Acta Math. Univ. Comenianae
Vol. LXIII, 2(1994), pp. 227–239

ON THE STRUCTURE OF MINIMAL ATTRACTION CENTERS OF RECURRENT TRAJECTORIES OF CONTINUOUS MAPS OF THE INTERVAL

A. G. SIVAK

Abstract. We study the structure of minimal attraction centers of recurrent trajectories of continuous maps of the interval, i.e. trajectories of points, which belong to their ω-limit sets. We establish sufficient conditions, under which a pair of closed sets is realizable as the pair of the ω-limit set and the minimal attraction center of a recurrent trajectory of a continuous map. The case when these conditions are necessary and are not sufficient is also discussed and corresponding examples are suggested.

1. Introduction

We study the dynamics of continuous maps $f\colon I \to I$ where I is the interval $[0,1]$. Each point $x \in I$ corresponds to an ordered sequence $\{f^n(x)\}_{n=1}^{\infty}$, which is called the trajectory of the point x. The limit behavior of a trajectory is usually described by its ω-limit set, i.e. by the set of limit points of the trajectory. The admissible topological structure of ω-limit sets of continuous maps and the dynamics of such maps on ω-limit sets were studied in sixties by A. N. Sharkovskiĭ ([**6**]–[**9**]). In particular, it has been shown in [**6**] that for continuous maps of the interval any ω-limit set is either a nowhere dense set or a finite collection of mutually disjoint nondegenerate intervals. Recently it was proved [**1**] that any nonempty closed set of the above mentioned structure is the ω-limit set of a trajectory of a continuous map of the interval.

Statistical peculiarities of the limit behavior of a trajectory are characterized by the minimal attraction center or statistical limit set (σ-limit set) of the trajectory, i.e. by the smallest closed set, near which the trajectory moves almost all time. The notion of minimal attraction center was first used in [**2**], [**4**] (see also [**5**]) in connection with the study of the existence problem for invariant measures of dynamical systems. We use the following definition of this set.

The trajectory of a point x is called to be statistically asymptotic [**4**] to a closed set F if for any open neighborhood U of F one has $\lim\limits_{n\to\infty} \frac{1}{n}\sum_{i=0}^{\infty} 1_U(f^i(x)) = 1$

Received November 12, 1993.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 58F12, 26A18; Secondary 58F03, 58F08, 54H20.

where 1_U is the indicator function on U, i.e. the real-valued function such that $1_U(x) = 1$ for $x \in U$ and $1_U(x) = 0$ for $x \notin U$. The σ-limit set $\sigma(x, f)$ is defined to be the smallest closed set, which the trajectory of x is statistically asymptotic to.

The set $\sigma(x, f)$ is characterized by the following two properties:

(i) the trajectory of x is statistically asymptotic to $\sigma(x, f)$,

(ii) for every $y \in \sigma(x, f)$ and for every open set U containing the point y, one has

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{\infty} 1_U(f^i(x)) > 0.$$

The admissible topological structure of minimal attraction centers of continuous maps of the interval was described in [**10**]: in order that a closed nonempty subset of the interval be the σ-limit set of a trajectory of a continuous map of the interval, it is necessary and sufficient that either this subset be nowhere dense or it be a finite collection of mutually disjoint nondegenerate closed intervals.

In this paper we study the structure of minimal attraction centers of recurrent (more exactly, ω-recurrent) trajectories of continuous maps of the interval, i.e. trajectories belonging to their ω-limit sets. Using simple arguments based on results of [**6**], [**1**] and [**10**], it is not difficult to understand the mutually admissible structure of ω- and σ-limit sets of recurrent trajectories having infinite ω-limit sets (if an ω-limit set is finite, then it is a cycle [**6**]): the ω-limit set must be a perfect set, which satisfies the above mentioned admissibility conditions for ω-limit sets, and the σ-limit set must either coincide with the ω-limit set or be a nonempty nowhere dense subset of the ω-limit set. We prove that if a pair of closed sets (P, S) satisfies these admissibility conditions for (ω, σ)-pairs of recurrent trajectories of continuous maps and if, in addition, the set P is not two or more closed intervals, then these conditions are sufficient for a pair of sets be realizable as the (ω, σ)-pair of a recurrent trajectory of a continuous map of the interval. If the set P is two or more intervals, the map must cyclically permute these intervals and this fact generates additional restrictions on the structure of corresponding σ-limit set in the ω-limit set. For this case we suggest corresponding examples and prove that such a pair of sets is the (ω, σ)-pair of a recurrent trajectory of a continuous map of the interval if and only if the set S can be continuously mapped onto itself in a suitable way, i.e. the problem under consideration is reduced to the problem of finding of a continuous map of a certain kind on S.

2. Admissibility Conditions

In what follows, P and S are supposed to be subsets of the interval I such that $S \subset P$. We say that a pair of sets (P, S) is the (ω, σ)-pair of a (recurrent) trajectory of a continuous map if for some continuous map of the interval, P is the ω-limit set and S is the minimal attraction center (i.e. the σ-limit set) of some, one and the same, (recurrent) trajectory of the map. In this section we establish some properties of pairs of sets, which are (ω, σ)-pairs of recurrent trajectories. Namely, we prove the following statement, which is implied by main properties of ω-limit sets [**6**] and minimal attraction centers [**10**].

Proposition 1. *If a pair (P, S) of sets is the (ω, σ)-pair of a recurrent trajectory of a continuous map of the interval, then*

a) P is a nonempty closed set, which is either a finite set, a perfect nowhere dense set or a finite collection of mutually disjoint nondegenerate closed intervals;

b) S is a nonempty closed subset of P, which is either equal to P or nowhere dense in P.

Proof. It has been proved in [**6**] that any finite ω-limit set is a cycle and that if an infinite ω-limit set contains a periodic point, then this periodic point is not isolated in the ω-limit set. Hence any infinite ω-limit set has no isolated points whenever it is the ω-limit set of a recurrent trajectory because of such a trajectory is dense in its ω-limit set. This implies periodicity of isolated points of any ω-limit set and contradicts above mentioned arguments. Therefore the ω-limit set of any recurrent trajectory is either finite or perfect. As we have mentioned above, by results of [**6**] any ω-limit set of continuous maps of the interval is either a nowhere dense set or a finite collection of mutually disjoint nondegenerate closed intervals. Now this proves property a).

If for a recurrent trajectory its σ-limit set is dense in some part of its ω-limit set, then evidently the trajectory hits into the σ-limit set after finitely many steps because of any recurrent trajectory is dense in its ω-limit set. Since the σ-limit set is invariant, it must coincide with the closure of the trajectory, which is equal to the ω-limit set in this case. This implies property b) and completes the proof. □

3. Main Results

For the sake of convenience and conciseness of the consequent explanations, we use the following definition.

Definition. We say that a pair (P, S) of sets is admissible if P and S satisfy respectively conditions a) and b) of Proposition 1.

The following theorem describes the cases, in which any admissible pair of sets is the (ω, σ)-pair of a recurrent trajectory of a continuous map.

Theorem 1. *Let a pair (P, S) of subsets of the interval I be admissible. If P is not two or more intervals, then the pair of sets (P, S) is the (ω, σ)-pair of some recurrent trajectory of some continuous map of the interval.*

Now let us consider the case, which is excluded by the conditions of the theorem, i.e. the case when P is two or more intervals. For this case, the following theorem provides some necessary and sufficient conditions, under which an admissible pair of sets is the (ω, σ)-pair of some trajectory of a continuous map of the interval. We use the notion of σ-recurrent point in the statement of this theorem: a point x is called to be σ-recurrent if it belongs to its minimal attraction center, i.e. $x \in \sigma(x, f)$.

Theorem 2. *Let a pair (P, S) of sets be admissible and let $P = \cup_{i=0}^{n-1} I_i$, $n \geq 1$, where $\{I_i\}_{i=0}^{n-1}$ is a finite collection of mutually disjoint nondegenerate closed subintervals of the interval I. Then the pair of sets (P, S) is the (ω, σ)-pair of some trajectory of a continuous map of the interval if and only if there exists a continuous map $f\colon S \to S$ such that $f(S \cap I_i) = S \cap I_{(i+1) \bmod n}$ and the set of σ-recurrent points of f is dense in S.*

Note that if P is two or more intervals, then there always exists a continuous map, some trajectory of which has both the ω-limit set and the σ-limit set are equal to P, i.e. the case $P = S$ in the theorem can be examined easily. If P is an interval (i.e. $n = 1$), then we can always set the map f is equal to the identity mapping. Therefore in this case the admissibility of a pair of sets is sufficient that the pair be an (ω, σ)-pair. However if P is two or more intervals (i.e. $n > 1$), the situation is some different. In this case we have $f(I_i) = I_{(i+1) \bmod n}$ for any trajectory, the ω-limit set of which is equal to P. Hence each of the sets $S \cap I_0, S \cap I_1, \ldots, S \cap I_{n-1}$ must be cyclically mapped by the continuous map f onto other one. Obviously there are a lot of closed nowhere dense sets, which can not be continuously mapped in such a way. The simplest example for $n = 2$ may be any set $S \subset P$ consisting of three points. Furthermore since any σ-limit set must contain a dense subset consisting of σ-recurrent points [**10**], we obtain some additional restrictions on the set S because of, in particular, this implies that any isolated point of S must be periodic. If we denote the set of all isolated points of S by S^0 then the periodicity of isolated points implies that the closure of the set $S^* = S \backslash \overline{S^0}$ as well as the closed set $S^1 = \overline{S^0} \backslash S^0$ must be invariant under f. Using these observations, we obtain new restrictions on the topological structure of the set S in P and so on. For example, let $n = 2$, S^1 consist of two points $s_0^1 \in I_0$ and $s_1^1 \in I_1$ and the closure of S^* consist of two Cantor sets $S_0^* \subset I_0$ and $S_1^* \subset I_1$. Note that in this case sets $S_0^0 = S^0 \cap I_0$ and $S_1^0 = S^0 \cap I_1$ are infinite sequences, which tend respectively to points s_0^1 and s_2^1. Since S^1 is invariant, we must have $f(s_0^1) = s_1^1$ and $f(s_1^1) = s_0^1$. Since $\overline{S^*}$ is invariant, we have to exclude the cases when just one of these points belongs to the set $\overline{S^*}$. Using similar arguments, one can construct a lot of more complicated examples of admissible pairs of sets, which can not be realized as (ω, σ)-pairs of trajectories of continuous maps of the interval.

4. Proofs of Main Results

Proof of Theorem 1. Let (P, S) be an admissible pair of subsets of the interval I. We consider some cases.

P is a finite set. In this case the admissibility of a pair (P, S) implies the equality $S = P$. It is obvious that any cyclic permutation of the finite set P can be continuously extended onto the whole interval I in order to obtain a map with required properties.

P is a Cantor set and $S = P$. In this case we construct a continuous map $f\colon P \to P$ generating an almost periodic dynamics on P. By a well known result of the theory of dynamical systems (see [**3**] or [**5**]), the set P is minimal and hence for all points of P, their ω-limit sets and minimal attraction centers are coinciding and equal to P. After this we extend f to the components of $I \backslash P$ by linearity.

Let us consider a binary representation of points in P, which is defined by the following "almost bisection procedure" for P. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a monotonically decreasing sequence of positive real numbers, which will be defined later. Let $a = \inf P$, $b = \sup P$ and $J = [a, b]$. Since P is nowhere dense in J, we can find a point $c \notin P$, for which $|\frac{1}{2}(a + b) - c| < \varepsilon_1$, and divide the set P into two disjoint closed subsets $P_0 = P \cap [a, c]$ and $P_1 = P \cap [c, b]$. For $i = 0, 1$ we denote $a_i = \inf P_i$, $b_i = \sup P_i$ and $J_0 = [a_0, b_0]$, $J_1 = [a_1, b_1]$. Note that $J_0 \cap J_1 = \emptyset$ and for any $\varepsilon_1 \le \frac{1}{4}(b - a)$, both P_0 and P_1 are nonempty and hence both J_0 and J_1 are nondegenerate.

After this both sets P_0 and P_1 have to be "almost bisected" again to within ε_2: we can find points $c_0 \notin P$ and $c_1 \notin P$, for which we have $|\frac{1}{2}(a_0 + b_0) - c_0| < \varepsilon_2$ and $|\frac{1}{2}(a_1 + b_1) - c_1| < \varepsilon_2$, and then define $P_{i0} = P_i \cap [a_i, c_i]$ and $P_{i1} = P_i \cap [c_i, b_i]$, $i = 0, 1$. For $i, j \in \{0, 1\}$, let $J_{ij} = [a_{ij}, b_{ij}]$ where $a_{ij} = \inf P_{ij}$, $b_{ij} = \sup P_{ij}$. In order that the procedure can be continued, it is sufficient that $\varepsilon_2 \le \frac{1}{4} \min\limits_{i \in \{0,1\}} |J_i|$ where $|J_i|$ denotes the length of the interval $|J_i|$.

After n steps we shall have 2^n mutually disjoint subsets P_α and 2^n corresponding intervals J_α, $\alpha \in \{0, 1\}^n = \{i_1 \dots i_{n-1} i_n : i_j \in \{0, 1\}\}$. For the next step we can define $\varepsilon_{n+1} = \frac{1}{4} \min\limits_{|\alpha| = n} |J_\alpha|$ where the symbol $|\alpha|$ denotes the number of elements in the finite chain α. Starting with an arbitrary small enough ε_1 and using this formula for ε_{n+1} successively for $n = 1, 2, \dots$, we shall have $\max\limits_{|\alpha| = n} |J_\alpha| \le 2^{-n}(1 + n/2)|J|$ and hence $\lim\limits_{n \to \infty} \max\limits_{|\alpha| = n} |J_\alpha| = 0$.

As a result we can set a one-to-one correspondence between the points of the Cantor set P and the infinite binary sequences: any $\alpha = \alpha_1 \alpha_2 \alpha_3 \cdots \in \{0, 1\}^{\aleph}$ corresponds to a unique point $x_\alpha \in P$, which is defined by the equality $x_\alpha = \bigcap\limits_{n \ge 1} J_{\alpha_1 \dots \alpha_n}$. Let us define the addition operation on the set $\{0, 1\}^{\aleph}$ as follows: starting with the lowest digit (α_1), we sum successively corresponding digits and add the overflow unit (if it occurs) to the next digit. For example, $111\cdots+100\cdots=000\ldots$, $100\cdots+000\cdots=100\ldots$, $100\cdots+100\cdots=010\ldots$ and so on.

Now we can define the map $f\colon P\to P$ by the equality $f(x_{\alpha_1\alpha_2\alpha_3\ldots})=x_{\alpha_1\alpha_2\alpha_3\cdots+100\ldots}$. It is clear that for any $n\geq 1$, 2^n sets of the family $\{P_\alpha\}_{|\alpha|=n}$ are cyclically permuted by the map f. Diameters of these sets tend to zero uniformly as $n\to\infty$. Hence the map $f\colon P\to P$ is continuous and the trajectory of any point $x\in P$ is dense in P and almost periodic under f. Extending f continuously to the components of $I\backslash P$ by linearity, we complete the proof for this case.

P is a Cantor set and S is nowhere dense in P. We use the following construction. At first we construct a continuous function $\varphi\colon I\to I$, for which $S\subset \mathrm{Fix}\,(\varphi)$ where $\mathrm{Fix}\,(\varphi)$ denotes the set of fixed points of φ. We prove that φ has an invariant Cantor set P^* containing S and that φ is expanding on P^* in some sense. These properties of φ imply the existence of a point $x^*\in P^*$, the ω-limit set and the minimal attraction center of which are respectively P^* and S. After this we construct a homeomorphism $h\colon I\to I$ for which we have $h(P)=P^*$ and $S\subset\mathrm{Fix}\,(h)$. At last considering the continuous map $f=h^{-1}\circ\varphi\circ h$ and the trajectory of the point $x=h^{-1}(x^*)$ under f, we prove that the ω-limit set of this trajectory is P and the minimal attraction center is S.

In order to avoid some difficulties, we consider some other set $S^*\supset S$ instead of S. The set S^* is defined as follows. At first we add to S points $\inf P$, $\sup P$ and if the set S contains no one-side limit points of P in the interval $(\inf P,\sup P)$, then we add one such a point to S. Let this new set be denoted by S_1. Any one-side limit point d of P in $(\inf P,\sup P)$ is an end of a unique interval from the family of components of the open set $(\inf P,\sup P)\backslash P$; for any given d, let d' denote the second end of this interval. The set S^* is defined by adding to the set S_1 all one-side limit points d of P in the interval $(\inf P,\sup P)$, for which $d'\in S_1$. Note that the set S^* is still a nonempty closed nowhere dense subset of P.

Let Δ denote the family of all open intervals $D=(d,d')$ such that $d\in S^*\cap(\inf P,\sup P)$ and d is a one-side limit point of P. Let Φ denote the family of all components of the open set $(\inf P,\sup P)\backslash\overline{\bigcup_{D\in\Delta}D}$. Note that the union of all intervals of Δ and Φ define an open dense subset of $(\inf P,\sup P)$. Moreover any two different intervals of Δ can not touch each other as well as any two different intervals of Φ can not touch each other.

Let us consider any interval $F\in\Phi$ and the closed set $S^*_F=\overline{F}\cap S^*$. Note that $\inf S^*_F=\inf F$ and $\sup S^*_F=\sup F$. Moreover the set $S^*_F\cap F$ contains no one-side limit points of P. It is clear that S^*_F is closed and nowhere dense in $\overline{F}$.

Let $\Gamma(F)$ denote the family of all components of the open set $F\backslash S^*_F$. Note that sets S^* and $\bigcup_{G\in\Gamma(F),\ F\in\Phi}G$ are disjoint.

We define $\varphi(x)=x$ for $x\notin\bigcup_{G\in\Gamma(F),\ F\in\Phi}G$.

Before we define φ on intervals of $\Gamma(F)$, we shall define images of the intervals $F \in \Phi$ first. Let $J_0 = [\inf P, \sup P]$ and F_0 be the largest of the intervals of Φ in J_0. We set $\varphi(F_0) = [\inf P, \sup P]$. Then we have $J_0 = L \cup \overline{F_0} \cup R$ where L and R are the left and the right components of $J_0 \setminus \overline{F_0}$ respectively. Let $J_{00} = [\inf L, \sup L]$ and $J_{10} = [\inf R, \sup R]$. If J_{00} is nondegenerate, then it contains some intervals from Δ. Let D_{00} be the largest of them. Then we have $J_{00} = J_{000} \cup D_{00} \cup J_{100}$ where J_{000} and J_{100} are the left and the right components of $J_{00} \setminus D_{00}$ respectively. By using the same arguments, we obtain $J_{10} = J_{010} \cup D_{10} \cup J_{110}$ (we should note that all degenerate intervals being once occurred are supposed to be excluded from the further consideration).

Each of the intervals J_{000}, J_{100}, J_{010}, J_{110} must contain some intervals of Φ, the largest of which will be denoted by F_{000}, F_{100}, F_{010} and F_{110} respectively. We define $\varphi(F_{000}) = [\inf J_0, \sup F_0]$, $\varphi(F_{100}) = [\inf J_{100}, \sup F_0]$, $\varphi(F_{010}) = [\inf F_0, \sup J_{010}]$, $\varphi(F_{110}) = [\inf F_0, \sup J_0]$.

Now for each of the intervals of the set $\{J_{ij0}\}$, $i, j \in \{0, 1\}$, we have obtained the conditions, which are similar to the initial conditions for J_0: the largest interval F_{ij0} of the family Φ on J_{ij0} and its image $\varphi(F_{ij0})$ are defined. Hence we can apply above described arguments to each interval J_{ij0} in order to define images of the next several intervals from Φ. Since we choose the largest intervals of Φ on each step, this way gives a possibility to define images of all intervals of the set Φ.

Remark. Note that for any $F \in \Phi$, points $\inf \varphi(F)$ and $\sup \varphi(F)$ are chosen to be nonisolated in P from the right and from the left respectively.

Lemma 1. *Let $\{F_n\}$ be a sequence of intervals from Φ. If $|F_n| \to 0$ as $n \to \infty$, then $|\varphi(F_n)| \to 0$ as $n \to \infty$.*

Proof. For each $F, F^* \in \Phi$ (and similarly for each $D \in \Delta$) we have either $F \subset \varphi(F^*)$ or $F \cap \varphi(F^*) = \emptyset$ (respectively $D \subset \varphi(F^*)$ or $D \cap \varphi(F^*) = \emptyset$). Moreover each step of the above used construction removes from the consideration some of the largest intervals of Φ and Δ. Therefore for any given $F \in \Phi$ (and also $D \in \Delta$) there exists finitely many intervals $F^* \in \Phi$ such that $F \cap \varphi(F^*) \neq \emptyset$ (respectively $D \cap \varphi(F^*) \neq \emptyset$).

Let us suppose that the lemma is not true. Then we can find a sequence $F_n = (u_n, v_n)$ of distinct intervals from Φ, for which we shall have $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = a$ for some a and $\bigcap_{n \geq 1} \varphi(F_n) \supset (b, c)$ for some b and c with $b < c$. Since the intervals of Φ and Δ form a dense set in $[\inf P, \sup P]$, we can find an interval $A \in \Phi \cup \Delta$ such that $A \cap \varphi(F_n) \neq \emptyset$ for all n. This contradiction proves the lemma. $\square$

Let us choose any interval $F \in \Phi$ and consider the nowhere dense set $S_F = S^* \cap \overline{F}$. Recall that $\Gamma(F)$ denotes the family of all components of $F \setminus S_F$. We define φ on each interval $G \in \Gamma(F)$ as a continuous piecewise linear function, which consists of three linear pieces: at first we divide the interval G into three equal parts and define φ at two division points inside of the interval; after this we expand φ on the whole interval G by linearity (recall that the ends of G must be fixed points of φ).

Let us consider the initial interval $F_0 = \overline{F}$, $F \in \Phi$, for which we know its image $\varphi(F_0)$. Let $G_0 = (a_0, b_0)$ be the largest of the intervals of $\Gamma(F)$ in F_0. Then $F_0 = F_{00} \cup G_0 \cup F_{10}$ where F_{00} and F_{10} are the left and the right components of the set $F_0 \backslash G_0$ respectively. According to the above mentioned reasoning, in order that the piecewise linear map $\varphi|_{G_0}$ be defined, it is sufficient that this map be defined at points $a_0 + \frac{1}{3}(b_0 - a_0)$ and $b_0 - \frac{1}{3}(b_0 - a_0)$. To this end we set $\varphi(a_0 + \frac{1}{3}(b_0 - a_0)) = \sup \varphi(F_0)$ and $\varphi(b_0 - \frac{1}{3}(b_0 - a_0)) = \inf \varphi(F_0)$. Having the map $\varphi|_{G_0}$ defined in such a way, we can also images of the intervals F_{00} and F_{10}: if F_{00} is nondegenerate, then we set $\varphi(F_{00}) = [\inf F_0, \sup G_0]$; if F_{10} is nondegenerate, then we set $\varphi(F_{10}) = [\inf G_0, \sup F_0]$.

Now since images of the intervals F_{00} and F_{10} are defined, we can choosing the largest intervals of $\Gamma(F)$ in F_{00} and F_{10} respectively and then repeat on F_{00} and F_{10} the above described construction for F_0. As a result, we define φ on the next several intervals of $\Gamma(F)$. Since we choose the largest intervals of $\Gamma(F)$ on each step, the map φ will be defined on all intervals of the set $\Gamma(F)$ in such a way. By applying this method to each $F \in \Phi$, we define the map φ on the whole interval I.

Lemma 2. *The map $\varphi\colon I \to I$ is continuous.*

Proof. Note that the constructive definition of the map φ on intervals of $\Gamma(F)$ is similar to the constructive definition of images of intervals from Φ in the case $\Delta = \emptyset$. Hence by using the arguments of the proof of Lemma 1, we can conclude that for any sequence $\{G_n\}$ of intervals from $\Gamma(F)$, where $F \in \Phi$, we have $|\varphi(G_n)| \to 0$ as $n \to \infty$ whenever $|G_n| \to 0$ as $n \to \infty$. This implies the continuity of φ. □

Lemma 3. *For any open interval $U \subset [\inf P, \sup P]$, one has either $\varphi^K(U) = [\inf P, \sup P]$ for some $K < \infty$ or $\varphi^K(U) \subset D$ for some $D \in \Delta$ and some $K < \infty$.*

Proof. At first let us suppose that for some interval $F \in \Phi$ we have $U \subset F$ and $\overline{U} \cap (S^* \cap \overline{F}) \neq \emptyset$. It is clear that in this case we can find an interval $G \in \Gamma(F)$ such that the interval $U_G = U \cap G$ is nondegenerate and U_G has at list one common end with G (recall that $\overline{G} \cap (S^* \cap \overline{F}) = \{\inf G, \sup G\}$). Since $\varphi|_G$ is expanding, there exists $k < \infty$ such that $\varphi^k(U_G) \supset G$. Furthermore, for any $G \in \Gamma(F)$, we have either $\varphi(G)$ contains F or it contains some other interval $G^* \in \Gamma(F)$ with $|G^*| \geq |G|$. Therefore there is $l < \infty$ such that $\varphi^l(G) \supset \varphi(F) \supset F$. Using similar arguments, we can prove that for some $m < \infty$, we shall have $\varphi^m(F) = [\inf P, \sup P]$. Hence for $K = k+l+m$, we obtain $\varphi^K(U) = [\inf P, \sup P]$.

If $U \not\subset \bigcup\limits_{D \in \Delta} D$ and $\overline{U} \cap S^* = \emptyset$, then $\overline{U} \subset G_0 \subset F_0$ for some $F_0 \in \Phi$ and $G_0 \in \Gamma(F_0)$. If $\varphi(\overline{U})$ contains an extreme value of $\varphi|_{G_0}$, then by the remark before

Lemma 1 we can find an interval $F \in \Phi$ such that $\varphi(U) \cap F$ is nondegenerate and $\overline{\varphi(U)}$ contains at least one point of $\overline{F} \cap S^*$, i.e. $\overline{\varphi(U)} \cap (\overline{F} \cap S^*) \neq \emptyset$ and we can apply above described arguments of the proof to $\varphi(U) \cap F$. If $\overline{\varphi(U)}$ contains no extreme values of $\varphi|_{G_0}$, then $\varphi(U)$ is an open interval, for which $|\varphi(U)| \geq 3|U|$ and for which we have either $\varphi(U) \subset \bigcup_{D \in \Delta} D$ or $\varphi(U) \not\subset \bigcup_{D \in \Delta} D$ and $\overline{\varphi(U)} \cap S^* = \emptyset$. The first case is trivial, and in the second one we can apply the above described reasoning to this new interval $\varphi(U)$. Since the interval $[\inf P, \sup P]$ is finite, the proof shall be completed after a finite number of iterations of U. □

Let us consider the set $P^* = [\inf P, \sup P] \setminus \bigcup_{n \geq 0} \varphi^{-n}(\bigcup_{D \in \Delta} D)$.

Since φ is continuous and equal to the identity mapping on the open set $\bigcup_{D \in \Delta} D$, the set P^* is closed and invariant. Therefore by Lemma 3 the set P^* is nowhere dense. Since $S^* \subset \mathrm{Fix}\,(\varphi)$ and $S^* \bigcap \bigcup_{D \in \Delta} D = \emptyset$, we have $S^* \subset P^*$. If a point $x \in P^*$ is isolated in P^*, then by the definition of P^* we can see that for some $K < \infty$, $\varepsilon > 0$ and $D \in \Delta$, we have $\varphi^K((x-\varepsilon, x)) \subset D$, $\varphi^K((x, x+\varepsilon)) \subset D$ and $\varphi^K(x)$ is an end of the interval D. This contradicts Lemma 3 because of all points of $\overline{D}$ are fixed points of φ. Hence P^* is perfect. Note also that for all $F \in \Phi$ and $G \in \Gamma(F)$, we have $\inf(G \cap P) = \inf(G \cap P^*) = \inf G$ and $\sup(G \cap P) = \sup(G \cap P^*) = \sup G$.

Using these properties of the set P^* and Lemma 3, it is not difficult to check that for any open interval U, for which $U \cap P^* \neq \emptyset$, we can find $K < \infty$ such that $\varphi^K(U) \supset [\inf P, \sup P]$.

Lemma 4. *For any nonempty closed subset S of S^*, there exists a point $x^* \in P^*$, the ω-limit set and the minimal attraction center of which under φ are P^* and S respectively.*

Proof. For $n = 1, 2, \ldots$ let us set $\varepsilon_n = 2^{-n}$ choose finite ε_n-nets $S_n = \{s_1^{(n)}, s_2^{(n)}, \ldots, s_{k_n}^{(n)}\}$, $P_n = \{p_1^{(n)}, p_2^{(n)}, \ldots, p_{k_n}^{(n)}\}$ of compact sets S and P^* respectively such that $S_n \subset S$, $P_n \subset P^*$ and such that the number of points in the set S_n is equal to the number of points in P_n.

For $n \geq 1$ and $k = 1, 2, \ldots, k_n$, we define $\gamma_k^{(n)} = 1 - 2^{-n}$. For $x \in P^*$ and $\varepsilon > 0$, let $\overline{B}(x, \varepsilon)$ denote the interval $[x - \varepsilon, x + \varepsilon] \cap [\inf P^*, \sup P^*]$. By above stated properties of P^*, for any interval $\overline{B}(s_k^n, \varepsilon_n)$ we can find $t = t(k, n)$ such that $\varphi^t(\overline{B}(s_k^n, \varepsilon_n)) = [\inf P^*, \sup P^*]$ and for any interval $\overline{B}(p_k^n, \varepsilon_n)$ we can find $\tau = \tau(k, n)$ such that $\varphi^\tau(\overline{B}(p_k^n, \varepsilon_n)) = [\inf P^*, \sup P^*]$. Having obtained the numbers $t(k, n)$ and $\tau(k, n)$, successively for $n \geq 1$ and $k = 1, 2, \ldots, k_n$ we can define numbers $T(k, n)$ such that $\frac{T(k,n)}{\Sigma(k,n)} > \gamma_k^{(n)}$ where

$$\Sigma(k, n) = \sum_{k' \leq k} (T(k', n) + t(k', n) + t(k', n)) \\ + \sum_{n' \leq n} \sum_{1 \leq k' \leq k_{n'}} (T(k', n') + t(k', n') + t(k', n')) .$$

After this by using the continuity of φ, Lemma 3 and corresponding properties of P^*, we can construct an infinite decreasing sequence of closed intervals

$$X_1^{(1)} \supset X_2^{(1)} \supset \dots \supset X_{k_1}^{(1)} \supset X_1^{(2)} \supset X_2^{(2)} \supset \dots \supset X_{k_2}^{(2)} \supset X_1^{(3)} \supset \dots$$

such that

i) $\varphi^K(X_k^{(n)}) \subset \overline{B}(s_k^{(n)}, \varepsilon_n)$ for $\Sigma(k-1,n) \le K \le \Sigma(k-1,n) + T(k,n)$,

ii) $\varphi^L(X_k^{(n)}) = \overline{B}(p_k^{(n)}, \varepsilon_n)$ for $L = \Sigma(k-1,n) + T(k,n) + t(k,n)$,

iii) $\varphi^M(X_k^{(n)}) = [\inf P^*, \sup P^*]$ for $M = \Sigma(k,n)$.

(We assume $\Sigma(0,1) = 0$ and $\Sigma(0,n) = \Sigma(k_{n-1}, n-1)$ for $n > 1$.)

The required point x^* is obtained as the intersection of all $X_k^{(n)}$. $\square$

Now we can complete the proof of the case under consideration. The map $h\colon I \to I$ is defined as follows. For an arbitrary interval $G \in \Gamma(F)$, $F \in \Phi$, let C and C^* denote Cantor sets $\overline{G} \cap P$ and $\overline{G} \cap P^*$ respectively. As it has been mentioned above, we have $\inf C = \inf C^* = \inf G$ and $\sup C = \sup C^* = \sup G$. Let $\alpha\colon C \to \{0,1\}^{\aleph}$ and $\alpha^*\colon C^* \to \{0,1\}^{\aleph}$ be one-to-one correspondences defined by binary representations of Cantor sets C and C^* respectively, which have been described in the proof of the previous case. Then we define $h_G|_C = (\alpha^*)^{-1} \circ \alpha$. It is not hard to prove that $h_G|_C$ is monotonically increasing and continuous. We extend h_G to the components of $\overline{G} \setminus C$ by linearity. Then h_G is an orientation preserving homeomorphism of $\overline{G}$, for which we have $h_G(C) = C^*$. For any $G \in \Gamma(F)$, $F \in \Phi$, we set $h|_G = h_G$. All other points of the interval I are supposed to be fixed points of the map h. Then $h\colon I \to I$ is a homeomorphism, for which $h(P) = P^*$ and $S^* \subset \mathrm{Fix}\,(h)$. It is clear that for the trajectory of the point $x = h^{-1}(x^*) \in P$ (the point x^* is determined by Lemma 4 of the map $f = h^{-1} \circ \varphi \circ h$, the ω-limit set is P and the minimal attraction center is S because of f and φ are topologically conjugate.

P **is an interval.** In the case $S = P$ we can consider the tent map on P, i.e. the continuous map, which is linearly conjugated to the map $T(x) = 1 - 2|x|$ on $[-1,1]$. It is well known that the Lebesgue measure is an invariant measure of the tent map and hence for almost all points (with respect to the Lebesgue measure) their ω-limit sets are equal to their minimal attraction centers and coincide with the whole interval.

If S is nowhere dense in P, then we consider the family Γ of open (in P) components of the set $P \setminus S$ and define a continuous map $f\colon P \to P$ by using the method, which is completely identical to the method of construction of the map φ on any interval $F \in \Phi$ in the above considered case. After this by using certain arguments of the mentioned case, we can similarly find a point in P, the trajectory of which under f has the ω-limit set is equal to P and the minimal attraction center is equal to S. This completes the proof of the case and the theorem.

Proof of Theorem 2. Let an admissible pair (P, S) be such that P is a finite collection of mutually disjoint nondegenerate closed intervals, $P = I_0 \cup I_1 \cup \cdots \cup I_{n-1}$, $n \geq 2$, and S is nowhere dense in P. Suppose that for a continuous map $f \colon S \to S$ we have $f(S \cap I_i) = S \cap I_{(i+1) \bmod n}$ for $i = 0, 1, \ldots, n-1$ and the set of σ-recurrent points of f is dense in S. We are going to extend the map f onto the whole set P in a way that provides the following properties of f:

i) $f(I_i) = I_{(i+1) \bmod n}$ for $i = 0, 1, \ldots, n-1$;

and

ii) for any open interval $U \subset P$, there exists $K = K(U)$ such that $f^K(U) = I_0$.

Due to the expansion property ii) of f, a statement similar to the statement of Lemma 4 can be proved for the map f and the pair of sets (P, S).

For any $i \in \{0, 1, \ldots, n-1\}$, let us set $J_0^{(i)} = [\inf S_i, \sup S_i]$ where $S_i = S \cap I_i$. Let $G_0^{(i)}$ denote the largest component of the open set $J_0^{(i)} \backslash S_i$ in $J_0^{(i)}$. The left and the right components of the set $J_0^{(i)} \backslash G_0^{(i)}$ are denoted by $J_{00}^{(i)}$ and $J_{10}^{(i)}$ respectively. Now for each $j \in \{1, 2\}$, if $J_{j0}^{(i)}$ is nondegenerate, we choose the largest component of the open set $J_{j0}^{(i)} \cap (J_0^{(i)} \backslash S_i)$ in $J_{j0}^{(i)}$ and denote this component by $G_{j0}^{(i)}$. The left and the right components of the set $J_{j0}^{(i)} \backslash G_{j0}^{(i)}$ are denoted by $J_{0j0}^{(i)}$ and $J_{1j0}^{(i)}$ respectively. After this, for each $k \in \{1, 2\}$ and $j \in \{1, 2\}$, if $J_{kj0}^{(i)}$ is nondegenerate, we choose the largest component of the open set $J_{kj0}^{(i)} \cap (J_0^{(i)} \backslash S_i)$ in $J_{kj0}^{(i)}$ and denote this component by $G_{j0}^{(i)}$. The left and the right components of the set $J_{j0}^{(i)} \backslash G_{j0}^{(i)}$ are denoted by $J_{0j0}^{(i)}$ and $J_{1j0}^{(i)}$ respectively. By repeating the arguments, we shall index all intervals of the complement of the set S_i in $J_0^{(i)}$: $J_0^{(i)} \backslash S_i = \bigcup\limits_{|\alpha| \geq 1} G_\alpha^{(i)}$ where α is a finite chain of 0 and 1 ending by 0, and $|\alpha|$ denote the number of elements in the chain.

Note that since the set S is nowhere dense in P, we have $|G_\alpha^{(i)}| \to 0$ as $|\alpha| \to \infty$ where $|G_\alpha^{(i)}|$ denotes the length of the interval $G_\alpha^{(i)}$. By the same reason $|J_\alpha^{(i)}| \to 0$ as $|\alpha| \to \infty$. Note also that the map f has already been defined at ends of the intervals $G_\alpha^{(i)}$.

Let us define images under f of the intervals $G_\alpha^{(i)}$ first. If $\alpha = 0$ (i.e. $|\alpha| = 1$), then we set $f(G_0^{(i)}) = I_{(i+1) \bmod n}$. If $|\alpha| > 1$, then the image of $G_\alpha^{(i)}$ is defined as follows. Let $G_\alpha^{(i)} = (a, b)$. If the interval $[f(a), f(b)] \subset I_{(i+1) \bmod n}$ contains an interval $G_\beta^{(i+1) \bmod n}$ with $|\beta| < |\alpha|$, then we set $f(G_\alpha^{(i)}) = [f(a), f(b)]$. Otherwise the interval $[f(a), f(b)]$ belongs to some interval $J_\beta^{(i+1) \bmod n}$ with $|\beta| = |\alpha|$. Let $G_\gamma^{(i+1) \bmod n}$ be an interval with $|\gamma| = |\alpha| - 1$, which is adjoining to the interval $J_\beta^{(i+1) \bmod n}$. Then we set $f(G_\alpha^{(i)}) = J_\beta^{(i+1) \bmod n} \bigcup \overline{G_\gamma^{(i+1) \bmod n}}$.

Remark. We denote the above considered interval with ends $f(a)$ and $f(b)$ by $[f(a), f(b)]$ in both cases $f(a) < f(b)$ and $f(b) < f(a)$.

For such defined images of the intervals G_α^i, we have $|f(G_\alpha^{(i)})| \to 0$ as $|\alpha| \to \infty$. Hence under these conditions the map f can still be continuously extended onto the whole set P.

Let $G_\alpha^{(i)} = (a, b)$ and $f(G_\alpha^{(i)}) = [a', b']$. We have $f(a) \in [a', b']$ and $f(b) \in [a', b']$. Let us consider a subdivision of the interval (a, b) by points $a \le c_1 < c_2 < \cdots < c_{m-1} < c_m \le b$, where $m \ge 6$ and even, such that points $c_2, \ldots, c_{m-1}$ divide the interval $[c_1, c_m]$ into $m-1$ equal parts. For each point c_k with an odd subscript k, we set $f(c_k) = b'$, and for each point c_k with an even subscript k, we set $f(c_k) = a'$. Then we extend the map f onto the whole interval (a, b) by linearity. It remains to define m and points c_1 and c_m such that the map f is expanding on each interval of its linearity.

Let us suppose that $b' - a' \ge b - a$ first. In this case we set $m = 6$, $c_1 = a + \frac{1}{5}(b-a)\frac{b'-f(a)}{b'-a'}$ and $c_6 = b - \frac{1}{5}(b-a)\frac{f(b)-a'}{b'-a'}$. For this choice, the absolute value of the derivative of f is not less than 5 on each interval of its continuity in (a, b).

If $b' - a' < b - a$, then we set $c_1 = a + \frac{1}{5}(b' - a')\frac{b'-f(a)}{b'-a'}$ and $c_m = b - \frac{1}{5}(b' - a')\frac{f(b)-a'}{b'-a'}$ and choose $m \ge \frac{5(b-a)}{b'-a'} + 1$. For this choice, the absolute value of the derivative of f is not less than 5 on each interval of its continuity in (a, b) also.

On the set $I_i \backslash J_0^{(i)}$, $i = 0, 1, \ldots, n-1$, the map f is defined in such a way that $f(I_i \backslash J_0^{(i)}) \subset J_0^{(i+1) \bmod n}$ if the set is not empty. In order to define the map on the whole interval I, we can extend f to the components of $I \backslash P$ by linearity.

We are going to prove that for any open interval $U \subset P$, there exists $K = K(U)$ such that $f^K(U) = I_0$. First we observe that if the interval U contains an interval $G_\alpha^{(i)}$, then the statement is obvious. Let us prove that any open interval will cover an interval $G_\alpha^{(i)}$ after a finite number of iterations. Without loss of generality, we can suppose that U contains no points of S, i.e. U belongs to an interval $G_\beta^{(j)} = (a, b)$. Let $c_1, \ldots c_m$ be the points, which define the above described subdivision of the interval (a, b). If U contains at least two of these points, then, obviously, $f(U)$ contains an interval $G_\alpha^{(i)}$ where $i = (j+1) \bmod n$. If U contains at most one of these points, then $|f(U)| \ge \frac{5}{2}|U|$. If in this case the interval $f(U)$ does not cover some interval of the required kind, then there is an interval $G_\gamma^{(k)}$, where $k = (j+1) \bmod n$, such that $|f(U) \cap G_\gamma^{(k)}| \ge \frac{1}{2}|f(U)| \ge \frac{5}{4}|U|$. Hence for the subinterval $U_1 = f(U) \cap G_\gamma^{(k)}$ of the interval $f(U)$, we have $U_1 \subset G_\gamma^{(k)}$ and $|U_1| \ge \frac{5}{4}|U|$. By applying the above used arguments to the interval U_1, we prove that either $f(U_1)$ covers a suitable interval or it contains an interval U_2, which contains no points of S and for which we have $|U_2| \ge \frac{5}{4}|U_1|$. It is obvious that for some finite K, the interval $f(U_K)$ will cover an interval $G_\alpha^{(i)}$.

Having established that the map f has this expansion property on P, we can prove (analogously to the proof of Lemma 4 above) the existence of a point $x \in P$, the trajectory of which under f has the ω-limit set equal to P and the minimal attraction center is equal to S (a detailed proof of a similar statement for expanding maps of the interval is contained in [**10**]). Thus the "if" part of the theorem is proved.

Since any σ-limit set contains a dense subset consisting of σ-recurrent points [**10**], the "only if" part of the theorem is trivial and the proof is completed. $\square$

References

1. Agronsky S., Bruckner A., Ceder J. and Pearson T., *The structure of ω-limit sets for continuous functions*, Real Analysis Exchange **15** (1989-1990), 483–510.

2. Hilmy H., *Sur les centres d'attraction minimaux dans les systèmes dynamiques*, Comp. Math. **3**, No. 2 (1936), 227–238.

3. Gottschalk W. H., *Powers of homeomorphisms with almost periodic properties*, Bull. Amer. Math. Soc. **50** (1944), 222–227.

4. Krylov N. and Bogoljubov N., *La téorie générale de la measure dans son application á l'étude des systèmes de la mécanique non lineaire*, Ann. of Math. (2) **38** (1937), 65–113.

5. Nemytskiĭ V. V. and Stepanov V. V., *Qualitative Theory of Differential Equations*, Princeton Univ. Press, Princeton, N.J., 1960.

6. Sharkovskiĭ A. N., *Attracting and attracted sets*, Dokl. AN SSSR **160** (1965), 1036–1038. (Russian)

7. ———, *On continuous map on ω-limit set*, Dopovidi AN URSR, No. 11 (1965), 1407–1410. (Ukrainian)

8. ———, *Behavior of map in a neighborhood of attracting set*, Ukrain. Mat. Zhurn. **18**, No. 2 (1966), 60–83. (Russian)

9. ———, *Attracting sets containing no cycles*, Ukrain. Mat. Zhurn. **20**, No. 1 (1968), 136–142. (Russian)

10. Sivak A. G., *The structure of σ-limit sets for continuous mappings*, Report Nr. 248, Institut für Dynamische Systeme, Universität Bremen, Bremen, 1991.

A. G. Sivak, Institute of Mathematics, Ukrainian Academy of Sciences, Tereschenkivska 3, 252601 Kiev-4, Ukraine