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THE FULL PERIODICITY KERNEL FOR A CLASS OF GRAPH MAPS

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ABSTRACT. Let X be a graph and let $\mathcal{C}$ be a class of X-maps (that is, of continuous maps from X into itself). A map $f \in \mathcal{C}$ is said to have full periodicity if $\mathrm{Per}\,(f) = \mathbb{N}$ (here, $\mathrm{Per}\,(f)$ denotes the set of periods of all periodic points of f and $\mathbb{N}$ the set of positive integers). The set $K \subseteq \mathbb{N}$ is a full periodicity kernel of $\mathcal{C}$ if it satisfies the following two conditions: (i) If $f \in \mathcal{C}$ and $K \subseteq \mathrm{Per}\,(f)$ then f has full periodicity and (ii) if $S \subset \mathbb{N}$ is a set such that, for every $f \in \mathcal{C}$, $S \subseteq \mathrm{Per}\,(f)$ implies $\mathrm{Per}\,(f) = \mathbb{N}$, then $K \subseteq S$. In this paper we show the existence and characterize the full periodicity kernel of the class of continuous maps from a graph with zero Euler characteristic to itself having all branching points fixed.

1. Introduction

A finite regular graph (or graph, for short) is a pair consisting of a connected Haussdorff space X and a finite subset $V \neq \emptyset$ (the points of V are called **vertices**) such that $X \setminus V$ is the disjoint union of a finite number of open subsets $e_1, e_2, \dots, e_n$ called **edges** with the property that each e_i is homeomorphic to an open interval of the real line. Given a vertex v, the number of edges having a v as endpoint (with the edges whose closures are homeomorphic to a circle counted twice) will be called the **valence** of v. A vertex of valence 1 is called an **endpoint** and a vertex of valence ≥ 3 is called a **branching point**. Given a graph X, let $e(X)$ and $b(X)$ denote the number of its endpoints and branching points, respectively. A **circuit** is a subset of X homeomorphic to a circle. Circuits do not contain endpoints.

The rational homology groups of a graph X are well-known. We have $H_0(X;\mathbb{Q}) \approx \mathbb{Q}$ and $H_1(X;\mathbb{Q}) \approx \mathbb{Q}^d$, where d is the number of independent circuits of X (for more details see [**10**]). The **Euler characteristic** $\chi\,(X)$ **of** X is defined to be $1-d$. If v and e are the number of vertices and edges of X, respectively, then $\chi\,(X) = v - e$. A graph without circuits is called **a tree**. The Euler characteristic of a tree is obviously 1.

A continuous map from a topological space X into itself will be called an X**-map**. A point $x \in X$ is called a **periodic point of period** n with respect to an X-map f if n is the smallest positive integer such that $f^n\,(x) = x$. The set

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$\{x, f(x), \ldots, f^{n-1}(x)\}$ is called the **periodic orbit of** x. We denote by $\text{Per}(f)$ the set of periods of all periodic orbits of f.

The set of all positive integers will be denoted by $\mathbb{N}$.

An X-map f has a **full periodicity** if $\text{Per}(f) = \mathbb{N}$. Let $\mathcal{C}$ be a class of X-maps. Then the set $K \subseteq \mathbb{N}$ is a **full periodicity kernel of** $\mathcal{C}$ (from now on FPK) if it satisfies the following conditions:

(a) If $f \in \mathcal{C}$ and $K \subseteq \text{Per}(f)$, then $\text{Per}(f) = \mathbb{N}$.
(b) If $S \subset \mathbb{N}$ is a set such that, for every $f \in \mathcal{C}$, $S \subseteq \text{Per}(f)$ implies $\text{Per}(f) = \mathbb{N}$, then $K \subseteq S$.

Note that for a given topological space X and a class $\mathcal{C}$ of X-maps, if there is a FPK, then it is unique. In such a case the FPK of $\mathcal{C}$ will be denoted by $K(\mathcal{C})$.

Let X be a graph and $\mathcal{C}$ be the class of all X-maps. The set $K(\mathcal{C})$ has been computed when X is one of the following graphs with positive Euler characteristic: The interval [**12**], [**7**] and the n-stars [**2**] (see also [**11**] and [**1**] for the particular case $n = 3$) — an n-star is the tree X_n which is most easily described as $\{z \in \mathbb{C} : z^n \in [0,1]\}$. Assume now that X is any tree with n endpoints and $\mathcal{G}$ is the class of all X-maps which leave all branching points of X fixed. It is known (see [**4**]) that the characterization of the set of periods of maps from $\mathcal{G}$ coincides with the one of X_n-maps. Therefore, [**2**] also gives $K(\mathcal{G})$. Indeed, it coincides with the FPK of the class of all X_n-maps.

The set $K(\mathcal{C})$ has been also computed when X is a circle [**5**] and when X is the simplest graph with zero Euler characteristic and $b(X) \neq 0$ (that is, the graph σ formed by the points $(x, y) \in \mathbb{R}^2$ such that either $x^2 + y^2 = 1$ or $0 \leq x \leq 1$ and $y = 1$) [**8**].

In what follows X will denote a graph with zero Euler characteristic. If $e(X) = n$ we will denote by $\mathcal{G}_n$ the class of all X-maps which leave the branching points of X fixed. We point out that $K(\mathcal{G}_1)$ found in [**8**] coincides with the FPK given in [**1**], [**2**], [**11**] for X_3-maps. The goal of this paper is to show the existence and characterize the set $K(\mathcal{G}_n)$ for each $n > 1$. To this end, since the FPK of the set of all X_n-maps was characterized in [**2**] (see Proposition 2.2 in the present paper) it suffices to prove the next result:

Theorem A. *The FPK of $\mathcal{G}_n$ exists and coincides with the FPK of the class of all X_{n+2}-maps.*

This paper is organized as follows. In Section 3 we prove Theorem A and we give a full characterization of the sets $K(\mathcal{G}_n)$. To do it, we need two results that give us the characterization of the sets of periods of the X_n-maps and the maps from $\mathcal{G}_n$. They are the n-star's Theorem, proved by Baldwin [**3**], and the Graph's Theorem, proved by Llibre, Paraños and Rodríguez [**9**]. They will be stated in Section 2 together with the necessary definitions. Also in Section 2 we give the full characterization of the FPK of the class of the X_n-maps.

2. Definitions and Preliminary Results

To state the n-star's and Graph's Theorems we shall use Sarkovskii's ordering and Baldwin's partial orderings on positive integers.

The Sarkovskii's ordering, $\leq_s$, on $\mathbb{N} \cup \{2^\infty\}$ (here we include the symbol 2^∞ for technical reasons that will be explained below) is

$$\begin{aligned} 1 \leq_s 2 \leq_s 2^2 \leq_s 2^3 \leq_s \ldots \leq_s 2^\infty \leq_s \ldots \leq_s 2^2 \cdot 9 \leq_s 2^2 \cdot 7 \leq_s 2^2 \cdot 5 \leq_s 2^2 \cdot 3 \\ \leq_s 2 \cdot 9 \leq_s 2 \cdot 7 \leq_s 2 \cdot 5 \leq_s 2 \cdot 3 \leq_s \ldots \leq_s 9 \leq_s 7 \leq_s 5 \leq_s 3 \end{aligned}$$

and, for $t \geq 2$, Baldwin's partial orderings are the orderings $\leq_t$ defined on the set

$$\mathbb{N}_t = (\mathbb{N} \cup \{t \cdot 2^\infty\}) \setminus \{2, 3, \ldots, t-1\}$$

as follows. Let $m, k \in \mathbb{N}_t$ with $m \neq k$ and $k \neq 1$. Then we write $m \leq_t k$ if one of the following cases hold (see Figure 1):

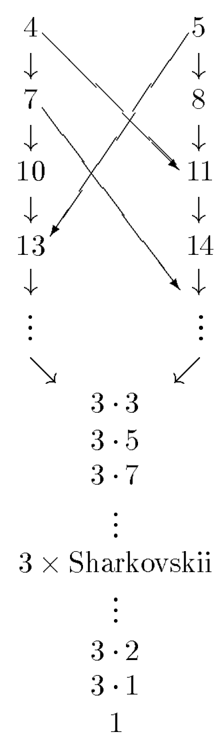

Figure 1. The ordering $\leq_3$ in $\mathbb{N}_3$.

(i) $m = 1$.
(ii) t divides m and k, and $(m/t) \leq_s (k/t)$.
(iii) t does not divide k, and either $m = t \cdot 2^\infty$ or $m = ik + jt$ with $i \in \mathbb{N} \cup \{0\}$ and $j \in \mathbb{N}$

(in case (ii) we use the formal arithmetic rule for $t \cdot 2^\infty : \frac{t \cdot 2^\infty}{t} = 2^\infty$).

The structure of the orderings $\leq_t$ is the following. The smallest element is 1. Next, all the multiples of t come (including $t \cdot 2^\infty$) in the ordering induced by Sarkovskii's ordering. Clearly, the largest element of this "segment" of $\leq_t$ is $3 \cdot t$. Finally, $\leq_t$ divides the set of naturals which are not multiples of t into $t-1$ "branches". The l-th branch with $l \in \{1, 2, \dots, t-1\}$ is formed by all naturals (except l) which are congruent with l modulo t in decreasing order. Note that all elements of these branches are larger than $3 \cdot t$ in the ordering $\leq_t$ (and, hence, larger than all multiples of t, $t \cdot 2^\infty$ and 1). In [**3**] some diagrams illustrating these partial orderings are given. Although Baldwin defines his partial orderings on the set of all natural numbers, it can be seen that the numbers which we take away from $\mathbb{N}$ when defining $\mathbb{N}_t$ can be removed without any loss of information. From the definition we have that $\leq_2$ is just the Sarkovskii's ordering if we identify the symbol $2 \cdot 2^\infty$ with 2^∞.

We note that, by means of the inclusion of the symbol $t \cdot 2^\infty$, each subset of $\mathbb{N} \setminus \{2, 3, \dots, t-1\}$ has a maximal element with respect to the ordering $\leq_t$ (i.e., $t \cdot 2^\infty$ plays the role of the maximum of the set $\left\{2^k \cdot t : k \in \mathbb{N}\right\}$).

A set $A \subseteq \mathbb{N}$ is an **initial segment of** $\leq_t$ if whenever k is an element of A and $m \leq_t k$ then m is also an element of A. That is, the set A is closed under $\leq_t$–predecessors. As an example note that the set $\mathbb{N} \setminus \{2, 4\}$ is an initial segment of the ordering $\leq_3$ while $\mathbb{N} \setminus \{2, 4, 13\}$ it is not. We point out that the union of initial segments of an ordering $\leq_t$ is an initial segment of $\leq_t$.

We now can state n-star's Theorem due to Baldwin [**3**].

n-star's Theorem. *Let f be an X_n-map. Then* $\operatorname{Per}(f)$ *is a nonempty finite union of initial segments of $\{\leq_t : 2 \leq t \leq n\}$. Conversely, if A is a nonempty finite union of initial segments of $\{\leq_t : 2 \leq t \leq n\}$, then there is a X_n-map f such that* $\operatorname{Per}(f) = A$.

To state Graph's Theorem we need to consider another further ordering. Let $\leq_1$ be the following ordering on $\mathbb{N}$

$$1 \leq_1 \dots \leq_1 7 \leq_1 6 \leq_1 5 \leq_1 4 \leq_1 3 \leq_1 2$$

introduced in [**Blo**]. The notion of initial segment is extended to the ordering $\leq_1$ in the natural way. Now we are ready to state the Graph's Theorem due to Llibre, Paraños and Rodríguez (see [**9**]).

Graph's Theorem. *Let $f \in \mathcal{G}_n$. Then* $\operatorname{Per}(f)$ *is a nonempty finite union of initial segments of $\{\leq_t : 1 \leq t \leq n+2\}$. Conversely, if A is a nonempty finite* *union of initial segments of* $\{\leq_t : 1 \leq t \leq n+2\}$*, then there is a map* $f \in \mathcal{G}_n$ *such that* $\mathrm{Per}\,(f) = A$.

In what follows K_n will denote the FPK of the set of all X_n-maps. The following two propositions, which characterize it, follow from [**2**] (see Corollary 4.3 and Theorem 4.15).

Proposition 2.1. *The set* K_n *exists for each* $n \geq 2$*. Moreover, it is the set of all* $m \in \mathbb{N}$ *for which there exists* $f \in X_n$*, such that* $\mathrm{Per}\,(f) = \mathbb{N} \setminus \{m\}$.

Given $n \geq 2$ we shall denote by p_n the smallest prime number which is greater than n and by δ_n the smallest prime divisor of n. Also, for $k \in \mathbb{N}$, we set $T_n(k) = \{m \in \mathbb{N} \setminus \{k\} : m \leq_t k \text{ for all } t = 2, 3, \dots, n\}$.

Proposition 2.2. *Assume that* $n \geq 3$*. Then*

(a) $\{2, 3, \dots, 2n-1\} \subset K_n \subset \{2, 3, \dots, p_n(n-2)\} \cup \{p_n(n-1)-n, p_n(n-1)\}$.
(b) If $m \in \{2n, 2n+1, \dots, p_n(n-2)\} \cup \{p_n(n-1)-n, p_n(n-1)\}$*, then* $m \notin K_n$ *if and only if* $m \in T_n(k)$ *for some odd* $k \in K_n$ *such that* $n < k \leq m-n+2$ *and* $k/\delta_k \leq n$.

3. Proof of Theorem A

We start by showing the existence of the FPK of $\mathcal{G}_n$. This result, at the same time, will provide a first characterization of $K(\mathcal{G}_n)$. To this end, we follow [**2**].

For $t \in \mathbb{N}$ and $\alpha \in \mathbb{N}_t$ we will denote by $S_t(\alpha)$ the set $\{n \in \mathbb{N} : n \leq_t \alpha\}$. Notice that each $S_t(\alpha)$ is an initial segment of the ordering $\leq_t$ and that $S_t(\beta) \subsetneq S_t(\alpha)$ for each $\beta \in \mathbb{N}_t$, $\beta <_t \alpha$. However, in general, an initial segment of the ordering $\leq_t$ is not of the form $S_t(\alpha)$ with $\alpha \in \mathbb{N}_t$. To see it consider for example the set $\mathbb{N} \setminus \{2, 4\}$. It is an initial segment of the ordering $\leq_3$ but it is not of the form $S_3(\alpha)$ with $\alpha \in \mathbb{N}_3$. Indeed, since $S_3(4) = \mathbb{N} \setminus \{2, 5, 8\}$, $S_3(5) = \mathbb{N} \setminus \{2, 4, 7, 10\}$ and $S_3(7) = \mathbb{N} \setminus \{2, 4, 5, 8, 11, 14\}$ we see that

$$S_3(\alpha) \subsetneq S_3(5) \cup S_3(7) = \mathbb{N} \setminus \{2, 4\} \subsetneq S_3(4) \cup S_3(5) = \mathbb{N}_3 \setminus \{3 \cdot 2^\infty\}$$

for each $\alpha \in \mathbb{N}_3 \setminus \{4\}$. So, the claim holds.

Proposition 3.1. *Let* $C \subset \mathbb{N}$ *be a minimal set (with respect to the inclusion relation) with the property that each* $f \in \mathcal{G}_n$ *such that* $C \subset \mathrm{Per}\,(f)$ *has full periodicity. Then* $m \in C$ *if and only if there exists* $f \in \mathcal{G}_n$ *such that* $\mathrm{Per}\,(f) = \mathbb{N} \setminus \{m\}$.

Proof. Suppose that there exists $f \in \mathcal{G}_n$ such that $\mathrm{Per}\,(f) = \mathbb{N} \setminus \{m\}$. If $m \notin C$, then $C \subset \mathrm{Per}\,(f)$ but $\mathrm{Per}\,(f) \neq \mathbb{N}$; a contradiction.

Now we assume that $m \in C$ and we prove that there exists a map $f \in \mathcal{G}_n$ such that $\mathrm{Per}\,(f) = \mathbb{N} \setminus \{m\}$. First we claim that there exists a map $g \in \mathcal{G}_n$ such that $m \notin \mathrm{Per}\,(g)$ but $C \setminus \{m\} \subset \mathrm{Per}\,(g)$. To prove this claim we note that $1 \in \mathrm{Per}\,(f)$

for each $f \in \mathcal{G}_n$. Therefore, by the minimality of C, $1 \notin C$. If $C = \{m\}$ we simply take $g \in \mathcal{G}_n$ such that $\mathrm{Per}\,(g) = \{1\}$. So, we assume that $C \neq \{m\}$.

If we suppose that there exist $k \in C \setminus \{m\}$ such that $m \leq_t k$ for each $t \in \{1, 2, \ldots, n+2\}$ then the Graph's Theorem contradicts the minimality of C. Hence, for each $k \in C \setminus \{m\}$ there exists $\widetilde{t}_k \in \{1, 2, \ldots, n+2\}$ such that $m \not\leq_{\widetilde{t}_k} k$. Now, we set

$$t_k = \begin{cases} \widetilde{t}_k & \text{if } k \in \mathbb{N}_{\widetilde{t}_k}, \\ k & \text{otherwise.} \end{cases}$$

Then, $k \in \mathbb{N}_{t_k}$ for each $k \in C \setminus \{m\}$ and, by the definition of the $\leq_t$ orderings, we still have that $m \not\leq_{t_k} k$ for each $k \in C \setminus \{m\}$. Let us consider

$$A = \bigcup_{k \in C \setminus \{m\}} S_{t_k}(k).$$

This set is a finite union of initial segments of the orderings $\leq_t$ and thus, the existence of $g \in \mathcal{G}_n$ such that $\mathrm{Per}\,(g) = A$ is guaranteed by Graph's Theorem. Since $m \notin A$ and $C \setminus \{m\} \subset A$, the claim is proved.

Now, let $B = \mathbb{N} \setminus \mathrm{Per}\,(g)$. If $B = \{m\}$ we are done. Otherwise we can consider two cases:

Case 1. $B \subset S_2(m)$.

Since $m \in \mathbb{N}$ there exists the maximum k of the set $S_2(m) \setminus \{m\}$ in the $\leq_2$ ordering. From Graph's Theorem there exists a map $f \in \mathcal{G}_n$ such that $\mathrm{Per}\,(f) = \mathrm{Per}\,(g) \cup S_2(k)$. Then, we have

$$\mathbb{N} \setminus \{m\} = (\mathbb{N} \setminus B) \cup (B \setminus \{m\}) \subset \mathrm{Per}\,(g) \cup S_2(k) \subset \mathbb{N} \setminus \{m\}\,.$$

Hence, $\mathrm{Per}\,(f) = \mathbb{N} \setminus \{m\}$.

Case 2. $B \not\subset S_2(m)$.

Then there exists $k \in B$ such that $m \leq_2 k$. Again by Graph's Theorem there exists a map $f \in \mathcal{G}_n$ such that $\mathrm{Per}\,(f) = \mathrm{Per}\,(g) \cup S_2(m)$. Clearly $\mathrm{Per}\,(f) \supset C$ but $k \notin \mathrm{Per}\,(f)$; a contradiction. □

From 3.1 we immediately obtain:

Corollary 3.2. *The FPK of* $\mathcal{G}_n$ *exists. Moreover, it is the set of all* $m \in \mathbb{N}$ *for which there exists* $f \in \mathcal{G}_n$ *such that* $\mathrm{Per}\,(f) = \mathbb{N} \setminus \{m\}$.

Finally we are ready to prove Theorem A.

Proof of Theorem A. First we prove that $K_{n+2} \subset K(\mathcal{G}_n)$. Let $m \in K_{n+2}$. Then, by Proposition 2.1, there exists an X_{n+2}-map f such that $\mathrm{Per}\,(f) = \mathbb{N} \setminus \{m\}$. By the n-star's and Graph's Theorems there exists a map $g \in \mathcal{G}_n$ such that $\mathrm{Per}\,(g) = \mathbb{N} \setminus \{m\}$ and so, by Corollary 3.2, $m \in K(\mathcal{G}_n)$. Now we will prove that $K(\mathcal{G}_n) \subset$

K_{n+2}. Let $m \in K(\mathcal{G}_n)$. If $m \leq 2(n+2) - 1$, by Proposition 2.2, $m \in K_{n+2}$. So, assume that $m \geq 2(n+2)$. By Corollary 3.2 there exists $f \in \mathcal{G}_n$ such that $\mathrm{Per}(f) = \mathbb{N} \setminus \{m\}$. Then, by Graph's Theorem, for each $t \in \{1, 2, \ldots, n+2\}$ there exists an initial segment B_t of $\leq_t$ such that

$$\mathbb{N} \setminus \{m\} = \mathrm{Per}(f) = \bigcup_{t=1}^{n+2} B_t.$$

We note that $B_1 \subset \{1, m+1, m+2, \ldots\}$ in view of the definition of the $\leq_1$ ordering. We claim that there exists $t_m \in \{2, 3, \ldots, n+2\}$ such that t_m does not divide m. To prove it suppose on the contrary that m is a multiple of t for $t = 2, 3, \ldots, n+2$. From [**6**; Bertrand's Postulate (Theorem 418)] it follows that $p_{n+2} < 2(n+2)$ (recall that p_{n+2} denotes the smallest prime number larger than $n+2$). So, $p_{n+2} < m$ and, hence, $p_{n+2} \notin B_1$. On the other hand, since m is multiple of t and $m \notin B_t$ for each $t \in \{2, 3, \ldots, n+2\}$, bearing the ordering definitions in mind, we get that B_t is contained in the set of all multiples of t union $\{1\}$. So, since p_{n+2} is prime and $p_{n+2} > n+2$, we see that $p_{n+2} \notin B_t$ for each $t = 2, 3, \ldots, n+2$. Hence

$$p_{n+2} \notin \bigcup_{t=1}^{n+2} B_t = \mathbb{N} \setminus \{m\}$$

which is a contradiction because $p_{n+2} \neq m$. This ends the proof of the claim.

Since t_m does not divide m and $m \geq 2(n+2)$ we have $m \in \mathbb{N}_{t_m}$ and there exists $\alpha \in \{1, 2, \ldots, t_m - 1\}$ and $\beta > 1$ such that $m = \beta t_m + \alpha$.

Set

$$B^*_{t_m} = \left[\bigcup_{j=1}^{t_m - 1} S_{t_m}(\beta t_m + j) \right] \setminus \{m\}.$$

Since the union of initial segments of $\leq_{t_m}$ is an initial segment of $\leq_{t_m}$, we get that $B^*_{t_m}$ is an initial segment of $\leq_{t_m}$. Clearly $\{1, m+1, m+2, \ldots\} \subset B^*_{t_m}$. So, the set $\widetilde{B}_{t_m} = B^*_{t_m} \cup B_{t_m}$ is an initial segment of $\leq_{t_m}$, $B_1 \cup B_{t_m} \subset \widetilde{B}_{t_m}$ and $m \notin \widetilde{B}_{t_m}$. In view of the n-star's Theorem there exists an X_{n+2}-map g such that

$$\mathrm{Per}(g) = \left(\bigcup_{\substack{t=2 \\ t \neq t_m}}^{n+2} B_t \right) \bigcup \widetilde{B}_{t_m}.$$

Hence

$$\mathbb{N} \setminus \{m\} = \bigcup_{t=1}^{n+2} B_t \subset \left(\bigcup_{\substack{t=2 \\ t \neq t_m}}^{n+2} B_t \right) \bigcup \widetilde{B}_{t_m} \subset \mathbb{N} \setminus \{m\}.$$

Therefore, $\mathrm{Per}(g) = \mathbb{N} \setminus \{m\}$ and, by Proposition 2.1, $m \in K_{n+2}$. $\square$

By Theorem A and Proposition 2.2 we immediately get the following:

Corollary B. *Assume that $n \geq 1$. Then*

(a) $\{2, 3, \dots, 2n+1\} \subset K(\mathcal{G}_n) \subset \{2, 3, \dots, p_{n+2}n\} \cup \{p_{n+2}(n+1) - (n+2), p_{n+2}(n+1)\}$.

(b) If $m \in \{2n+2, 2n+3, \dots, p_{n+2}n\} \cup \{p_{n+2}(n+1) - (n+2), p_{n+2}(n+1)\}$, then $m \notin K(\mathcal{G}_n)$ if and only if $m \in T_{n+2}(k)$ for some odd $k \in K(\mathcal{G}_n)$ such that $n + 2 < k \leq m - n$ and $k/\delta_k \leq n + 2$.

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