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ON DIRECT DECOMPOSITIONS OF CERTAIN ORTHOMODULAR LATTICES

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ABSTRACT. Let L be an orthomodular lattice. For $a, b \in L$ define $a \leftrightarrow^c b$ if either a and b both belong to the centre $C(L)$ of L or if $\{a, b\} \cap C(L) = \emptyset$ and $a \leftrightarrow b$ (i.e. a is compatible with b). Let R be the transitive closure of the relation $\leftrightarrow^c$. Then there exist at least three equivalence classes of the relation R in L if and only if either L is a horizontal sum (if $C(L) = \{0, 1\}$) or L is a direct product of a Boolean algebra and a horizontal sum.

In the theory of orhomodular lattices (abbreviated: OML) a classical theorem states that every finitely generated OML decomposes into a direct product of a Boolean algebra and an OML without nontrivial Boolean factor [**7, 1**]. This classical decomposition theorem has obtained several generalizations [**4, 6, 8**]. In the present paper, we will characterize certain type of OML's admitting the above direct decomposition by means of so-called c-compatibility.

Basic definitions and facts about orthomodular lattices can be found in [**1**] and [**2**]. We recall that an orthomodular lattice $L(0, 1, ', \vee, \wedge)$ is a horizontal sum of orthomodular lattices $L_i,\ i \in I$, if $L = \cup_{i \in I} L_i,\ L_i \cap L_j = \{0, 1\}$ for $i \neq j,\ i, j \in I$ ($0, 1$ are the zero and unit element in both L_i and L_j) and every L_i is contained in L as a subalgebra.

Two elements a, b of an orthomodular lattice L are called compatible (written $a \leftrightarrow b$) if $a = (a \wedge b) \vee (a \wedge b')$, $b = (a \wedge b) \vee (a' \wedge b)$ (one of the latter equalities is sufficient). The compatibility relation $\leftrightarrow$ is clearly reflexive and symmetric, but not transitive. If we introduced the transitive closure of $\leftrightarrow$, the result would be trivial, because all the elements of L would belong to the same class via the centre $C(L)$ of L. Therefore we suggest to change the compatibility relation as follows: We say that a and b ($a, b \in L$) are c-compatible ($a \leftrightarrow^c b$) if one of (i) and (ii) is satisfied, where

(i) $a \in C(L)$ and $b \in C(L)$,
(ii) $a \notin C(L), b \notin C(L)$ and $a \leftrightarrow b$.

Now we define the transitive closure R of c-compatibility: aRb if there are d_1, $d_2, \dots, d_n \in L$ such that $d_1 = a, d_n = b$ and $d_i \leftrightarrow^c d_{i+1},\ i < n$. Evidently, R is an equivalence relation and the centre $C(L)$ of L is one of the equivalence classes. Let us denote by $\mathcal{T}$ the family of all equivalence classes of R different from $C(L)$,

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i.e. $L = \cup\{T \mid T \in \mathcal{T}\} \cup C(L)$. It is easy to see that if L is a horizontal sum of L_i, $i \in I$, then for every $T \in \mathcal{T}$ there is a (unique) $i \in I$ such that $T \subset L_i$.

In what follows, we shall need the notion of a commutator. We recall that the commutator of a finite subset F of L is defined by

$$\operatorname{com} F = \bigvee_{f\in\{0,1\}^n} \bigwedge_{i\le n} a_i^{f(i)},$$

where $a^{f(i)} = a$ if $f(i) = 1$, and $a^{f(i)} = a'$ if $f(i) = 0$ (see [**1**], [**2**], [**3**], [**4**]). For an arbitrary subset $M \subset L$ we put

$$\operatorname{com} M = \bigwedge\{\operatorname{com} F \mid F \text{ is a finite subset of } M\}$$

if the infimum on the right-hand side exists (see [**6**]).

For a subset M of L, we denote by $C(M)$ the commutant of M, i.e.

$$C(M) = \{b \in L \mid b \leftrightarrow a \text{ for all } a \in M\}.$$

We shall need the following results.

Lemma 1. *Let M be a subset of an orthomodular lattice L such that* $\operatorname{com} M$ *exists in L and $C(C(M)) = L$. Then $c = \operatorname{com} M$ belongs to $C(L)$ and $L = [0,c] \times [0,c']$, where $[0,c]$ is a Boolean algebra and $[0,c']$ has no nontrivial Boolean factor.*

Proof. The proof can be easily obtained from [**6**] by the combination of Corollary 1, Theorem 10 and Corollary 4. □

Lemma 2. *Let $L = B \times L_1$, where B is a Boolean algebra. Let $\mathcal{T}$ be the family of all R-equivalence classes in L different from $C(L)$ and let T_1 be the family of all R-equivalence classes in L_1 different from $C(L_1)$. Then $\mathcal{T}_1 = \{T \wedge (0,1) \mid T \in \mathcal{T}\}$ (here $T \wedge (0,1) = \{t \wedge (0,1) \mid t \in T\}$).*

Proof. It follows by the simple observation that $(a,b)R(c,d)$ in L if and only if bRd in L_1. □

Lemma 3. *Let $\mathcal{T} \cup C(L)$ be the partition of L by the relation R. Then*

(i) *$T \cup C(L)$ is a subalgebra of L for any $T \in \mathcal{T}$,*

(ii) *if for any $a \in T_1$, $b \in T_2$ ($T_1 \neq T_2$) we have $a \wedge b = 0$ (dually, $a \vee b = 1$) then $C(L) = \{0,1\}$. Consequently, L is a horizontal sum of the logics $T \cup C(L)$, $T \in \mathcal{T}$.*

(iii) *$T \cup C(L)$ ($T \in \mathcal{T}$) cannot be expressed in the form of any horizontal sum.*

Proof. (i) Clearly, $0, 1 \in T \cup C(L)$, and $a \in T \cup C(L)$ implies $a' \in T \cup C(L)$. Suppose that $a, b \in T \cup C(L)$. If $a, b \in C(L)$, then $a \wedge b \in C(L)$. Suppose that $a \in T$. If $a \wedge b \notin C(L)$, then $a \leftrightarrow a \wedge b$ implies that $aRa \wedge b$, hence $a \wedge b \in T$.

(ii) Suppose that $a \in T_1$, $b \in T_2$ is where $T_1 \neq T_2$. Let there be $c \in C(L)$, $c \neq 0, c \neq 1$. Then $a \wedge c \in T_1 \cup C(L), b \wedge c \in T_2 \cup C(L)$ by (i). The following situations can occur:

(a) $a \wedge c,\ b \wedge c \in C(L)$. Then $a \wedge c',\ b \wedge c' \notin C(L)$, for otherwise $a, b \in C(L)$. Therefore is $a \wedge c' \in T_1,\ b \wedge c' \in T_2$. But then $(a \wedge c') \vee (b \wedge c') = 1 = (a \vee b) \wedge c' = c'$, a contradiction.
(b) $a \wedge c \in C(L),\ b \wedge c \in T_2$. Then $a \wedge c' \in T_1$, and $(a \wedge c') \vee (b \wedge c) = 1$ implies that $c = c \wedge ((a \wedge c') \vee (b \wedge c)) = b \wedge c, c' \wedge ((a \wedge c') \vee (b \wedge c)) = a \wedge c'$. But then $c \leq b,\ c' \leq a$ imply $a' \leq b$, i.e. $aRa \wedge b$ a contradiction.
(c) $a \wedge c \in T_1,\ b \wedge c \in T_2$. Then $1 = (a \wedge c) \vee (b \wedge c) = (a \vee b) \wedge c = c$, a contradiction.

All the remaining cases are symmetric.

(iii) is clear.

Now we are ready to prove our main result. □

Theorem 1. *Let L be an orthomodular lattice and let card$\mathcal{T} \geq 2$. Let $a \in T_1,\ b \in T_2$, $(T_1, T_2 \in \mathcal{T},\ T_1 \neq T_2)$ and let $c = \operatorname{com}\{a, b\}$. Then $c \in C(L)$ and $L = [0, c] \times [0, c']$, where $[0, c]$ is a Boolean algebra and $[0, c']$ is the horizontal sum of the orthomodular lattices $(T \cup C(L)) \wedge c'$, and none of the latter lattices is a horizontal sum.*

Proof. Let $a \in T_1,\ b \in T_2$ $(T_1 \neq T_2)$. Then $C\{a, b\} = C(L)$, hence $C(C(\{a, b\}) = L$. By Lemma 1, $L = [0, c] \times [0, c']$, where $[0, c]$ is a Boolean algebra and $[0, c']$ has no nontrivial Boolean factor. By Lemma 2, $[0, c'] = \cup\{T \wedge c' \mid T \in \mathcal{T}\} \cup C(L) \wedge c' = \cup\{(T \cup C(L)) \wedge c' \mid T \in \mathcal{T}\}$ and every $(T \cup C(L)) \wedge c'$ is a subalgebra of $[0, c']$ by Lemma 3 (i). If $a \in T_1 \wedge c',\ b \in T_2 \wedge c',\ T_1 \neq T_2$, then $\operatorname{com}_{[0,c']}\{a, b\} = 0$ (where $\operatorname{com}_{[0,c']}$ means the commutator in $[0, c']$), for otherwise there would exist a nontrivial Boolean factor in $[0, c']$. Hence $a \wedge b = 0$ for any $a \in T_1 \wedge c'$ and $b \in T_2 \wedge c'$, $T_1 \neq T_2$. Therefore by Lemma 3 (ii), $C([0, c']) = \{0, c'\}$, and hence $[0, c']$ is the horizontal sum of $(T \cup C(L)) \wedge c', t \in \mathcal{T}$. Finally, by Lemma 3 (iii), none of $(T \cup C(L)) \wedge c'$ is a horizontal sum. □

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