ON EXISTENCE OF POSITIVE SOLUTION FOR INITIAL VALUE PROBLEM OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS OF ORDER $1 < \alpha \leq 2$

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Abstract. The existence of positive solution for a class of nonlinear fractional differential equations are investigated by the method of upper and lower solutions and using Schauder and Banach fixed point theorems.

1. Introduction

The fractional differential equations (FDE) are considered as alternative models to nonlinear differential equations which induced extensive researches in various applicable fields such as physics, mechanics, chemistry, engineering, etc. (see [4], [6], [15]). In recent years, the theory of fractional differential equations has been given a great interest, especially to finding sufficient conditions for existence and uniqueness of the solutions of nonlinear FDE ([7]–[11], [13], and references therein). Many researchers (see [1], [2], [5], [12] and [14]) investigated the positivity of such solutions for FDE. More precisely, D. Delbosco and L. Rodino [3] proved the existence of the solutions to FDE using Banach and Schauder fixed point theorems; Zhang [12] investigated the existence and

Received January 30, 2014.
2010 *Mathematics Subject Classification.* Primary 26A33; Secondary 34A12, 34G20.
Key words and phrases. fractional differential equations; positive solution; upper and lower solutions; existence and uniqueness; Banach and Schauder fixed point theorems.

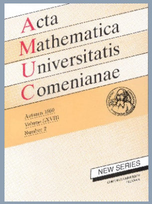
uniqueness of positive solution using the method of the upper and lower solution and cone fixed-point theorem; Lakshmikantham [13] obtained the existence of the local and global solutions using classical differential equation theorem. However, in the previous works, the nonlinear function in the FDE has to satisfy a monotonous characteristic or some control conditions. In fact, the FDEs with nonmonotone function can respond better to impersonal law, so it is very important to weaken monotone condition. Moreover, the cone fixed point theorems are used to get the existence of positive a solution.

Motivated by these works, in this paper, we mainly investigate the existence of solution to FDE of order $1 < \alpha \leq 2$ without any monotonic conditions nor using cone fixed theorem, but by considering the so-called upper and lower control functions. These functions can be used in the technique of upper and lower solutions in connection with Schauder and Banach fixed-point theorems.

2. Preliminaries

Let $X = C(J)$, $J = [0,1]$ be the Banach space of all real-valued continuous functions defined on the compact interval J, endowed with the maximum norm. Define the subspace $A = \{x \in X \,:\, x(t) \geq 0,\ t \in J\}$ of X. By a positive solution $x \in X$, we mean a function $x(t) > 0$, $0 < t \leq 1$ and $x(0) = 0$.

Let $a, b \in \mathbb{R}^+$ such that $b > a$. For any $x \in [a,b]$, we define the upper-control function $U(t,x) = \sup\{f(t,\lambda) \,:\, a \leq \lambda \leq x\}$, and lower-control function $L(t,x) = \inf\{f(t,\lambda) \,:\, x \leq \lambda \leq b\}$. Obviously, $U(t,x)$, and $L(t,x)$ are monotonous non-decreasing on the argument x and $L(t,x) \leq f(t,x) \leq U(t,x)$.



We assume hereafter that $f\colon J \times X \to X$ is a continuous function such that the fractional integral

$$I^{\alpha} f(t, x(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) \mathrm{d}s$$

exists for any order $0 < \alpha \leq 2$. Moreover, the Caputo fractional derivative $D^{\alpha} x = I^{2-\alpha} x^{(2)}$, $x \in X$ exists for any order $1 < \alpha \leq 2$.

Consider the following nonlinear fractional differential equation

$$\begin{cases} D^{\alpha} x(t) = f(t, x(t)), & 0 < t \leq 1, \\ x(0) = 0, & x'(0) = \theta > 0, \end{cases} \tag{1}$$

where $1 < \alpha \leq 2$. Equation (1) is the equivalent to the integral equation (see [7])

$$x(t) = \theta t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) \mathrm{d}s. \tag{2}$$

To transform equation (2) to be applicable to Schauder fixed point, we define an operator $\Phi\colon A \to A$ by

$$(\Phi x)(t) = \theta t + \frac{1}{\Gamma}(\alpha) \int_0^t (t-s)^{\alpha-1} f(s, x(s)) \mathrm{d}s, \qquad t \in J, \tag{3}$$

where the figured fixed point must satisfy the identity operator equation $\Phi x = x$.

The following assumptions are needed for the next results.

H1 Let $x^*(t)$, $x_*(t) \in A$, such that $a \leq x_*(t) \leq x^*(t) \leq b$ and

$$\begin{cases} D^\alpha x^*(t) \geq U(t, x^*(t)), \\ D^\alpha x_*(t) \leq L(t, x_*(t)) \end{cases}$$

for any $t \in J$.

H2 For $t \in J$ and $x, y \in X$, there exists a positive real number $\beta < 1$ such that

$$|f(t,y) - f(t,x)| \leq \beta \|y - x\|.$$

The functions $x^*(t)$ and $x_*(t)$ are respectively called the pair of upper and lower solutions for Equation (1).

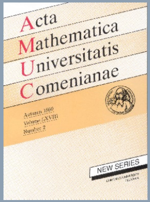
3. Existence of Positive Solution

In this section, we consider the results of existence problem for many cases of the FDE (1). Moreover, we introduce the sufficient conditions of the uniqueness problem of (1).

Theorem 3.1. *Assume that* (**H1**) *is satisfied, then the FDE* (1) *has at least one solution* $x \in X$ *satisfying* $x_*(t) \leq x(t) \leq x^*(t)$, $t \in J$.

Proof. Let $C = \{x \in A : x_*(t) \leq x(t) \leq x^*(t),\ t \in J\}$, endowed with the norm $\|x\| = \max_{t \in J} |x(t)|$, then we have $\|x\| \leq b$. Hence, C is a convex, bounded, and closed subset of the Banach space X. Moreover, the continuity of f implies the continuity of the operator Φ on C defined by (3). Now, if $x \in C$, there exists a positive constant c such that $\max\{f(t, x(t)) : t \in J, x(t) \leq b\} < c$. Then

$$|(\Phi x)(t)| \leq \theta t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s))| \, \mathrm{d}s \leq \theta + \frac{ct^\alpha}{\Gamma(\alpha+1)}.$$

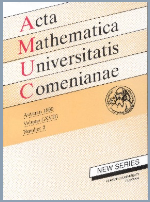


Thus,

$$\|\Phi x\| \leq \theta + \frac{c}{\Gamma(\alpha+1)}.$$

Hence, $\Phi(C)$ is uniformly bounded. Next, we prove the equicontinuity of Φ. Let $x \in C$, $\varepsilon > 0$, $\delta > 0$, and $0 \leq t_1 < t_2 \leq 1$ such that $|t_2 - t_1| < \delta$. If $\delta = \min\left\{1, \frac{\varepsilon\Gamma(\alpha+1)}{2(\theta\Gamma(\alpha+1)+2c)}, \left(\frac{\varepsilon\Gamma(\alpha+1)}{4c}\right)^{\frac{1}{\alpha}}\right\}$, then

$$\begin{aligned}
&|(\Phi x)(t_1) - (\Phi x)(t_2)| \\
&\leq \theta\,(t_2 - t_1) \\
&\quad + \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1 - s)^{\alpha-1} f(s, x(s))\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2 - s)^{\alpha-1} f(s, x(s))\mathrm{d}s\right| \\
&\leq \theta\,(t_2 - t_1) + \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\right) f(s, x(s))\mathrm{d}s\right| \\
&\quad + \left|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2 - s)^{\alpha-1} f(s, x(s))\mathrm{d}s\right| \\
&\leq \theta\,(t_2 - t_1) + \frac{c}{\Gamma(\alpha+1)}\left(t_2^\alpha - t_1^\alpha + 2\,(t_2 - t_1)^\alpha\right) \\
&\leq \left(\theta + \frac{2c}{\Gamma(\alpha+1)}\right)\delta + \frac{2c\delta^\alpha}{\Gamma(\alpha+1)} \\
&< \varepsilon.
\end{aligned}$$



Therefore, $\Phi(C)$ is equicontinuous. The Arzelè-Ascoli Theorem implies that $\Phi\colon A \to A$ is compact. The only thing to apply Schauder fixed point is to prove that $\Phi(C) \subseteq C$. Let $x \in C$, then by hypotheses, we have

$$\begin{aligned}(\Phi x)(t) &= \theta t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\mathrm{d}s\\ &\le \theta t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(s, x(s))\mathrm{d}s\\ &\le \theta t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(s, x^*(s))\mathrm{d}s \le x^*(t),\end{aligned}$$

and

$$\begin{aligned}(\Phi x)(t) &= \theta t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\mathrm{d}s\\ &\ge \theta t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L(s, x(s))\mathrm{d}s\\ &\ge \theta t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L(s, x_*(s))\mathrm{d}s \ge x_*(t).\end{aligned}$$

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Hence, $x_*(t) \leq (\Phi x)(t) \leq x^*(t)$, $t \in J$, that is, $\Phi(C) \subseteq C$. According to Schauder fixed point theorem, the operator Φ has at least one fixed point $x \in C$. Therefore, the FDE (1) has at least one positive solution $x \in X$ and $x_*(t) \leq x(t) \leq x^*(t)$, $t \in J$. □

Next, we consider many particular cases of the previous theorem.

Corollary 3.2. *Assume that there exist continuous functions $k_1(t)$ and $k_2(t)$ such that $0 < k_1(t) \leq f(t, x(t)) \leq k_2(t) < \infty$, $(t, x(t)) \in J \times [0, +\infty)$. Then, the FDE (1) has at least one positive solution $x \in X$. Moreover,*

$$\theta t + I^\alpha k_1(t) \leq x(t) \leq \theta t + I^\alpha k_2(t). \tag{4}$$

Proof. By the given assumption and the definition of control function, we have $k_1(t) \leq L(t,x) \leq U(t,x) \leq k_2(t)$, $(t, x(t)) \in J \times [a, b]$. Now, we consider the equations

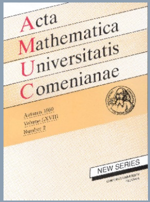
$$\begin{aligned} D^\alpha x(t) &= k_1(t), \quad x(0) = 0, \quad x'(0) = \theta \\ D^\alpha x(t) &= k_2(t), \quad x(0) = 0, \quad x'(0) = \theta. \end{aligned} \tag{5}$$

Obviously, equations (5) are equivalent to

$$\begin{aligned} x(t) &= \theta t + I^\alpha k_1(t), \\ x(t) &= \theta t + I^\alpha k_2(t). \end{aligned}$$

Hence, the first implies $x(t) - \theta t = I^\alpha k_1(t) \leq I^\alpha(L(t, x(t)))$, and the second implies $x(t) - \theta t = I^\alpha k_2(t) \geq I^\alpha(U(t, x(t)))$, which are the upper and lower solutions of Equation (5), respectively. An application of Theorem 3.1 yields that the FDE (1) has at least one solution $x \in X$ and satisfies Equation (4). □

Corollary 3.3. *Assume that $0 < \sigma < k(t) = \lim_{x \to \infty} f(t, x) < \infty$ for $t \in J$. Then the FDE (1) has at least a positive solution $x \in X$.*



Proof. By assumption, if $x > \rho > 0$, then $0 \leq |f(t,x) - k(t)| < \sigma$ for any $t \in J$. Hence, $0 < k(t) - \sigma \leq f(t,x) \leq k(t) + \sigma$ for $t \in J$ and $\rho < x < +\infty$. Now if $\max\{f(t,x) : t \in J, x \leq \rho\} \leq \nu$, then $k(t) - \sigma \leq f(t,x) \leq k(t) + \sigma + \nu$ for $t \in J$, and $0 < x < +\infty$. By Corollary 3.2, the FDE (1) has at least one positive solution $x \in X$ satisfying

$$\theta t + I^{\alpha} k(t) - \frac{\sigma t^{\alpha}}{\Gamma(\alpha+1)} \leq x(t) \leq \theta t + I^{\alpha} k(t) + \frac{(\sigma+\nu) t^{\alpha}}{\Gamma(\alpha+1)}.$$

$\square$

Corollary 3.4. *Assume that $0 < \sigma \leq f(t, x(t)) \leq \gamma x(t) + \eta < \infty$ for $t \in J$, and σ, η and γ are positive constants. Then, the FDE (1) has at least one positive solution $x \in C[0,\delta]$, where $0 < \delta < 1$.*

Proof. Consider the equation

$$\begin{aligned} D^{\alpha} x(t) &= \gamma x(t) + \eta, \qquad & 0 < t \leq 1, \\ x(0) &= 0, & x'(0) = \theta > 0. \end{aligned} \tag{6}$$

Equation (6) is equivalent to integral equation

$$\begin{aligned} x(t) &= \theta t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\gamma x(s) + \eta)\, \mathrm{d}s \\ &= \theta t + \frac{\eta t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\gamma}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) \mathrm{d}s. \end{aligned}$$

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Let ω and ϕ be positive real numbers. Choose an appropriate $\delta \in (0,1)$ such that $0 < \frac{\gamma\delta^\alpha}{\Gamma(\alpha+1)} < \phi < 1$ and $\omega > (1-\phi)^{-1}\left(\theta\delta + \frac{\eta\delta^\alpha}{\Gamma(\alpha+1)}\right)$. Then if $0 \leq t \leq \delta$, the set $B_\omega = \{x \in X : |x(t)| \leq \omega,\ 0 \leq t \leq \delta\}$ is convex, closed, and bounded subset of $C[0,\delta]$. The operator $\digamma : B_\omega \to B_\omega$ given by

$$(\digamma x)(t) = \theta t + \frac{\eta t^\alpha}{\Gamma(\alpha+1)} + \frac{\gamma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)\mathrm{d}s$$

is compact as in the proof of Theorem 3.1. Moreover,

$$|(\digamma x)(t)| \leq \theta t + \frac{\eta t^\alpha}{\Gamma(\alpha+1)} + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}\|x\|.$$

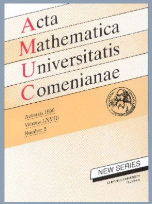
If $x \in B_\omega$, then

$$|(\digamma x)(t)| \leq (1-\phi)\omega + \phi\omega = \omega,$$

that is $\|\digamma x\| \leq \omega$. Hence, the Schauder fixed theorem ensures that the operator $\digamma$ has at least one fixed point in B_ω, and then Equation (6) has at least one positive solution $x^*(t)$, where $0 < t < \delta$. Therefore, if $t \in J$ one can asserts that

$$x^*(t) = \theta t + \frac{\eta t^\alpha}{\Gamma(\alpha+1)} + \frac{\gamma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x^*(s)\mathrm{d}s.$$

The definition of control function implies $U(t, x^*(t)) \leq \gamma x^*(t) + \eta = D^\alpha x^*(t)$, then x^* is an upper positive solution of the FDE (1). Moreover, one can consider $x_*(t) = \theta t + \frac{\sigma t^\alpha}{\Gamma(\alpha+1)}$ as a lower positive solution of Equation (1). By Theorem 3.1, the FDE (1) has at least one positive solution $x \in C[0,\delta]$, where $0 < \delta < 1$ and $x_*(t) \leq x(t) \leq x^*(t)$. $\square$



The last result is the uniqueness of the positive solution of (1) using Banach contraction principle.

Theorem 3.5. *Assume that* ($\mathbf{H}1$) *and* ($\mathbf{H}2$) *are satisfied. Then the FDE* (1) *has a unique positive solution* $x \in X$.

Proof. From Theorem 3.1, it follows that the FDE (1) has at least one positive solution in C. Hence, we need only to prove that the operator Φ defined in (3) is a contraction on X. In fact, for any $x, y \in X$, we have

$$\begin{aligned} |(\Phi x)(t) - (\Phi y)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(s, x(s)) - f(s, y(s)) \right| \mathrm{d}s \\ &\leq \frac{\beta t^{\alpha}}{\Gamma(\alpha+1)} \left\| x - y \right\|. \end{aligned}$$

If $1 < \alpha \leq 2$, then $1 < \Gamma(\alpha+1) \leq 2$ implies $\frac{\beta t^{\alpha}}{\Gamma(\alpha+1)} < 1$. Hence, the operator Φ is a contraction mapping. Therefore, the FDE (1) has a unique positive solution $x \in X$. $\square$

Finally, we give an example to illustrate our results.

Example 3.6. *We consider the fractional equation*

$$\begin{cases} D^{\frac{3}{2}} x(t) = 1 + \frac{t\,\mathrm{e}^{-tx(t)}}{1+\cos t}, & 0 < t \leq 1 \\ x(0) = 0, & x'(0) = \theta > 0, \end{cases} \tag{7}$$

where $f(t,x) = 1 + \frac{t\,\mathrm{e}^{-tx}}{1+\cos t}$. *Since* $\lim_{x\to\infty}(1 + \frac{t\,\mathrm{e}^{-tx}}{1+\cos t}) = 1$ *and* $1 \leq 1 + \frac{1}{2} t\,\mathrm{e}^{-tx} \leq f(t,x) \leq 1 + t\,\mathrm{e}^{-tx} \leq 1 + t \leq 2$ *for* $(t,x) \in [0,1] \times [0,+\infty)$, *hence by any of the above Corollaries, the* *equation* (7) *has a positive solution. We lost the uniqueness property of the existed solution due to the contraction principle is not applicable on the function* $f(t, x)$.

1. Wang C., Zhang H., Wang S. *Positive solution of a nonlinear fractional differential equation involving Caputo derivative*, Discrete Dynamics in Natural and Society (2012), Art ID425408.

2. Wang C., Wang R., Wang S., Yang C., *Positive Solution of Singular Boundary Value Problem for a Nonlinear Fractional Differential Equation*, Bound. Value Probl. (2011), Art ID 297026.

3. Delbosco D., Rodino L., *Existence and uniqueness for a nonlinear fractional differential equation*, J. Math. Anal. Appl. **204** (1996), 609–625.

4. Buckwar E., Luchko Y., *Invariance of a partial differential equation of fractional order under lie group of scaling trabsformations*, J. Math. Anal. Appl. **227** (1998), 81–97.

5. Kaufmann E., Mboumi E., *Positive solutions of a boundary value problem for a nonlinear fractional differential equation*, Electron. J. Qual. Theory Differ. Equ. **3** (2008), 1–11.

6. Mainardi F., *The fundamental solutions for the fractional diffusion-wave equation*, Appl. Math. Letters **9** (1996), 23–28.

7. Miller K. S., Ross B., *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, 1993.

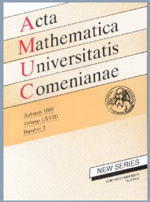
8. Matar M., *Existence and uniqueness of solutions to fractional semilinear mixed Volterra-Fredholm integrodifferential equations with nonlocal conditions*, Electronic Journal of Differential Equations, 155 (2009), pp.1-7.

9. Matar M., *Boundary value problem for fractional integro-differential equations with nonlocal conditions*, Int. J. Open Problems Compt. Math. **3** (2010), 481–489.

10. ______, *On existence and uniqueness of the mild solution for fractional semilinear integro-differential equations*, J. Integral Equations Appl., **23** (2011), 457–466.

11. ______, *On existence of solution to nonlinear fractional differential equations for* $0 < \alpha \leq 3$, Journl of Fractional Calculus and Applications, **3** (2012), 1–7.

12. Zhang S., *The existence of a positive solution for a fractional differential equation*, J. Math. Anal. Appl. **252** (2000), 804–812.



13. Ladshmikantham V., Vatsals A. S., *Basic theory of fractional dfffferential equations*, Nonlinear Anal. **60** (2008), 2677–2682.

14. Bai Z. B., Qiu T. T., *Existence of positive solution for singular fractional differential equation*, Appl. Math. Comput. **215** (2009), 2761–2767.

15. Zhu Z., Li G., Cheng C., *Quasi-static and dynamical analysis for viscoelastic Timoshenko beam with fractional derivative constitutive relation*, Appl. Math. Mech. **23** (2002), 1–15.

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