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COMPLETE SPACE-LIKE SUBMANIFOLDS IN DE SITTER SPACE

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Abstract. In this paper, we characterize the complete space-like submanifolds with parallel mean curvature vector satisfying $H^2 = \frac{4(n-1)c}{n^2}$ in the de Sitter space completely.

1. Introduction

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of constant curvature c whose index is p. It is called an indefinite space form of index p and simply a space form when $p = 0$. If $c > 0$, we call it as a de Sitter space of index p, denote it by $S_p^{n+p}(c)$. The study of space-like hypersurfaces in de Sitter space has been recently of substantial interest from both physics and mathematical points of view. Akutagawa [1] and Ramanathan [**10**] investigated space-like hypersurfaces in a de Sitter space and proved independently that a complete space-like hypersurface in a de Sitter space with constant mean curvature is totally umbilical if the mean curvature H satisfies $H^2 \leq c$ when $n = 2$ and $n^2H^2 < 4(n-1)c$ when $n \geq 3$. Later, Cheng [**3**] generalized this result to general submanifolds in a de Sitter space.

On the other hand, the well-known examples with $H^2 = \frac{4(n-1)c}{n^2}$ when $n > 2$ are umbilical sphere $S^n(\frac{(n-2)^2}{n^2}c)$ and the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$, $c_1 = (2-n)c$ and $c_2 = \frac{n-2}{n-1}c$. Hence it is natural to study if complete space-like hypersurfaces with $H^2 = \frac{4(n-1)c}{n^2}$ $(n > 2)$ are only the above examples. In [**4**], Cheng gave an affirmative answer if M^n is compact and gave some characterizations when M^n is complete and noncompact. In this paper, we consider the case of space-like submanifolds with parallel mean curvature vector satisfying $H^2 = \frac{4(n-1)c}{n^2}$ in the de Sitter space and prove the following theorem

Theorem. *Let M^n be an n-dimensional $(n \geq 3)$ complete space-like submanifold in the de Sitter space $S_p^{n+p}(c)$ with parallel mean curvature vector. If*

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$H^2 = \frac{4(n-1)}{n^2}c$, then M^n is totally umbilical, or M^n is the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$ in $S_1^{n+1}(c)$, or M^n has unbounded volume and positive Ricci curvature and $\int_{M^n} S^m\,dv = \infty$ for any m, where S is the norm square of the second fundamental form of M^n.

2. Preliminaries

Let $S_p^{n+p}(c)$ be an $(n+p)$-dimensional de Sitter space of constant curvature c whose index is p. Let M^n be an n-dimensional Riemannian manifold immersed in $S_p^{n+p}(c)$. As the semi-Riemannian metric of $S_p^{n+p}(c)$ induces the Riemannian metric of M^n, M^n is called a space-like submanifold. We choose a local field of semi-Riemannian orthonormal frames $e_1, \dots, e_{n+p}$ in $S_p^{n+p}(c)$ such that at each point of M^n, $e_1, \dots, e_n$ span the tangent space of M^n and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n+p; \quad 1 \leq i, j, k, \dots \leq n; \quad n+1 \leq \alpha, \beta, \gamma \leq n+p.$$

Let $\omega_1, \dots, \omega_{n+p}$ be its dual frame field so that the semi-Riemannian metric of $S_p^{n+p}(c)$ is given by $d\bar{s}^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \epsilon_A \omega_A^2$, where $\epsilon_i = 1$ and $\epsilon_\alpha = -1$. Then the structure equations of $S_p^{n+p}(c)$ are given by

$$d\omega_A = \sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{1}$$

$$d\omega_{AB} = \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D, \tag{2}$$

$$K_{ABCD} = c\,\epsilon_A \epsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{3}$$

Restrict these form to M^n, we have

$$\omega_\alpha = 0, \qquad n+1 \leq \alpha \leq n+p, \tag{4}$$

the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. From Cartan's lemma we can write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{5}$$

From these formulas, we obtain the structure equations of M^n:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{6}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{7}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{8}$$

where R_{ijkl} are the components of the curvature tensor of M^n and

$$h = \sum_{\alpha} h_\alpha e_\alpha = \sum_{i,j,\alpha} h^\alpha_{ij} \omega_i \otimes \omega_j \otimes e_\alpha \tag{9}$$

is the second fundamental form of M^n.

For indefinite Riemannian manifolds in detail, refer to O'Neill [**7**].

Let S be the norm square of the second fundamental form of M^n, ξ denote the mean curvature vector field of M^n and H the mean curvature of M^n, that is

$$\xi = \frac{1}{n} \sum_{\alpha} \Big(\sum_i h^\alpha_{ii} \Big) e_\alpha, \ \ H = |\xi|, \ \ S = \sum_{\alpha,i,j} (h^\alpha_{ij})^2.$$

Moreover, the normal curvature tensor $\{R_{\alpha\beta kl}\}$, the Ricci curvature tensor $\{R_{ik}\}$ and the scalar curvature R are expressed as

$$R_{\alpha\beta kl} = \sum_m (h^\alpha_{km} h^\beta_{ml} - h^\alpha_{lm} h^\beta_{mk}),$$

$$R_{ik} = (n-1)\, c\, \delta_{ik} - \sum_\alpha (\sum_l h^\alpha_{ll}) h^\alpha_{ik} + \sum_{\alpha,j} h^\alpha_{ij} h^\alpha_{jk}, \tag{10}$$

$$R = n(n-1)c + (S - n^2 H^2). \tag{11}$$

Define the first and the second covariant derivatives of $\{h^\alpha_{ij}\}$, say $\{h^\alpha_{ijk}\}$ and $\{h^\alpha_{ijkl}\}$ by

$$\sum_k h^\alpha_{ijk} \omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{kj} \omega_{ki} + \sum_k h^\alpha_{ik} \omega_{kj} + \sum_\beta h^\beta_{ij} \omega_{\beta\alpha}, \tag{12}$$

(13)

$$\sum_l h^\alpha_{ijkl} \omega_l = dh^\alpha_{ijk} + \sum_m h^\alpha_{mjk} \omega_{mi} + \sum_m h^\alpha_{imk} \omega_{mj} + \sum_m h^\alpha_{ijm} \omega_{mk} + \sum_\beta h^\beta_{ijk} \omega_{\beta\alpha}.$$

Then, by exterior differentiation of (5), we obtain the Codazzi equation

$$h^\alpha_{ijk} = h^\alpha_{ikj}. \tag{14}$$

It follows that the Ricci identities hold

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj} R_{mikl} + \sum_m h^\alpha_{im} R_{mjkl} + \sum_\beta h^\beta_{ij} R_{\beta\alpha kl}. \tag{15}$$

The Laplacian Δh^α_{ij} of the fundamental form h^α_{ij} is defined to be $\sum_k h^\alpha_{ijkk}$, from (15) we have

$$\Delta h^\alpha_{ij} = \sum_{m,k} h^\alpha_{im} R_{mkjk} + \sum_{m,k} h^\alpha_{mk} R_{mijk} + \sum_k h^\alpha_{kkij}. \tag{16}$$

We need the following generalized maximum principle due to Omori [**9**] and Yau [**11**]:

Lemma 2.1. *Let M^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and $F\colon M^n \to R$ a smooth function bounded from below. Then there is a sequence of points $\{p_k\}$ in M^n such that*

$$\lim_{k\to\infty} F(p_k) = \inf(F), \quad \lim_{k\to\infty} |\nabla F(p_k)| = 0, \quad \lim_{k\to\infty} \inf \Delta F(p_k) \geq 0.$$

We also need the following algebraic lemma due to M. Okumura [**8**] (see also [**2**]).

Lemma 2.2. *Let μ_i, $i = 1, \dots, n$, be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta = \text{constant} \geq 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{17}$$

and the equality holds in (17) *if and only if at least* $(n-1)$ *of the* μ_i *are equal.*

Now we assume that the mean curvature vector ξ is parallel and $H^2 = \frac{4(n-1)}{n^2}c$. We can choose $e_{n+1} = \xi/H$. Then

$$\sum_k k^\alpha_{kki} = 0, \ \ \omega_{\alpha,n+1} = 0, \ \ H^\alpha H^{n+1} = H^{n+1}H^\alpha, \tag{18}$$

$$\operatorname{tr} H^{n+1} = nH, \ \ \operatorname{tr} H^\alpha = 0, \ \alpha \neq n+1, \tag{19}$$

where H^α denote the matrix (h^α_{ij}).

Putting

$$\mu_{ij} = h^{n+1}_{ij} - H\delta_{ij}, \ \ \tau^\alpha_{ij} = h^\alpha_{ij}, \ \alpha \neq n+1, \tag{20}$$

we have

$$|\mu|^2 = \operatorname{tr}(\mu)^2 = \sum \mu^2_{ij} = \operatorname{tr}(H^{n+1})^2 - nH^2, \tag{21}$$

$$|\tau|^2 = \sum_{\beta\neq n+1} (h^\beta_{ij})^2, \tag{22}$$

$$\operatorname{tr}\mu = 0, \ \operatorname{tr}(\tau^\beta) = 0, \ \beta \neq n+1, \tag{23}$$

$$S = |\mu|^2 + |\tau|^2 + nH^2. \tag{24}$$

A submanifold M^n is said to be pseudo-umbilical if it is umbilical with respect to the direction of the mean curvature vector ξ, i.e., $h^{n+1}_{ij} = H\delta_{ij}$. From (21)-(24) we know that M^n is pseudo-umbilical if and only if $|\mu|^2 = 0$, M^n is totally umbilical if and only if $|\mu|^2 = 0$ and $|\tau|^2 = 0$.

$$\begin{aligned}\Delta h^{n+1}_{ij} = nch^{n+1}_{ij} - nHc\delta_{ij} + \sum h^{n+1}_{km}h^\beta_{mk}h^\beta_{ij} - \sum h^{n+1}_{km}h^\beta_{mj}h^\beta_{ik} \\ + \sum h^{n+1}_{mi}h^\beta_{mk}h^\beta_{kj} - nH\sum h^{n+1}_{mi}h^{n+1}_{mj}.\end{aligned} \tag{25}$$

Thus

$$\begin{aligned}\frac{1}{2}\Delta(|\mu|^2) = \sum(h^{n+1}_{ijk})^2 + nc\sum(h^{n+1}_{ij})^2 - n^2cH^2 \\ - nH\operatorname{tr}(H^{n+1})^3 + \sum_{\beta\neq n+1} \operatorname{tr}(H^{n+1}H^\beta)^2 + [\operatorname{tr}(H^{n+1})^2]^2.\end{aligned} \tag{26}$$

On the other hand

$$\operatorname{tr}(H^{n+1})^3 = \operatorname{tr}\mu^3 + 3H[\operatorname{tr}(H^{n+1})^2 - nH^2] + nH^3. \tag{27}$$

By using (23), (27) and Lemma 2.2, we have from (26)

$$\begin{aligned} \frac{1}{2}\Delta(|\mu|^2) &\geq (|\mu|^2 + nH^2)^2 - nH[\operatorname{tr}(\mu)^3 + 3H|\mu|^2 + nH^3] + nc|\mu|^2 \\ &= |\mu|^2(|\mu|^2 + nc - nH^2) - nH\operatorname{tr}(\mu)^3 \\ &\geq |\mu|^2\left(|\mu|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\mu| + nc - nH^2\right) \\ &= |\mu|^2\left(|\mu| - \frac{(n-2)}{\sqrt{n}}\sqrt{c}\right)^2, \end{aligned} \tag{28}$$

where we used $H^2 = \frac{4(n-1)}{n^2}c$.

Now consider the positive smooth function f on M^n defined by

$$f = \frac{1}{\sqrt{1+|\mu|^2}}.$$

It is easy to check that

$$|\nabla f|^2 = \frac{1}{4}\frac{|\nabla(|\mu|^2)|^2}{(1+|\mu|^2)^3} \tag{29}$$

and that

$$f\Delta f = -\frac{1}{2}\frac{\Delta(|\mu|^2)}{(1+|\mu|^2)^2} + 3|\nabla f|^2. \tag{30}$$

From (28) and (30), we have

$$f\Delta f \leq -|\mu|^2(|\mu| - (n-2)\sqrt{c}/\sqrt{n})^2/(1+|\mu|^2)^2 + 3|\nabla f|^2. \tag{31}$$

From (10) and $H^2 = \frac{4(n-1)}{n^2}c$, we have

$$Ric(e_i) \geq (n-1)c - nHh_{ii}^{n+1} + \sum_k (h_{ik}^{n+1})^2 = (\lambda_i - \sqrt{(n-1)c}) \geq 0, \tag{32}$$

where $h_{ij}^{n+1} = \lambda_i\delta_{ij}$. So the Ricci curvature of M^n is non-negative, we may apply Lemma 2.1 to the smooth function f. Then there is a sequence of points p_k in M^n such that

$$\lim_{k\to\infty} f(p_k) = \inf f, \;\; \lim_{k\to\infty} |\nabla f(p_k)| = 0, \;\; \lim_{k\to\infty} \inf \Delta f(p_k) \geq 0.$$

From (31), we have $\inf(f) \neq 0$, so $\lim_{k\to\infty} |\mu|^2(p_k) = \sup|\mu|^2 < \infty$. Approaching the limit of both sides of inequality (31), we obtain $\sup|\mu|^2 = 0$, or $\sup|\mu|^2 = \frac{(n-2)^2}{n}c$.

If $|\mu|^2$ reaches its supremum on M^n, from (28) we know that $|\mu|^2$ is subharmonic. Thus $|\mu|^2$ would be constant because of the maximum principle. So we have the following proposition

Proposition 2.1. *Let M^n be an n-dimensional ($n \geq 3$) complete space-like submanifold in the de Sitter space $S_p^{n+p}(c)$ with parallel mean curvature vector. If $H^2 = \frac{4(n-1)}{n^2}c$, then either M^n is pseudo-umbilical or* $\sup|\mu|^2 = \frac{(n-2)^2}{n}c$, *and this supremum is attained if and only if $|\mu|^2 \equiv \frac{(n-2)^2}{n}c$.*

3. The Proof of Theorem

By use of (18), we have from (16) for $\alpha \neq n+1$

$$\begin{aligned}\Delta h_{ij}^\alpha = {} & nch_{ij}^\alpha + \sum h_{km}^\alpha h_{mk}^\beta h_{ij}^\beta - 2\sum h_{km}^\alpha h_{mj}^\beta h_{ik}^\beta \\ & + \sum h_{mi}^\alpha h_{mk}^\beta h_{kj}^\beta + \sum h_{jm}^\alpha h_{mk}^\beta h_{ki}^\beta - nH\sum h_{mi}^\alpha h_{mj}^{n+1}.\end{aligned} \tag{33}$$

Thus

$$\begin{aligned}\frac{1}{2}\Delta(|\tau|^2) = {} & \sum_{\alpha\neq n+1}(h_{ijk}^\alpha)^2 + nc|\tau|^2 + \sum_{\alpha\neq n+1} h_{km}^\alpha h_{mk}^\beta h_{ij}^\beta h_{ij}^\alpha \\ & - 2\sum_{\alpha\neq n+1} h_{km}^\alpha h_{mj}^\beta h_{ik}^\beta h_{ij}^\alpha + \sum_{\alpha\neq n+1} h_{mi}^\alpha h_{mk}^\beta h_{kj}^\beta h_{ij}^\alpha \\ & + \sum_{\alpha\neq n+1} h_{jm}^\alpha h_{mk}^\beta h_{ki}^\beta h_{ij}^\alpha - nH\sum_{\alpha\neq n+1} h_{mi}^\alpha h_{ij}^\alpha h_{mj}^{n+1}.\end{aligned} \tag{34}$$

By use of (18) and (19), we have from (34)

$$\frac{1}{2}\Delta(|\tau|^2) = \sum_{\alpha\neq n+1}(h_{ijk}^\alpha)^2 + nc|\tau|^2 + I + II, \tag{35}$$

where

$$\begin{aligned}I = {} & \sum_{\alpha,\beta\neq n+1}[\mathrm{tr}(H^\alpha H^\beta)]^2 - 2\sum_{\alpha,\beta\neq n+1} h_{km}^\alpha h_{mj}^\beta h_{ik}^\beta h_{ij}^\alpha \\ & + \sum_{\alpha,\beta\neq n+1} h_{mi}^\alpha h_{mk}^\beta h_{kj}^\beta h_{ij}^\alpha + \sum_{\alpha,\beta\neq n+1} h_{jm}^\alpha h_{mk}^\beta h_{ki}^\beta h_{ij}^\alpha,\end{aligned} \tag{36}$$

$$\begin{aligned}II = {} & \sum_{\alpha\neq n+1} h_{km}^\alpha h_{mk}^{n+1} h_{ij}^{n+1} h_{ij}^\alpha - 2\sum_{\alpha\neq n+1} h_{km}^\alpha h_{mj}^{n+1} h_{ik}^{n+1} h_{ij}^\alpha \\ & + \sum_{\alpha\neq n+1} h_{mi}^\alpha h_{mk}^{n+1} h_{kj}^{n+1} h_{ij}^\alpha + \sum_{\alpha\neq n+1} h_{jm}^\alpha h_{mk}^{n+1} h_{ki}^{n+1} h_{ij}^\alpha \\ & - nH\sum_{\alpha\neq n+1} h_{mi}^\alpha h_{ij}^\alpha h_{mj}^{n+1} \\ = {} & \sum_{\alpha\neq n+1} h_{km}^\alpha h_{mk}^{n+1} h_{ii}^{n+1} h_{ij}^\alpha - nH\sum_{\alpha\neq n+1} h_{mi}^\alpha h_{ij}^\alpha h_{mj}^{n+1}.\end{aligned} \tag{37}$$

We put $S_{\alpha\beta} = \sum h_{ij}^\alpha h_{ij}^\beta$ for $\alpha, \beta \neq n+1$, then $(S_{\alpha\beta})$ is a $(p-1)\times(p-1)$ symmetric matrix. It can be assumed to be diagonal for a suitable choice of

$e_{n+2}, \dots, e_{n+p}$. Set $S_\alpha = S_{\alpha\alpha}$ and we have $|\tau|^2 = \sum_{\alpha \neq n+1} S_\alpha$. In general, for a matrix $A = (a_{ij})$, we put $N(A) = \mathrm{tr}(A^t A)$. Now we have from (36),

$$\begin{aligned} I &= \sum_{\alpha \neq n+1} S_\alpha + \sum_{\alpha,\beta \neq n+1} N(H^\alpha H^\beta - H^\beta H^\alpha) \\ &\geq \sum_{\alpha \neq n+1} S_\alpha^2 \geq \left(\sum_{\alpha \neq n+1} S_\alpha \right)^2 /(p-1) = |\tau|^4/(p-1). \end{aligned} \tag{38}$$

By Proposition 2.1, we need to divide the proof of Theorem into the following three cases.

Case (i): M^n is pseudo-umbilical, that is $|\mu|^2 = 0$ or $h_{ij}^{n+1} = H\delta_{ij}$ on M^n, from (37) we get

$$II = -nH^2|\tau|^2. \tag{39}$$

Thus, in this case, we have

$$\frac{1}{2}\Delta(|\tau|^2) \geq (nc - nH^2)|\tau|^2 + |\tau|^4/(p-1) = \frac{(n-2)^2}{n}|\tau|^2 c + |\tau|^4/(p-1). \tag{40}$$

Let $f = 1/\sqrt{1+|\tau|^2}$, by use of the similar methods of proof of $|\mu|^2$ in section 2, we have $|\tau|^2 = 0$. Hence M^n is totally umbilical.

Case (ii): $\sup |\mu|^2 = \frac{(n-2)^2}{n}c$ and supremum of $|\mu|^2$ is attained, then $|\mu|^2 \equiv \frac{(n-2)^2}{n}c$. From Lemma 2.2, we have

$$\lambda_1 = \sqrt{(n-1)c}, \quad \lambda_2 = \cdots = \lambda_n = \frac{\sqrt{c}}{\sqrt{n-1}}. \tag{41}$$

For any fixed $\alpha \neq n+1$, let $h_{ij}^\alpha = \alpha_i \delta_{ij}$, noting $\alpha_1 + \cdots + \alpha_n = 0$, by use of (41), we have for any $\alpha \neq n+1$

$$\sum h_{km}^\alpha h_{mk}^{n+1} h_{ii}^{n+1} h_{ij}^\alpha = \left(\sum_m \lambda_m \alpha_m \right)^2 = c\left(\sqrt{n-1} - \frac{1}{\sqrt{n-1}} \right)^2 \alpha_1^2. \tag{42}$$

$$\begin{aligned} -nH \sum h_{mi}^\alpha h_{ij}^\alpha h_{mj}^{n+1} &= -nH \sum_m \lambda_m \alpha_m^2 \\ &= -nH\sqrt{c}[\sqrt{n-1}\alpha_1^2 + (\alpha_2^2 + \cdots + \alpha_n^2)/\sqrt{n-1}] \\ &\geq -2c(n-1)\alpha_1^2 - 2c(\alpha_2^2 + \cdots + \alpha_n^2) \\ &= -2c(n-1)(x + (1-x))\alpha_1^2 - 2c(\alpha_2^2 + \cdots + \alpha_n^2) \\ &\geq -2c(n-1)x\alpha_1^2 - 2c(n-1)(1-x)(n-1)(\alpha_2^2 + \cdots + \alpha_n^2) \\ &\quad - 2c(\alpha_2^2 + \cdots + \alpha_n^2) \\ &= -2c(n-1)x\alpha_1^2 - 2c[1 + (n-1)^2(1-x)](\alpha_2^2 + \cdots + \alpha_n^2), \end{aligned} \tag{43}$$

where x is a real number satisfying $0 \leq x \leq 1$.

Choosing $x = \frac{2n^2-5n+4}{2(n-1)^2}$, for fixed $\alpha \neq n+1$, from (42) and (43) we have

$$\begin{aligned}(44)\qquad &\sum h^{\alpha}_{km}h^{n+1}_{mk}h^{n+1}_{ii}h^{\alpha}_{ij} - nH\sum h^{\alpha}_{mi}h^{\alpha}_{ij}h^{n+1}_{mj} \\ &\geq [\frac{(n-2)^2}{n-1} - 2(n-1)x]c\alpha_1^2 - 2c[1+(n-1)^2(1-x)](\alpha_2^2+\cdots+\alpha_n^2) \\ &= -nc(\alpha_1^2+\cdots+\alpha_n^2) = -nc\sum_{i,j}(h^{\alpha}_{ij})^2.\end{aligned}$$

From (37), (43) and (44) we have

$$II \geq -nc\sum_{i,j,\alpha\neq n+1}(h^{\alpha}_{ij})^2 = -nc|\tau|^2. \tag{45}$$

Combining (35) and (38) with (45), we get

$$\frac{1}{2}\Delta(|\tau|^2) \geq \frac{|\tau|^4}{p-1}. \tag{46}$$

Let $f = 1/\sqrt{1+|\tau|^2}$, by use of the similar methods of proof of $|\mu|^2$ in Section 2, we have $|\tau|^2 = 0$. Hence M^n is a hyperbolic cylinder $H^1(c_1)\times S^{n-1}(c_2)$ in $S_1^{n+1}(c)$.

Case (iii): $\sup|\mu|^2 = \frac{(n-2)^2}{n}c$ and $|\mu|^2 < \frac{(n-2)^2}{n}c$. By (32), we know that M^n has non-negative Ricci curvature. If there is a point p in M^n and a unit vector $v \in T_pM^n$ such that $\mathrm{Ric}(v,v)(p) = 0$, then taking $e_1 = v$, we obtain $\lambda_i = \frac{nH}{2}$. Hence

$$|\mu|^2 = \frac{n^2H^2}{4} + \lambda_2^2 + \cdots + \lambda_n^2 - nH^2 < \frac{(n-2)^2}{n}c,$$

namely

$$\lambda_2^2 + \cdots + \lambda_n^2 < c.$$

Since

$$(n-1)c = \frac{n^2H^2}{4} = (\lambda_2 + \cdots + \lambda_n)^2,$$

we get

$$(n-1)c > (n-1)(\lambda_2^2 + \cdots + \lambda_n^2) \geq (\lambda_2 + \cdots + \lambda_n)^2 = (n-1)c.$$

This is a contradiction. Hence the Ricci curvature is positive. From the result due to Yau [**12**], we know that M^n has unbounded volume and $\int_{M^n} S^m dv = \infty$ for any m. This completes the proof of Theorem.

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