta < 2\alpha - 1$ *and* $b \neq 0$, *then*

$$a_n = \frac{1}{[a(\alpha-1)]^{\frac{1}{\alpha-1}}}\left(\frac{1}{n}\right)^{1/(\alpha-1)} + \frac{b\,[a(\alpha-1)]^{\frac{\alpha-\beta-1}{\alpha-1}}}{a(2\alpha-1-\beta)}\left(\frac{1}{n}\right)^{\frac{\beta-1}{\alpha-1}} + o\left(\left(\frac{1}{n}\right)^{\frac{\beta-1}{\alpha-1}}\right).$$

Proof. We give the idea of the proof only in the case (i). Since $f(x) < x$, for x in a small neighborhood of zero, the sequence a_n is decreasing to zero if we assume also that a_0 is positive. Then we apply Cesáro's lemma for the sequences $u_n = \frac{1}{a_n^{\alpha-1}}$, and $v_n = n$:

$$\lim_n \frac{1}{na_n^{\alpha-1}} = \lim_n \left(\frac{1}{a_{n+1}^{\alpha-1}} - \frac{1}{a_n^{\alpha-1}}\right) = \lim_n \left(\frac{1}{f(a_n)^{\alpha-1}} - \frac{1}{a_n^{\alpha-1}}\right).$$

Using the well-known formula from calculus $\lim_{x\to 0} \dfrac{1-(1-x)^\gamma}{x} = \gamma$, we obtain

$$\begin{aligned}
\lim_n \frac{1}{na_n^{\alpha-1}} &= \lim_n \frac{1}{a_n^{\alpha-1}} \frac{\left(1-\left(1-aa_n^{\alpha-1}+ba_n^{\beta-1}+o(a_n^{\beta-1})\right)^{\alpha-1}\right)}{\left(1-aa_n^{\alpha-1}+ba_n^{\beta-1}+o(a_n^{\beta-1})\right)^{\alpha-1}} \\
&= \lim_n \frac{1-\left(1-aa_n^{\alpha-1}+ba_n^{\beta-1}+o(a_n^{\beta-1})\right)^{\alpha-1}}{aa_n^{\alpha-1}-ba_n^{\beta-1}-o(a_n^{\beta-1})} \frac{aa_n^{\alpha-1}-ba_n^{\beta-1}-o(a_n^{\beta-1})}{a_n^{\alpha-1}} \\
&= (\alpha-1)a.
\end{aligned}$$

Equivalently, this means that $a_n = \dfrac{1}{[a(\alpha-1)]^{\frac{1}{\alpha-1}}}\left(\dfrac{1}{n}\right)^{1/(\alpha-1)} + o\left(\left(\dfrac{1}{n}\right)^{1/(\alpha-1)}\right)$, which is the first approximation in the statements (i)–(iii). Now let us assume that $\beta = 2\alpha - 1$. To simplify the computations we will denote $c = a(\alpha-1)$, and $y_n = aa_n^{\alpha-1} - ba_n^{\beta-1} - o(a_n^{\beta-1})$, which under the above assumption becomes $y_n = aa_n^{\alpha-1} - ba_n^{2(\alpha-1)} - o(a_n^{2(\alpha-1)})$. We want to apply Cesàro's lemma again for $u_n = cn - \frac{1}{a_n^{\alpha-1}}$ and $v_n = \ln\ n$:

$$\begin{aligned}
\lim_n \frac{cn - \frac{1}{a_n^{\alpha-1}}}{\ln n} &= \lim_n \frac{c - \frac{1}{a_{n+1}^{\alpha-1}} + \frac{1}{a_n^{\alpha-1}}}{\ln(1+\frac{1}{n})} \\
&= \lim_n\ n\frac{(1-y_n)^{\alpha-1} + ca_n^{\alpha-1}(1-y_n)^{\alpha-1} - 1}{a_n^{\alpha-1}(1-y_n)^{\alpha-1}} \\
&= c\lim_n\ n^2y_n^2\frac{(1-y_n)^{\alpha-1} - 1 + (\alpha-1)y_n}{y_n^2} + n^2\left(ca_n^{\alpha-1}(1-y_n)^{\alpha-1} - (\alpha-1)y_n\right).
\end{aligned}$$

Taking into account that $\lim_{n\to\infty} ny_n = \frac{a}{c}$ and $\lim_{y\to 0} \frac{(1-y)^\gamma - 1 + \gamma y}{y^2} = \frac{\gamma(\gamma-1)}{2}$, we may continue the above computation as follows:

$$\begin{aligned}\lim_n \frac{cn - \frac{1}{a_n^{\alpha-1}}}{\ln n} &= \frac{a(\alpha-2)}{2} + c\lim_n (\alpha-1)n^2[a a_n^{\alpha-1}(1-y_n)^{\alpha-1} \\ &\quad - a a_n^{\alpha-1} + b a_n^{2(\alpha-1)} + o(a_n^{2(\alpha-1)})] = \frac{a(\alpha-2)}{2} + \frac{b}{a} \\ &\quad + a(\alpha-1)\lim_n \; n^2 a_n^{\alpha-1}\left((1-y_n)^{\alpha-1} - 1\right) \\ &= \frac{b - \frac{a^2\alpha}{2}}{a}\end{aligned}$$

This finally says that

$$\lim_n \frac{\left[(cn)^{1/(\alpha-1)}a_n - 1\right] n}{\ln n} = \frac{b - \frac{a^2\alpha}{2}}{c^2},$$

from which (i) can be easily derived. The rest of the cases are treated similarly. □

In Odlyzko's excellent paper [**5**], a few methods are studied for approximating nonlinear recurrences by linear ones. If $f(x) = x - x^2$, the following method for determining an approximation of a_n is presented. Let $x_n = 1/a_n$. By iteration we obtain (cf. [**2**])

$$x_n = x_{n-1} + 1 + \frac{a_{n-1}}{1 - a_{n-1}} = \cdots = \frac{1}{a_0} + n + \sum_{j=0}^{n-1} \frac{a_j}{1 - a_j}.$$

If $0 < a_0 < 1$, then we get that

$$n \le x_n \le n + O(\log n),$$

therefore $x_n = n + \log n + o(\log n)$. In our next theorem, we push further the technique (by a somewhat similar method). We would like to mention that the function of which orbit is studied here constitutes an important case of an one-dimensional dynamical system (see Theorem 10.1, Chap. II of [**4**]).

Theorem 3. *Assume $a_{n+1} = f(a_n)$, where $f(x) = x - x^2$. For each $a_1 \in I = (0,1)$, the function g defined by*

$$g(a_1) = \lim_{n\to\infty} \left(\frac{1}{a_n} - n - \ln n\right), \tag{2}$$

has the properties:

(i) *g is continuously differentiable on I, and for all $x \in I$ we have $g(x) = g(1-x)$, and $g(f(x)) = g(x) + 1$;*

(ii) *g is strictly decreasing on $(0, 1/2)$, strictly increasing on $(1/2, 1)$, and its minimum value $g(1/2)$ is a positive number;*

(iii) *the measure $d\xi(x) = g'(x)dx$ is invariant under the action of f on $(0, 1/2)$, i.e., for any measurable subset A of $(0, 1/2)$ we have $\xi(A) = \xi(f(A))$;*

(iv) *if we denote* $G_k(a_1) = \sum_{n\geq 1}^{\infty} \left(\frac{a_n}{1-a_n}\right)^k$, $k \geq 2$, *then for* $x \in (0, 1/2)$

$$g(x) = \ln\left(C + \int_x^{1/2} \frac{1}{t} \exp\left(\frac{1}{t} - 1 - \sum_{k=2}^{\infty} \frac{1}{k} G_k(t)\right) dt\right), \tag{3}$$

where $C = \exp(g(1/2))$ *is a constant approximately equal to* $2.15768...$.

(v) *the following expansions hold:*

$$a_n = \frac{1}{n} - \frac{\ln n}{n^2} - \frac{g(a_0)}{n^2} + \frac{(\ln n)^2}{n^3} + \frac{(2g(a_0)-1)\ln n}{n^3} + o\left(\frac{\ln n}{n^3}\right), \tag{4}$$

$$\begin{aligned} \frac{1}{a_n} &= n + \ln n + g(x) + \frac{\ln n}{n} + \frac{(-\frac{1}{2}+g)}{n} - \frac{1}{2}\frac{(\ln n)^2}{n^2} \\ &\quad + \frac{(\frac{3}{2}-g)\ln n}{n^2} + \left(\frac{3}{2}g - \frac{1}{2}g^2 - \frac{5}{6}\right)\frac{1}{n^2} + \frac{1}{3}\frac{(\ln n)^3}{n^3} \\ &\quad + (-2+g)\frac{(\ln n)^2}{n^3} + \left(\frac{19}{6} - 4g + g^2\right)\frac{\ln n}{n^3} + o\left(\frac{\ln n}{n^3}\right). \end{aligned}$$

Proof. The sequence $x_n = \frac{1}{a_n}$, $n \geq 1$, satisfies the recurrence relation $x_{n+1} = h(x_n)$, where $h(x) = x + 1 + \frac{1}{x-1}$, for $x \in (1, \infty)$. If we define $r(x_1) = \lim_{n\to\infty} y_n$ with $y_n = x_n - n - ln\ n$, clearly $g(x) = r(1/x)$ for all $x \in I$. Since all the properties of r transfer to g in a corresponding way, we prefer to work with the function r instead of g. Directly from the recurrence relation for x_n we easily see that x_n is a strictly increasing sequence, $x_2 \geq 4, (h(1,\infty)) = [4,\infty))$, and we get

$$x_{n+1} = x_2 + n - 1 + \sum_{k=2}^{n} \frac{1}{x_k - 1},\ n \geq 2. \tag{5}$$

From this we obtain that $x_n \geq n+2$ for all $n \geq 2$. This shows, in particular, that the limit defining r exists, since y_n is a decreasing sequence:

$$y_n - y_{n+1} = \ln(1 + \frac{1}{n}) - \frac{1}{x_n - 1} > \frac{1}{n+1} - \frac{1}{x_n - 1} \geq 0,\ n \geq 2.$$

Secondly, going back to (5), the next better estimation from above of x_n results:

$$x_{n+1} \leq x_2 + n - 1 + \sum_{k=2}^{n} \frac{1}{k+1} < x_2 + n - 1 + \ln(n+1) - \ln 2,\ n \geq 2. \tag{6}$$

Since for $u > v \geq 2$ or $1 < u < v \leq 2$, we get $h(u) > h(v) \geq 4$, and then $h(h(u)) > h(h(v) \geq 4$, a simple induction argument shows that r is decreasing on $(1, 2]$ and increasing on $[2, \infty)$. Therefore, in order to prove that r has finite values, it is enough to show that $r(2) > 0$. Hence, if $x_1 = 2$, (6) becomes

$$x_n \leq n + \omega + \ln n,\quad n \geq 2, \tag{7}$$

where $\omega = 2 - \ln 2 > 1$. Using (7) in (5), we obtain

$$x_{n+1} \geq n + 3 + \sum_{k=2}^{n} \frac{1}{k - 1 + \omega + \ln k},\ n \geq 2.$$

This implies that for $n \geq 2$

$$\begin{aligned} y_{n+1} &\geq 2 - \ln(n+1) + \sum_{k=1}^{n-1} \frac{1}{k+\omega+\ln(k+1)} \\ &> 2 - \ln(n+1) + \int_1^n \frac{dx}{x+\omega+\ln(x+1)}. \end{aligned}$$

Since $\dfrac{1}{x+\omega+\ln(x+1)} > \dfrac{1}{(x+\omega)} - \dfrac{\ln(x+1)}{(x+\omega)^2}$ on the interval $[1,\infty)$, we can continue the above sequence of inequalities as follows:

$$\begin{aligned} y_{n+1} &\geq 2 - \ln(n+1) + \int_1^n \frac{dx}{(x+\omega)} - \int_1^n \frac{\ln(x+1)dx}{(x+\omega)^2} \\ &= 2 - \ln(1+\omega) + + \ln(\frac{n+\omega}{n+1}) - \int_1^n \frac{\ln(x+1)dx}{(x+\omega)^2} \\ &> 2 - \ln(1+\omega) - \int_1^\infty \frac{\ln(x+1)dx}{(x+\omega)^2} \\ &= 2 + \frac{2\ln 2}{\omega^2-1} - \frac{\omega}{\omega-1}\ln(1+\omega). \end{aligned}$$

Since $\ln(1+\omega) = \ln\ 2(1+\frac{\omega-1}{2}) < \ln 2 + \frac{\omega-1}{2} = \frac{3-\omega}{2}$, we obtain from the above computation that

$$r(2) = \lim_n y_{n+1}(2) \geq \frac{(\omega-1)(\omega+4)}{2(\omega+1)} > 0.$$

Hence we have proved the second part of the statement (ii) in Theorem 3.

We next look at the sequence of the derivatives of the functions $x_n(x) = h^n(x)(x_1 = x)$, where $h^{n+1}(x) = h(h^n(x))$, $n \geq 1$. Since $h'(x) = 1 - \frac{1}{(x-1)^2}$, and $(h^n)'(x) = h'(h^{n-1}(x))h'(h^{n-2}(x))\dots h'(x)$, we get

$$y_n' = x_n' = \prod_{k=1}^{n-1}\left(1 - \frac{1}{(x_k-1)^2}\right), \quad n \geq 2. \tag{8}$$

Using the inequality $x_n \geq n+2$, $n \geq 2$, the product appearing in (8) is absolutely convergent. Therefore the sequence $y_n(x) = y_n(2) + \int_2^x y_n'(t)dt$ converges to $r(x) = r(2) + \int_2^x \prod_{k=1}^{\infty}(1 - \frac{1}{(x_k(t)-1)^2})dt$. In particular, this shows that r is continuously differentiable. In order to complete the proof of (i), let us observe that

$$\begin{aligned} r(h(x)) &= \lim_n\ y_n(h(x)) = \lim_n\ x_{n+1}(x) - n - \ln n = \\ &= \lim_n\ x_n(x) + 1 + \frac{1}{x_n-1} - n - \ln n \\ &= r(x) + 1. \end{aligned}$$

Hence $g(f(x)) = r(1/f(x)) = r(h(1/x)) = r(1/x)+1$ and $g(1-x) = g(f(1-x))-1 = g(f(x))-1 = g(x)$, for $x \in I$, which completes the proof of (i). Because

$$r'(x) = \prod_{k=1}^{\infty}\left(1-\frac{1}{(x_k(x)-1)^2}\right) = \frac{x(x-2)}{(x-1)^2}\prod_{k=2}^{\infty}\left(1-\frac{1}{(x_k(x)-1)^2}\right), \tag{9}$$

it is easy to see that $r'(x) > 0$ for $x > 2$ and $r'(x) < 0$ for $1 < x < 2$. This completes the proof of (ii).

To get (iii) we can use (i) to obtain $g'(f(x))f'(x) = g'(x)$, and hence by the change of variable formula,

$$\begin{aligned}
\xi(f(A)) &= \int_{f(A)} d\xi(x) = \\
&= \int_{f(A)} g'(x)dx = \int_{f(A)} g'(f(x))f'(x)dx \\
&= \int_{f(A)} g'(f(x))f'(x)dx = \int_A g'(x)dx \\
&= \int_A d\xi(x) = \xi(A).
\end{aligned}$$

In order to prove (iv), let us compute $\ln(r'(x))$ for $x > 2$, using formula (9) and the recursive relation:

$$\begin{aligned}
\ln(r'(x)) &= \ln\left(\prod_{k=1}^{\infty}\left(1-\frac{1}{(x_k(x)-1)^2}\right)\right) \\
&= \ln\left(\lim_n \prod_{k=1}^{n}\left(1-\frac{1}{x_k(x)-1}\right)\prod_{k=1}^{n}\left(1+\frac{1}{x_k(x)-1}\right)\right) \\
&= \lim_n \left(\sum_{k=1}^{n}\ln\left(1-\frac{1}{x_k(x)-1}\right) + \ln\left(\prod_{k=1}^{n}\frac{x_k(x)}{x_k(x)-1}\right)\right) \\
&= \lim_n \left(-\sum_{k=1}^{n}\sum_{j=1}^{\infty}\frac{1}{j}\left(\frac{1}{x_k(x)-1}\right)^j + \ln\left(\prod_{k=1}^{n}\frac{x_{k+1}(x)}{x_k(x)}\right)\right).
\end{aligned}$$

Here, we used the definition of $\{x_k\}_k$, that is, $x_{k+1} = h(x_k) = x_k + 1 + \frac{1}{x_k-1}$, therefore $\frac{x_k}{x_k-1} = \frac{x_{k+1}}{x_k}$, hence the last equality. After we interchange the sums, using (5) we can continue the above computation as follows:

$$
\begin{aligned}
&\ln(r'(x))\\
&= \lim_n \left(\ln(x_{n+1}(x)) - \ln\ x - \sum_{k=1}^{n} \frac{1}{x_k(x)-1} - \sum_{j=2}^{\infty}\sum_{k=1}^{n} \frac{1}{j}\left(\frac{1}{x_k(x)-1}\right)^j \right)\\
&= -\ln\ x + \lim_n \left(\ln(x_{n+1}(x)) - x_{n+1}(x) + n + x - \sum_{j=2}^{\infty}\sum_{k=1}^{n} \frac{1}{j}\left(\frac{1}{x_k(x)-1}\right)^j \right)\\
&= x - 1 - \ln x - \lim_n \left(x_{n+1}(x) - (n+1) - \ln(n+1) + \ln\left(\frac{n+1}{x_{n+1}(x)}\right)\right)\\
&\quad - \lim_n \left(\sum_{j=2}^{\infty}\sum_{k=1}^{n} \frac{1}{j}\left(\frac{1}{x_k(x)-1}\right)^j \right).
\end{aligned}
$$

Since the double sum $\displaystyle\sum_{j=2}^{\infty}\sum_{k=1}^{\infty} \frac{1}{j}\left(\frac{1}{x_k(x)-1}\right)^j$ is absolutely convergent we can interchange the limit sign with the sum sign in the above computation, and using the definition of r we obtain the following differential equation in r:

$$\ln(r'(x)) = x - 1 - \ln x - r(x) - \sum_{j=2}^{\infty}\sum_{k=1}^{\infty} \frac{1}{j}\left(\frac{1}{x_k(x)-1}\right)^j,$$

or

$$r'(x)\exp(r(x)) = \frac{1}{x}\exp(x - 1 - R(x)), \tag{10}$$

where $R(x) = \displaystyle\sum_{j=2}^{\infty}\sum_{k=1}^{\infty} \frac{1}{j}\left(\frac{1}{x_k(x)-1}\right)^j$. Integrating (10), we obtain a formula which gives us another way of approximating the values of r:

$$r(x) = \ln\left(C + \int_2^x \frac{1}{t}\exp\left(t - 1 - R(t)\right) dt\right), \quad x > 2. \tag{11}$$

In terms of the function g and the sequence $\{a_n\}$, after a change of variable, the formula (11) becomes

$$g(x) = \ln\left(C + \int_x^{\frac{1}{2}} \frac{1}{u}\exp\left(\frac{1}{u} - 1 - G(u)\right) du\right), \quad x \in (0, 1/2),$$

where $G(u) = R(1/u) = \displaystyle\sum_{j=2}^{\infty}\sum_{k=1}^{\infty} \frac{1}{j}\left(\frac{1}{x_k(1/u)-1}\right)^j = \sum_{j=2}^{\infty}\sum_{k=1}^{\infty} \frac{1}{j}\left(\frac{a_k(u)}{1-a_k(u)}\right)^j$,

and (iv) is proved.

To prove (v), we apply several times part (ii) of Cesàro's Lemma. First we take $u_n = x_n(x) - n - \ln\ n - r(x)$ and $v_n = (1/n)\ln n$:

$$\lim_n \frac{n(x_n(x) - n - \ln n - r(x))}{\ln n} = \lim_n \frac{\frac{1}{x_n - 1} - \ln(1 + \frac{1}{n})}{\frac{\ln(n+1)}{n+1} - \frac{\ln n}{n}} = 1.$$

Using the same technique we can compute the other terms in (11), and (10) is easily obtained from (11). □

We point out that there are cases when it is easy to determine expansions as in (4) for all $k \geq 2$. For example, if $f(x) = x/(1+x)$, then $\{a_n\}_n$ has the expansion of the form

$$a_n = \sum_{j=0}^{m} \frac{(-1)^j}{n^{j+1} a_0^j} + o\left(\frac{1}{n^{m+1}}\right),\ n \geq 1,\quad a_0 \in (0, \infty).$$

That can be seen easily by linearizing the recurrence $a_{n+1} = f(a_n)$ replacing $\dfrac{1}{a_n}$ by b_n. We obtain the linear equation $b_{n+1} = b_n + 1$, which obviously produces $a_n = \dfrac{1}{n + a_0^{-1}}$, from which we infer the previous approximation.

On the other hand, if $f(x) = \sin x$, we computed using Theorem 2 the following expansion:

$$a_n = \frac{\sqrt{3}}{\sqrt{n}} - \frac{3\sqrt{3}}{10}\frac{\ln n}{n\sqrt{n}} + \frac{9\sqrt{3}}{50}\frac{\ln n}{n^2\sqrt{n}} + o\left(\frac{\ln n}{n^2\sqrt{n}}\right),$$

where the coefficients do not seem to depend on the initial value of the sequence.

3. Almost Doubly-Exponential Sequences

Aho and Sloane [**1**] considered the sequences of the form $a_{n+1} = a_n^2 + g_n$, where $|g_n| \leq a_n/4$, $a_n \geq 1$ and $|\log(a_{n+1}a_n^{-2})|$ is decreasing, for $n \geq n_0$. They proved that under these conditions, there exists a constant k such that $a_n =$ *nearest integer to* k^{2^n}. Obviously, the sequence of Theorem 3 is not among the ones considered by Aho and Sloane, since it does not satisfy the mentioned conditions. In the spirit of [**1**], relaxing the conditions, using a somewhat different method, we prove the next theorem, involving what we call *almost doubly-exponential* recurrences. We denote by $exp(x)$ the exponential function e^x with Euler's constant base.

Theorem 4. *Let the sequence of positive integers* $a_{n+1} = a_n^2 + g_n$*, satisfying* $-a_n + 1 < g_n < a_n$*,* $a_n > 1$ *and* $\left|\log\left(a_{n+1}a_n^{-2}\right)\right|$ *is decreasing (for* $n \geq n_0$*). Then there exists* α *such that* $a_n = \lfloor \exp(2^n\alpha) \rfloor$*, or* $a_n = \lfloor \exp(2^n\alpha) \rfloor + 1$ *(for* $n \geq n_0$*).*

Proof. Since the entire proof refers to $n \geq n_0$, we may as well assume that $n_0 = 0$. The proof uses some ideas of [**1**] and [**5**]. Let $u_n := \log a_n$, and

$\delta_n := \log(g_n a_n^{-2} + 1)$. Thus $u_{n+1} = 2u_n + \delta_n$. Iterating we get

$$u_n = 2^n u_0 + 2^n \sum_{k=0}^{n-1} \delta_k 2^{-k-1}.$$

The series $\alpha := u_0 + \sum_{k=0}^{\infty} \delta_k 2^{-k-1}$ is absolutely convergent since $|\delta_k| < \log(1 + a_k^{-1}) < \log 2$. Taking $r_n := 2^n\alpha - u_n$, we get that $a_n = \exp(u_n) = \exp(2^n\alpha - r_n)$. Now,

$$\begin{aligned} \exp(2^n\alpha) &= a_n \exp(r_n), \quad \text{and} \\ r_n &= 2^n \sum_{k=n}^{\infty} \delta_k 2^{-k-1} = \sum_{k=0}^{\infty} \delta_{k+n} 2^{-k-1}. \end{aligned} \tag{12}$$

Since $\left|\log(a_{n+1}a_n^{-2})\right| = \left|\log(g_n a_n^{-2} + 1)\right| = |\delta_n|$ is decreasing, we get

$$|r_n| \le \sum_{k=0}^{\infty} |\delta_{k+n}| 2^{-k-1} \le |\delta_n| \sum_{k=0}^{\infty} 2^{-k-1} = |\delta_n|$$

which implies

$$a_n \exp(-|\delta_n|) \le \exp(2^n\alpha) \le a_n \exp(|\delta_n|). \tag{13}$$

We use now the definition of δ_n, and deduce

$$\begin{aligned} \exp(\delta_n) &= g_n a_n^{-2} + 1, \\ \exp(-\delta_n) &= (g_n a_n^{-2} + 1)^{-1}. \end{aligned} \tag{14}$$

Therefore, using (13) and (14), if $\delta_n > 0$, then

$$a_n - \exp(2^n\alpha) \le a_n - a_n \exp(-\delta_n) = a_n \left(1 - (g_n a_n^{-2} + 1)^{-1}\right), \tag{15}$$

$$a_n - \exp(2^n\alpha) \ge a_n - a_n \exp(\delta_n) = a_n \left(1 - (g_n a_n^{-2} + 1)\right) = -g_n a_n^{-1}. \tag{16}$$

Now, in (15) to have $a_n\left(1 - (g_n a_n^{-2} + 1)^{-1}\right) < 1$, it is necessary to have $(g_n a_n^{-2} + 1)^{-1} > 1 - 1/a_n$ which in turn is equivalent to $g_n < \dfrac{a_n^2}{a_n - 1} = a_n + 1 + \dfrac{1}{a_n - 1}$. The last inequality is true since $g_n < a_n$. In (16) to have $-g_n a_n^{-1} > -1$, it is necessary to have $g_n < a_n$.

If $\delta_n < 0$, by (13) and (14), then

$$a_n - \exp(2^n\alpha) \le a_n - a_n \exp(\delta_n) = a_n \left(1 - (g_n a_n^{-2} + 1)\right) = -g_n a_n^{-1}, \tag{17}$$

$$a_n - \exp(2^n\alpha) \ge a_n - a_n \exp(-\delta_n) = a_n \left(1 - (g_n a_n^{-2} + 1)^{-1}\right). \tag{18}$$

Now, in (17), $-g_n a_n^{-1} < 1$ is equivalent to $g_n > -a_n$, and the last inequality is certainly true, since $g_n > -a_n + 1$. In (18) to have $a_n\left(1 - (g_n a_n^{-2} + 1)^{-1}\right) > -1$, it is necessary to have $g_n a_n^{-2} + 1 > \dfrac{a_n}{a_n + 1} = 1 - \dfrac{1}{a_n + 1}$. That is equivalent to

$g_n > \dfrac{-a_n^2}{a_n+1} = -a_n + 1 - \dfrac{1}{a_n+1}$, which is certainly true, as g_n is an integer, $a_n > 1$ and $g_n > -a_n + 1$.

Thus, we obtain, in any case, that $|a_n - \exp(2^n\alpha)| < 1$, which implies (since a_n is an integer) that $a_n = \lfloor \exp(2^n\alpha) \rfloor$, or $a_n = \lfloor \exp(2^n\alpha) \rfloor + 1$. □

Remark 5. The previous theorem does not consider the case of $g_n = -a_n + 1$ (the lower bound). However, in that case we get $a_{n+1} = a_n^2 - a_n + 1$, which was dealt with by Aho and Sloane (Recurrence 2.4), if $a_1 = 2$, being transformed into a recurrence satisfying their conditions, deriving the solution $\lfloor k^{2^n} + \frac{1}{2} \rfloor$, for some real number k.

Consider now that case of $a_n < g_n < 2a_n$ in the recurrence $a_{n+1} = a_n^2 + g_n$, $a_n > 1$ positive integers. Let $g_n' = g_n - a_n$. Thus, $0 < g_n' < a_n$ and the recurrence can be written as

$$a_{n+1} = a_n^2 + a_n + g_n'.$$

Let $b_n = a_n + \frac{1}{2}$ and $h_n = g_n' - \frac{3}{4} = g_n - a_n - \frac{3}{4}$. It follows that

$$b_{n+1} = b_n^2 + h_n, \text{ with } -\frac{3}{4} < h_n < a_n - \frac{3}{4} < a_n,$$

which is of the first type, but (beware!) this sequence does not consist of integers. We start with one observation: since $a_n < g_n$, it follows that $g_n - a_n \geq 1$, therefore $h_n \geq \frac{1}{4}$, so h_n satisfies $0 < h_n < a_n$.

Let $u_n := \log b_n$, and $\delta_n := \log(h_n b_n^{-2} + 1)$. If $|\log(b_{n+1} b_n^{-2}|$ is decreasing, the same technique as before renders, since $h_n > 0$,

$$\begin{aligned} b_n - \exp(2^n\beta) &\leq b_n(1 - (h_n b_n^{-2} + 1)^{-1}), \\ b_n - \exp(2^n\beta) &\geq -h_n b_n^{-1}, \end{aligned}$$

where $\beta := u_0 + \sum_{k=0}^{\infty} \delta_k 2^{-k-1}$. Moreover, $b_n(1 - (h_n b_n^{-2} + 1)^{-1}) < 1$ if and only if $\dfrac{b_n - 1}{b_n} < \dfrac{1}{h_n b_n^{-2} + 1}$. This is equivalent to $h_n < \dfrac{b_n^2}{b_n - 1} = b_n + 1 + \dfrac{1}{b_n - 1}$, which is certainly true as $h_n < a_n < a_n + \frac{1}{2} = b_n$. Furthermore, since $-h_n b_n^{-1} > -1$, then

$$-\frac{3}{2} < a_n - \exp(2^n\beta) < \frac{1}{2}.$$

The right hand side inequality is improved by the simple observation that since $\delta_k > 0$, then $2^n\beta > u_n$, therefore, $\exp(2^n\beta) > b_n = a_n + \frac{1}{2}$, which implies

$$-\frac{3}{2} < a_n - \exp(2^n\beta) < -\frac{1}{2}$$

and so,

$$a_n < \exp(2^n\beta) - \frac{1}{2} < a_n + 1.$$

To cover the whole range $-a_n + 1 < g_n < 2a_n$, it suffices to study the case of $g_n = a_n$. In that case, we get the recurrence of positive integers $a_{n+1} = a_n^2 + a_n$. Taking $b_n = a_n + 1/2$, we get

$$b_{n+1} = b_n^2 - \frac{3}{4},$$

which was dealt with by Aho and Sloane, if $b_1 = \frac{3}{2}$, obtaining $b_n = \frac{3}{2} + \left\lfloor k^{2^n} + \frac{3}{2} \right\rfloor$, $n \geq 3$, for some real k.

Thus, we have proved

Theorem 6. *Let the recurrence of positive integers $a_{n+1} = a_n^2 + g_n$, where $a_n < g_n < 2a_n$, $a_n > 1$ (if $n \geq n_0$). Also assume that $|\log\big((a_{n+1} + 1/2)(a_n + 1/2)^{-2}\big)|$ is decreasing. Then there exists a real number β such that*

$$a_n = \left\lfloor \exp(2^n \beta) - \frac{1}{2} \right\rfloor, \ \textit{if } n \geq n_0.$$

If $a_{n+1} = a_n^2 + a_n$ and $a_1 = 1$, then

$$a_n = \left\lfloor \exp(2^n \beta) + \frac{5}{2} \right\rfloor, \ \textit{if } n \geq 3.$$

Certainly the theorem can be further extended by taking various other intervals for g_n and imposing the restrictive decreasing property on a_n.

The sequence g_n may or may not depend on a_n. If $g_n = a_n - 2a_n^2$, we end up with a recurrence of the form $a_{n+1} = f(a_n)$, where $f(x) = x - x^2$. Obviously, in this case Theorem 4 is not true, since the inequality imposed on g_n does not hold. But this case was dealt with by Theorem 3.

Can we relax the conditions of Theorem 4 and Theorem 6 even further? The answer is yes, but the result is not that accurate. Let the recurrence of positive integers $a_{n+1} = a_n^2 + h_n$ with $|h_n| < (1+\epsilon)a_n$, $a_n \geq 1$, where $\epsilon > 0$ is a fixed parameter. In the same manner as before, we denote by $\delta_n(\epsilon) = \log(h_n a_n^{-2} + 1)$ and $u_n = \log a_n$. The series $\alpha(\epsilon) = u_0 + \sum_{k=0}^{\infty} \delta_k(\epsilon) 2^{-k-1}$ is convergent since $-\log(2+\epsilon) \leq \log(1 - \frac{1+\epsilon}{a_k}) < \delta_k(\epsilon) < \log(1 + \frac{1+\epsilon}{a_k}) < \log(2+\epsilon)$, for k sufficiently large so that $a_k > 1 + \epsilon$. Taking $r_n = 2^n\alpha - u_n$, we get that $a_n = \exp(u_n) = \exp(2^n\alpha - r_n)$. We did not impose the decreasing property on $|\delta_n(\epsilon)|$, so we can only infer at this stage that

$$-\log(2+\epsilon) \leq r_n = \sum_{k=0}^{\infty} \delta_{k+n}(\epsilon) 2^{-k-1} \leq \log(2+\epsilon), \tag{19}$$

using the double inequality on $\delta_n(\epsilon)$.

With a bit more work, we conclude

Proposition 7. *Let $a_{n+1} = a_n^2 + h_n$ with $|h_n| < (1+\epsilon)a_n$, $a_n \geq 1$, where $\epsilon \geq 0$ is a fixed parameter. Then there exists a constant α such that*

$$\frac{1}{2+\epsilon}\exp(2^n\alpha) \leq a_n \leq (2+\epsilon)\exp(2^n\alpha),$$

if n is sufficiently large so that $a_n > 1+\epsilon$.

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