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A MONOTONE LINEAR APPROXIMATION OF A NONLINEAR ELLIPTIC PROBLEM WITH A NON-STANDARD BOUNDARY CONDITION*

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Abstract. We consider a 2nd order nonlinear elliptic boundary value problem (BVP) in a bounded domain $\Omega \subset \mathbb{R}^N$, with nonlocal boundary condition. More precisely, at some boundary part Γ_n, we impose a Dirichlet BC containing an unknown additive constant, accompanied of a nonlocal (integral) Neumann side condition. The rest of the boundary is equipped with Dirichlet or nonlinear Robin type BC. The problem is solved in the variational framework by linearization. The solution of the linearized problem converges to the exact weak solution in the $H^1(\Omega)$-norm.

Key words. nonlinear elliptic BVP, nonstandard boundary condition, linearization

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1. Introduction. Nonstandard boundary conditions (BCs) reflect sometimes the reality better than the standard ones do. To explain this, we start with a simple example. Consider the movement of gas through a porous medium. Boundary conditions reflect the behavior of a solution u (the squarred pressure) or the flux $\boldsymbol{q}$ at the boundary. Gerke at. al. [2, §4.1] considered a nonlocal BC, where a Dirichlet BC contains an unknown additive real constant and is accompanied of an integral Neumann side condition describing the total flux through the boundary, i.e.,

$$u = c \in \mathbb{R} \text{ (unknown) on } \Gamma, \qquad \int_\Gamma \boldsymbol{q} \cdot \boldsymbol{\nu} = s \text{ (given)}.$$

This means, that the gas pressure along Γ is supposed to be constant but unknown, whereas the discharge (total flux through Γ) is prescribed. Let us note, that the normal component of the flux cannot be measured point-wise.

Clearly, such a type of BC can be involved in various kinds of problem settings. Slodička and De Schepper [5] studied the following nonlinear elliptic BVP

$$\text{(1.1)} \quad \text{Find } u \in C^2(\overline{\Omega}) : \quad \begin{cases} -\Delta u + g(u) & = f \quad \text{in} \quad \Omega \\ u & = g_{Dir} \quad \text{on} \quad \Gamma_D \\ u & = g_n + const \quad \text{on} \quad \Gamma_n \\ G(u) \equiv \displaystyle\int_{\Gamma_n} -\nabla u \cdot \boldsymbol{\nu} & = s \in \mathbb{R} \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with sufficiently smooth boundary consisting of two complementary parts Γ_D and Γ_n, such that $\overline{\Gamma_D} \cap \overline{\Gamma_n} = \emptyset$. A typical example of such Ω is a domain with a hole in it. The function $g \in C^1(\mathbb{R})$ is supposed to be monotonically increasing, Lipschitz continuous and its graph should vary within two parallel increasing lines.

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The authors have shown the existence and uniqueness of a solution. The proofs are based on the continuous dependence of the total flux function $G(u_\alpha)$ on the real parameter α, where u_α solves the following auxiliary nonlinear BVP

$$\text{Find } u_\alpha \in C^2(\overline{\Omega}) : \quad \begin{cases} -\Delta u_\alpha + g(u_\alpha) & = f & \text{in} & \Omega \\ u_\alpha & = g_{Dir} & \text{on} & \Gamma_D \\ u_\alpha & = g_n + \alpha & \text{on} & \Gamma_n. \end{cases} \tag{1.2}$$

The proof relies on the comparison principle – see Gilbarg-Trudinger [3, Theorem 9.2], therefore the authors have dealt with classical solutions. The continuous dependence of $G(u_\alpha)$ on α implies, that the solution of (1.1) can be obtained in an iterative way by solving a sequence of nonlinear problems of the type (1.2).

Slodička and Van Keer [6] extended the problem setting (1.1) by a linear Robin type BC. Moreover, the authors have used the variational framework in the proofs, so they have weakened the assumptions on the data, e.g, on the nonlinear function g. More exactly, the assumption of the continuity of the first derivative of g has been removed.

The purpose of this paper is to involve two new aspects into the problem setting (1.1)

- by adding a linear convection term $\boldsymbol{a}_{con}$
- by considering a nonlinear Robin type BC.

We find a solution of this more general nonlinear problem using a special linearization method, cf. in [6]. Here, we start from a super- or sub-solution and in an iterative way we approach the weak solution of the nonlinear BVP. The proof of convergence is based on the monotonicity of the approximations. The approximate solution converges in the $H^1(\Omega)$-norm to the exact one.

In the last section we present a numerical experiment in order to show the efficiency of the proposed linearization method.

2. Assumptions and variational formulation. In this paper we study a nonlinear BVP in divergence form

$$\begin{cases} \nabla \cdot (-\boldsymbol{A}_{dif} \nabla u - \boldsymbol{a}_{con} u) + g(u) & = f & \text{in} & \Omega \\ u & = g_{Dir} & \text{on} & \Gamma_D \\ (-\boldsymbol{A}_{dif} \nabla u - \boldsymbol{a}_{con} u) \cdot \boldsymbol{\nu} - g_{Rob}(u) & = g_{Neu} & \text{on} & \Gamma_N \\ u & = g_n + const & \text{on} & \Gamma_n \\ G(u) \equiv \displaystyle\int_{\Gamma_n} (-\boldsymbol{A}_{dif} \nabla u - \boldsymbol{a}_{con} u) \cdot \boldsymbol{\nu} & = s \in \mathbb{R} & & \end{cases} \tag{2.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$. The total boundary Γ is supposed to be Lipschitz continuous and it is splited into three parts Γ_D, Γ_N and Γ_n, corresponding to a Dirichlet, Neumann and nonlocal part, respectively. Throughout the whole paper we assume

$$|\Gamma_D| > 0, \qquad \overline{\Gamma_n} \cap \overline{\Gamma_D} = \emptyset, \qquad |\Gamma_n| > 0. \tag{2.2}$$

The last inequality means that we are dealing with a nonlocal BC on Γ_n, otherwise the problem is standard. The matrix $\boldsymbol{A}_{dif}$ fulfills the inequality

$$C_0 \, |w|^2_{1,\Omega} \leq (\boldsymbol{A}_{dif} \nabla w, \nabla w)_\Omega \leq C \, |w|^2_{1,\Omega} \,, \qquad \forall w \in H^1(\Omega) \tag{2.3}$$

for some positive constants C_0 and C. Here, $(w,z)_M$ stands for the usual L_2-inner product of any real or vector-valued functions w, z on a set M, i.e., $(w,z)_M = \displaystyle\int_M wz$.

The fact that $|\Gamma_D| > 0$ implies that the seminorm $|\cdot|_{1,\Omega}$ represents a norm in $H^1(\Omega)$ equivalent to the usual norm $\|\cdot\|_{1,\Omega}$.

We assume the following relations for the convection term $\boldsymbol{a}_{con}$

$$\begin{array}{rcll} |\boldsymbol{a}_{con}| & \leq & C & \text{a.e. in } \Omega \\ \boldsymbol{a}_{con} \cdot \boldsymbol{\nu} & \geq & 0 & \text{a.e. on } \Gamma_N \cup \Gamma_n \\ \nabla \cdot \boldsymbol{a}_{con} & = & 0 & \text{a.e. in } \Omega. \end{array} \tag{2.4}$$

The functions g and g_{Rob} are globally Lipschitz continuous and obey the following growth conditions

$$\begin{array}{lrcccll} & |g(x) - g(y)| & \leq & L|x-y|, & & & \forall x, y \in \mathbb{R}, \\ & |g_{Rob}(x) - g_{Rob}(y)| & \leq & L|x-y|, & & & \forall x, y \in \mathbb{R}, \\ \exists k_1, q_1 \in \mathbb{R}_+ : & k_1 s - q_1 & \leq & g(s) & \leq & k_1 s + q_1, & \forall s \in \mathbb{R}, \\ \exists k_2, q_2 \in \mathbb{R}_+ : & k_2 s - q_2 & \leq & g_{Rob}(s) & \leq & k_2 s + q_2, & \forall s \in \mathbb{R}. \end{array} \tag{2.5}$$

The boundary data f, g_{Neu}, g_{Dir} and g_n fulfill the assumptions

$$f \in L_2(\Omega), \qquad g_{Neu} \in L_2(\Gamma_N) \tag{2.6}$$

and there exists a function $\tilde{g} \in H^1(\Omega)$ such that

$$\tilde{g} = \begin{cases} g_n & \text{on} \quad \Gamma_n \\ g_{Dir} & \text{on} \quad \Gamma_D. \end{cases} \tag{2.7}$$

When dealing with such a general setting as (2.1), then one cannot expect that the solution (if exists) will be classical. The lack of regularity can be caused by properties of the data entering (2.1), even in the case when $\Gamma_n = \emptyset$. Therefore, we pass to a variational framework. We give a reasonable definition of a weak solution in $H^1(\Omega)$. Moreover, we obtain this solution by a suitable linearization, i.e., we define a sequence of linear elliptic BVP which gives rise to a weak solution of (2.1).

Let us introduce the following subspace V of $H^1(\Omega)$

$$V = \{\varphi \in H^1(\Omega);\ \varphi = 0 \text{ on } \Gamma_D,\ \varphi = const \text{ on } \Gamma_n\}, \tag{2.8}$$

which is clearly a Hilbert space with the induced innerproduct and norm of $H^1(\Omega)$. Now, we define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(u, \varphi) = (\boldsymbol{A}_{dif} \nabla u + \boldsymbol{a}_{con} u, \nabla \varphi)_\Omega \qquad \forall u, \varphi \in H^1(\Omega)$$

and the linear functional $F : V \to \mathbb{R}$ by

$$\langle F, \varphi \rangle = (f, \varphi)_\Omega - (g_{Neu}, \varphi)_{\Gamma_N} - s\varphi|_{\Gamma_n} \qquad \forall \varphi \in V. \tag{2.9}$$

The following simple inequality together with (2.6) yield the continuity of the functional F

$$\left| s\varphi_{|\Gamma_n} \right| = \frac{|s|}{|\Gamma_n|} \int_{\Gamma_n} |\varphi| \leq C \|\varphi\|_{0,\Gamma_n} \leq C \|\varphi\|_{1,\Omega}. \tag{2.10}$$

The appropriate variational formulation of (2.1) reads as:
Find $u \in H^1(\Omega)$ such that $u - \tilde{g} \in V$ and

$$a(u, \varphi) + (g(u), \varphi)_\Omega + (g_{Rob}(u), \varphi)_{\Gamma_N} = \langle F, \varphi \rangle \qquad \forall \varphi \in V. \tag{2.11}$$

3. Linearization. Now, we are going to prove the existence of a weak solution to the BVP (2.11). To do this we use a linearization of the nonlinear problem exploiting ordering properties for the solutions. We define recurrent sequences starting from a *sub-solution* and a *super-solution*, respectively. In fact there will exist a solution to the BVP (2.11) lying between the sub- and super-solution.

We denote by v_0 the weak solution of the following linear BVP:
Find $v_0 \in H^1(\Omega)$ such that $v_0 - \tilde{g} \in V$ and

$$a(v_0, \varphi) + (k_1 v_0, \varphi)_\Omega + (k_2 v_0, \varphi)_{\Gamma_N} = \langle F, \varphi \rangle - (q_1, \varphi)_\Omega - (q_2, \varphi)_{\Gamma_N} \qquad \forall \varphi \in V. \tag{3.1}$$

Further, v_j for $j = 1, 2, \ldots$ are defined recursively as:
Find $v_j \in H^1(\Omega)$ such that $v_j - \tilde{g} \in V$ and

$$\begin{aligned} a(v_j, \varphi) + (Lv_j, \varphi)_\Omega + (Lv_j, \varphi)_{\Gamma_N} &= \langle F, \varphi \rangle + (Lv_{j-1}, \varphi)_\Omega - (g(v_{j-1}), \varphi)_\Omega \\ &\quad + (Lv_{j-1}, \varphi)_{\Gamma_N} - (g_{Rob}(v_{j-1}), \varphi)_{\Gamma_N} \qquad \forall \varphi \in V. \end{aligned} \tag{3.2}$$

We recall that the sequence $\{v_j\}_{j=0}^\infty$ is defined by means of linear BVPs. Our first step is to prove the existence of this sequence.

LEMMA 3.1. *Let the assumptions (2.2)-(2.7) be satisfied. Then the sequence $\{v_j\}_{j=0}^\infty \subset H^1(\Omega)$ is well defined.*

Proof. Let w be any function from V. The relation (2.4) together with the Friedrich's inequality and Green's theorem imply

$$\begin{aligned} C\,|w|_{1,\Omega}^2 \;\ge\; (\boldsymbol{a}_{con} w, \nabla w)_\Omega &= \frac{1}{2}\left(\boldsymbol{a}_{con}, \nabla w^2\right)_\Omega \\ &= -\frac{1}{2}\left(\nabla \cdot \boldsymbol{a}_{con}, w^2\right)_\Omega + \frac{1}{2}\left(\boldsymbol{a}_{con}\boldsymbol{\nu}, w^2\right)_\Gamma \\ &= \frac{1}{2}\left(\boldsymbol{a}_{con}\boldsymbol{\nu}, w^2\right)_{\Gamma_N \cup \Gamma_n} \\ &\ge 0. \end{aligned} \tag{3.3}$$

Hence, in view of (2.3) we have

$$C\,|w|_{1,\Omega}^2 \ge a(w, w) \ge C_0\,|w|_{1,\Omega}^2 \qquad \forall w \in V. \tag{3.4}$$

Therefore the left-hand sides of (3.1) and (3.2) are V-elliptic continuous bilinear forms. According to (2.6) and (2.10), the right-hand side of (3.1) is a bounded linear functional on V. Thus, there exists a unique solution $v_0 \in H^1(\Omega)$ of (3.1). If v_{j-1} belongs to $H^1(\Omega)$, then the right-hand side of (3.2) is also a bounded linear functional on V. Hence, there exists a unique $v_j \in H^1(\Omega)$ satisfying (3.2). $\square$

We intend to let $j \to \infty$ in (3.2). As usual for this purpose we need some uniform estimates for v_j. First, we prove the monotonicity of the sequence $\{v_j\}_{j=0}^\infty$.

LEMMA 3.2. *Let the assumptions of Lemma 3.1 be fulfilled. Then $v_{i-1}(\boldsymbol{x}) \le v_i(\boldsymbol{x})$ for all $i = 1, 2, \ldots$ a.e. in Ω and a.e. on Γ_N.*

Proof. We use mathematical induction with respect to i. Let $i = 1$. We subtract (3.2) for $j = 1$ from (3.1) and get

$$\begin{aligned} a(v_0 - v_1, \varphi) + (L(v_0 - v_1), \varphi)_\Omega + (L(v_0 - v_1), \varphi)_{\Gamma_N} &= (g(v_0) - k_1 v_0 - q_1, \varphi)_\Omega \\ &\quad + (g_{Rob}(v_0) - k_2 v_0 - q_2, \varphi)_{\Gamma_N}. \end{aligned}$$

Now, we choose $\varphi = (v_0 - v_1)^+ \in V$, where f^+ stands for the usual cut-off function

$$f^+(s) = \begin{cases} f(s) & \text{if } f(s) \ge 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Invoking the growth conditions for g and g_{Rob} (see (2.5)) we have

$$\begin{aligned} a\left(v_0 - v_1, (v_0 - v_1)^+\right) \quad &+ \left(L(v_0 - v_1), (v_0 - v_1)^+\right)_\Omega + \left(L(v_0 - v_1), (v_0 - v_1)^+\right)_{\Gamma_N} \\ &= \left(g(v_0) - k_1 v_0 - q_1, (v_0 - v_1)^+\right)_\Omega \\ &+ \left(g_{Rob}(v_0) - k_2 v_0 - q_2, (v_0 - v_1)^+\right)_{\Gamma_N} \\ &\leq 0. \end{aligned}$$

This and the ellipticity of the bilinear form a, see (3.4), imply

$$\int_{\substack{\Omega \\ v_0 \geq v_1}} \left\{ C_0 \left|\nabla(v_0 - v_1)\right|^2 + L(v_0 - v_1)^2 \right\} + \int_{\substack{\Gamma_N \\ v_0 \geq v_1}} L(v_0 - v_1)^2 \leq 0.$$

From this we conclude that $v_0 \leq v_1$ a.e. in Ω and a.e. on Γ_N.

Let us suppose that $v_{i-1} \leq v_i$ a.e. in Ω. We subtract (3.2) for $j = i+1$ from the same identity for $j = i$ and obtain

$$\begin{aligned} a\left(v_i - v_{i+1}, \varphi\right) \quad &+ \left(L(v_i - v_{i+1}), \varphi\right)_\Omega + \left(L(v_i - v_{i+1}), \varphi\right)_{\Gamma_N} \\ &= \left([g(v_i) - Lv_i] - [g(v_{i-1}) - Lv_{i-1}], \varphi\right)_\Omega \\ &+ \left([g_{Rob}(v_i) - Lv_i] - [g_{Rob}(v_{i-1}) - Lv_{i-1}], \varphi\right)_{\Gamma_N}. \end{aligned}$$

The function g is globally Lipschitz continuous with the Lipschitz constant L. Therefore, the function $h(s) := g(s) - Ls$ is monotonically decreasing because of

$$h'(s) = g'(s) - L \leq 0.$$

The same argumentation can be applied to the function $\tilde{h}(s) := g_{Rob}(s) - Ls$ which is also monotonically decreasing. Now, we put $\varphi = (v_i - v_{i+1})^+ \in V$ and deduce

$$\begin{aligned} a\left(v_i - v_{i+1}, (v_i - v_{i+1})^+\right) \quad &+ \left(L(v_i - v_{i+1}), (v_i - v_{i+1})^+\right)_\Omega + \left(L(v_i - v_{i+1}), (v_i - v_{i+1})^+\right)_{\Gamma_N} \\ &= \left(\underbrace{[g(v_i) - Lv_i] - [g(v_{i-1}) - Lv_{i-1}]}_{\leq 0}, (v_i - v_{i+1})^+ \right)_\Omega \\ &+ \left(\underbrace{[g_{Rob}(v_i) - Lv_i] - [g_{Rob}(v_{i-1}) - Lv_{i-1}]}_{\leq 0}, (v_i - v_{i+1})^+ \right)_{\Gamma_N} \\ &\leq 0. \end{aligned}$$

The ellipticity of the bilinear form a (cf. (3.4)) yields

$$\int_{\substack{\Omega \\ v_i \geq v_{i+1}}} \left\{ C_0 \left|\nabla(v_i - v_{i+1})\right|^2 + L(v_i - v_{i+1})^2 \right\} + \int_{\substack{\Gamma_N \\ v_i \geq v_{i+1}}} L(v_i - v_{i+1})^2 \leq 0,$$

which gives $v_i \leq v_{i+1}$ a.e. in Ω and a.e. on Γ_N. $\square$

Further, we inductively define an another sequence $\{z_j\}_{j=0}^{\infty}$. First, z_0 is the solution to the following linear BVP:
Find $z_0 \in H^1(\Omega)$ such that $z_0 - \tilde{g} \in V$ and

$$(3.5)\quad a(z_0, \varphi) + (k_1 z_0, \varphi)_\Omega + (k_2 z_0, \varphi)_{\Gamma_N} = \langle F, \varphi \rangle + (q_1, \varphi)_\Omega + (q_2, \varphi)_{\Gamma_N} \qquad \forall \varphi \in V.$$

Next, for a given z_{j-1} $(j = 1, 2, \ldots)$ subsequently define z_j as:
Find $z_j \in H^1(\Omega)$ such that $z_j - \tilde{g} \in V$ and

$$
\begin{aligned}
a(z_j,\varphi) + (Lz_j,\varphi)_\Omega + (Lz_j,\varphi)_{\Gamma_N} &= \langle F,\varphi\rangle \\
&+ (Lz_{j-1},\varphi)_\Omega - (g(z_{j-1}),\varphi)_\Omega \\
&+ (Lz_{j-1},\varphi)_{\Gamma_N} - (g_{Rob}(z_{j-1}),\varphi)_{\Gamma_N} \qquad \forall\varphi \in V.
\end{aligned}
\tag{3.6}
$$

Analogously as before we can prove the existence, uniqueness and the monotonicity of $\{z_j\}_{j=0}^\infty$. We omit the proof for short.

LEMMA 3.3. *Let the assumptions of Lemma 3.1 be satisfied. Then the sequence $\{z_j\}_{j=0}^\infty \subset H^1(\Omega)$ is well defined and $z_{i-1}(\boldsymbol{x}) \geq z_i(\boldsymbol{x})$ for all $i = 1, 2, \ldots$ a.e. in Ω and a.e. on Γ_N.*

The functions v_j and z_j for $j = 0, 1, \ldots$ can also be ordered. The arguments for this rely on the same procedure as in Lemma 3.2, therefore we state the following lemma without proof.

LEMMA 3.4. *Let the assumptions of Lemma 3.1 be fulfilled. Then $z_i(\boldsymbol{x}) \geq v_i(\boldsymbol{x})$ for all $i = 0, 1, 2, \ldots$ a.e. in Ω and a.e. in Γ_N.*

3.1. Energy estimates. The previous section was devoted to the study of the sequences $\{v_j\}_{j=0}^\infty$ and $\{z_j\}_{j=0}^\infty$. The results can be summarized as follows

$$
v_0 \leq v_1 \leq v_2 \leq \ldots \leq z_2 \leq z_1 \leq z_0, \qquad \text{a.e. in } \Omega \text{ and a.e. in } \Gamma_N. \tag{3.7}
$$

According to this we are able to prove the uniform stability of $\{v_j\}_{j=0}^\infty$ and $\{z_j\}_{j=0}^\infty$ in the space $H^1(\Omega)$.

LEMMA 3.5. *Let the assumptions of Lemma 3.1 be fulfilled. Then, there exists a positive constant C such that*

$$
\|v_j\|_{1,\Omega} + \|z_j\|_{1,\Omega} \leq C
$$

for all $j = 0, 1, \ldots$.

Proof. We start from the relation (3.1). Choose $\varphi = v_0 - \tilde{g} \in V$ into (3.1) and get

$$
\begin{aligned}
a(v_0,v_0) + (k_1 v_0,v_0)_\Omega + (k_2 v_0,v_0)_{\Gamma_N} &= \langle F, v_0 - \tilde{g}\rangle + a(v_0,\tilde{g}) \\
&- (q_1, v_0 - \tilde{g})_\Omega - (q_2, v_0 - \tilde{g})_{\Gamma_N} \\
&+ (k_1 v_0,\tilde{g})_\Omega + (k_2 v_0,\tilde{g})_{\Gamma_N} .
\end{aligned}
\tag{3.8}
$$

The terms on the right-hand side can be estimated as follows. For the first one we deduce

$$
\begin{aligned}
|\langle F, v_0 - \tilde{g}\rangle| &\leq |(f,v_0)_\Omega| + |(f,\tilde{g})_\Omega| + \left|(g_{Neu},v_0)_{\Gamma_N}\right| \\
&+ \left|(g_{Neu},\tilde{g})_{\Gamma_N}\right| + \frac{1}{|\Gamma_n|}\left(\left|(s,v_0)_{\Gamma_N}\right| + \left|(s,\tilde{g})_{\Gamma_N}\right|\right).
\end{aligned}
$$

Let ε be any positive real number. Applying successively the well known

- Cauchy Schwarz inequality
- Young inequality: $|ab| \leq \varepsilon a^2 + C_\varepsilon b^2$, with $C_\varepsilon = C\left(\frac{1}{\varepsilon}\right)$
- trace inequality $\|u\|_{0,\Gamma} \leq C\,\|u\|_{1,\Omega}$

we readily obtain

$$
|\langle F, v_0 - \tilde{g}\rangle| \leq \varepsilon\,\|v_0\|_{1,\Omega}^2 + C_\varepsilon .
$$

The other terms in the right-hand side of (3.8) can be estimated analogously with the same upper bound $\varepsilon \|v_0\|^2_{1,\Omega} + C_\varepsilon$. In view of the ellipticity of the bilinear form a (see (3.4)), the left-hand side of (3.8) can be estimated from below by

$$\tilde{C} \|v_0\|^2_{1,\Omega}, \qquad \tilde{C} > 0.$$

Now, choosing a sufficiently small positive ε we conclude

$$\|v_0\|_{1,\Omega} \leq C.$$

Exactly in the same way we get

$$\|z_0\|_{1,\Omega} \leq C.$$

According to the relation (3.7) we immediately get

$$\|v_j\|_{0,\Omega} + \|z_j\|_{0,\Omega} \leq C \qquad j = 0, 1, \ldots .$$

The rest of the proof can be easily obtained from the recursion formulas (3.2) and (3.6). □

3.2. Existence and uniqueness. We now let tent j to infinity. We show that a subsequence of the solution u_j (resp. z_j) of the linearized problem (3.2) (resp. (3.6)) converges to a weak solution of (2.11). To this end, we apply more or less standard results from functional analysis to build up a weak solution.

THEOREM 3.6 (existence of weak solution). *Let the assumptions of Lemma 3.1 be fulfilled. Then, there exists a weak solution u of (2.11). Moreover,*

$$\begin{array}{ll} z_j, v_j \rightharpoonup u & \text{in } H^1(\Omega) \\ z_j, v_j \to u & \text{in } L_2(\Gamma) \\ z_j, v_j \to u & \text{in } L_2(\Omega) \\ z_j, v_j \to u & \text{a.e. in } \Omega \end{array}$$

hold in the sense of subsequences.

Proof. According to the energy estimates from Lemma 3.5, there exists a subsequence of $\{v_j\}_{j=0}^{\infty}$ (denoted by the same symbol again), such that

$$\begin{array}{ll} v_j \rightharpoonup u & \text{in } H^1(\Omega) \\ v_j \to u & \text{in } L_2(\Omega) \\ v_j \to u & \text{a.e. in } \Omega. \end{array} \tag{3.9}$$

To prove the strong convergence (for a subsequence) $v_j \to u$ in $L_2(\Gamma)$ we apply the well known inequality (see Nečas [4])

$$\|w\|^2_{0,\Gamma} \leq \varepsilon \|w\|^2_{1,\Omega} + C_\varepsilon \|w\|^2_{0,\Omega}, \qquad C_\varepsilon = C\left(\frac{1}{\varepsilon}\right),$$

for small positive ε. Therefore, for small but fixed ε we deduce that

$$\begin{aligned} \|v_j - u\|^2_{0,\Gamma} &\leq \varepsilon \|v_j - u\|^2_{1,\Omega} + C_\varepsilon \|v_j - u\|^2_{0,\Omega} \\ &\leq C\varepsilon + C_\varepsilon \|v_j - u\|^2_{0,\Omega}. \end{aligned}$$

Thus

$$\lim_{j\to\infty} \|v_j - u\|^2_{0,\Gamma} \le C\varepsilon.$$

This holds for arbitrary small positive ε, therefore

$$\lim_{j\to\infty} \|v_j - u\|^2_{0,\Gamma} = 0. \tag{3.10}$$

Now, we would like passing to the limit in (3.2) (for a subsequence) as $j \to \infty$. On account of (3.9), (3.10) and the assumptions on the data functions, we arrive at

$$a(u,\varphi)+(Lu,\varphi)_\Omega+(Lu,\varphi)_{\Gamma_N} = (Lu - g(u),\varphi)_\Omega+(Lu - g_{Rob}(u),\varphi)_{\Gamma_N}+\langle F,\varphi\rangle, \qquad \forall \varphi \in V.$$

Canceling the terms involving L, we get

$$a(u,\varphi) + (g(u),\varphi)_\Omega + (g_{Rob}(u),\varphi)_{\Gamma_N} = \langle F,\varphi\rangle, \qquad \forall \varphi \in V$$

which means, that u is the weak solution of (2.11).

Analogous considerations can be made for the sequence $\{z_j\}_{j=0}^\infty$. □

To prove the uniqueness of the solution to the BVP (2.11) we need the following *strict monotonicity* assumption

$$\begin{aligned} \theta |v-w|^2_{1,\Omega} &\le a(v-w,v-w) + (g(v)-g(w),v-w)_\Omega \\ &+ (g_{Rob}(v)-g_{Rob}(w),v-w)_{\Gamma_N}, \qquad \forall v,w \in V, \end{aligned} \tag{3.11}$$

where θ is some positive constant.

THEOREM 3.7 (uniqueness of weak solution). *Let the assumptions of Lemma 3.1 be fulfilled. Moreover, assume that the strict monotonicity property (3.11) holds. Then there exists at most one weak solution of (2.11).*

Proof. The assertion immediately follows from (3.11). □

Note that the just proved uniqueness of a weak solution to the BVP (2.11) implies that the convergence statement valid for subsequences of $\{v_j\}_{j=0}^\infty$ and $\{z_j\}_{j=0}^\infty$ in Theorem 3.6 holds for the whole sequences.

4. Numerical experiment. Let Ω be the unit square in $\mathbb{R}^2$. Its boundary is splited into three parts: Γ_D (right), Γ_N (top and bottom) and Γ_n (left part of Γ), see Fig. 4.1.

For simplicity we consider the same nonlinear function in the domain and on Γ_n, i.e., $g \equiv g_{Rob}$, and this is defined by

$$g(s) = \begin{cases} s & \text{for } s \ge 4 \\ (s-2)\,|s-2| & \text{for } s \in [0,4] \\ s-4 & \text{for } s \le 0. \end{cases}$$

One can easily verify the inequality

$$s - 4 \le g(s) \le s + 4 \qquad \forall s \in \mathbb{R}.$$

Hence, the relation (2.5) is satisfied.

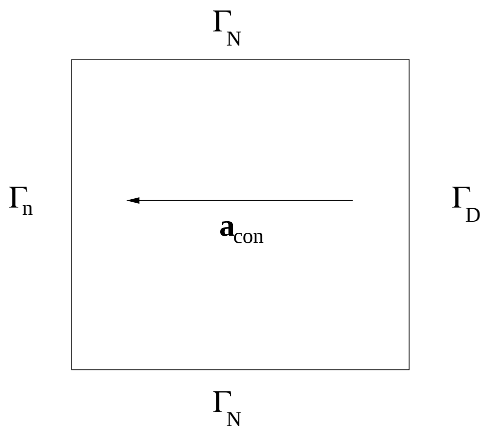


FIG. 4.1. *Domain Ω with the convection $\boldsymbol{a}_{con}$*

We have chosen the convection term $\boldsymbol{a}_{con} = (-1, 0)$, which clearly fulfills the assumption (2.4). We consider the following nonlinear elliptic BVP:
Find $(u, \alpha) \in (H^1(\Omega), \mathbb{R})$ such that

$$\begin{cases} \nabla \cdot (-\nabla u - \boldsymbol{a}_{con} u) + g(u) & = f & \text{in} & \Omega \\ u & = g_{Dir} & \text{on} & \Gamma_D \\ (-\nabla u - \boldsymbol{a}_{con} u) \cdot \boldsymbol{\nu} - g(u) & = g_{Neu} & \text{on} & \Gamma_N \\ u(x,y) & = y^2 + \alpha & \text{on} & \Gamma_n \\ \int_{\Gamma_n} (-\nabla u - \boldsymbol{a}_{con} u) \cdot \boldsymbol{\nu} & = -\frac{1}{3}, & & \end{cases}$$

where the data functions f, g_{Dir} and g_{Neu} are defined in such a way that the exact solution of the BVP is

$$\begin{aligned} u(x,y) & = 1 + x^3 + y^2 + x \\ \alpha & = 1. \end{aligned}$$

The linearization process has already been described in Section 3. We start from the super-solution z_0 – see (3.5), where the coefficients $k_1 = k_2 = 1$ and $q_1 = q_2 = 4$ have been used for the computation. Next, z_j for $j = 1, \ldots, 30$ are defined by (3.6), with $L = 4$. Thus, we have to solve a linear BVP with a nonlocal BC at Γ_n at each iteration. The choice of the space V, of all admissible test functions with constant traces on Γ_n, is not standard. Therefore, an application of a standard FE package for the numerical solution of such a problem is not straightforward. Here, we have followed the ideas from [5] and [6] to avoid this difficulty. We briefly explain the main idea.

Let us consider the following linear elliptic BVP:
Find $(u, \alpha) \in (H^1(\Omega), \mathbb{R})$ such that

$$\begin{cases} \nabla \cdot \left(-\tilde{\boldsymbol{A}}_{dif} \nabla u - \tilde{\boldsymbol{a}}_{con} u\right) + \tilde{a}_{sou} u & = \tilde{f} & \text{in} & \Omega \\ u & = \tilde{g}_{Dir} & \text{on} & \Gamma_D \\ \left(-\tilde{\boldsymbol{A}}_{dif} \nabla u - \tilde{\boldsymbol{a}}_{con} u\right) \cdot \boldsymbol{\nu} - \tilde{g}_{Rob} u & = \tilde{g}_{Neu} & \text{on} & \Gamma_N \\ u & = \tilde{g}_n + \alpha & \text{on} & \Gamma_n \\ G(u) = \int_{\Gamma_n} \left(-\tilde{\boldsymbol{A}}_{dif} \nabla u - \tilde{\boldsymbol{a}}_{con} u\right) \cdot \boldsymbol{\nu} & = \tilde{s}. & & \end{cases} \tag{4.1}$$

The solution will be constructed in three steps. We introduce two auxiliary problems (4.2) and (4.3), the solutions of which will give rise to a solution of (4.1) by the principle of superposition. First, we solve the BVP

$$
\left\{\begin{array}{rlll}
\nabla\cdot\left(-\tilde{\boldsymbol{A}}_{dif}\nabla v-\tilde{\boldsymbol{a}}_{con}v\right)+\tilde{a}_{sou}v & =\tilde{f} & \text{in} & \Omega\\
v & =\tilde{g}_{Dir} & \text{on} & \Gamma_D\\
\left(-\tilde{\boldsymbol{A}}_{dif}\nabla v-\tilde{\boldsymbol{a}}_{con}v\right)\cdot\boldsymbol{\nu}-\tilde{g}_{Rob}v & =\tilde{g}_{Neu} & \text{on} & \Gamma_N\\
v & =\tilde{g}_n & \text{on} & \Gamma_n.
\end{array}\right. \tag{4.2}
$$

The second BVP to be solved is

$$
\left\{\begin{array}{rlll}
\nabla\cdot\left(-\tilde{\boldsymbol{A}}_{dif}\nabla z-\tilde{\boldsymbol{a}}_{con}z\right)+\tilde{a}_{sou}z & =0 & \text{in} & \Omega\\
z & =0 & \text{on} & \Gamma_D\\
\left(-\tilde{\boldsymbol{A}}_{dif}\nabla z-\tilde{\boldsymbol{a}}_{con}z\right)\cdot\boldsymbol{\nu}-\tilde{g}_{Rob}z & =0 & \text{on} & \Gamma_N\\
z & =1 & \text{on} & \Gamma_n.
\end{array}\right. \tag{4.3}
$$

In the third step we choose the appropriate value of a real parameter α, for which the total flux through Γ_n fulfills the condition

$$G(u_\alpha)\equiv G(v+\alpha z)=G(v)+\alpha G(z)=s,$$

i.e.,

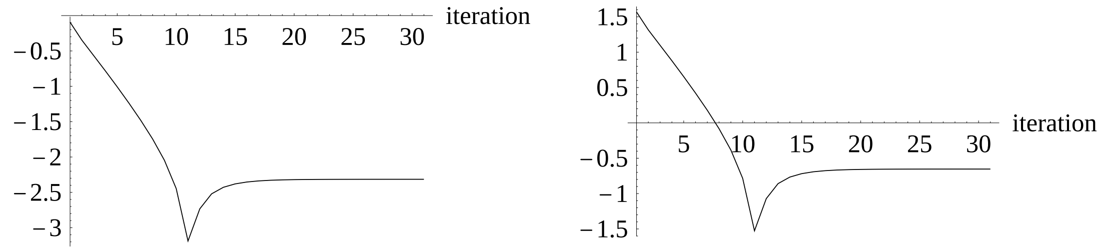
$$\alpha=\frac{s-G(v)}{G(z)}. \tag{4.4}$$

Here, $G(z)\neq 0$ – see [6]. One can easily see that the function u_α, for α given by (4.4), solves the BVP (4.1).

We have used the mixed non-conforming finite element formulation for the numerical solution of each linear elliptic BVP. This is equivalent to the mixed-hybrid method (see [1]). We explain briefly the main idea of this approximation.

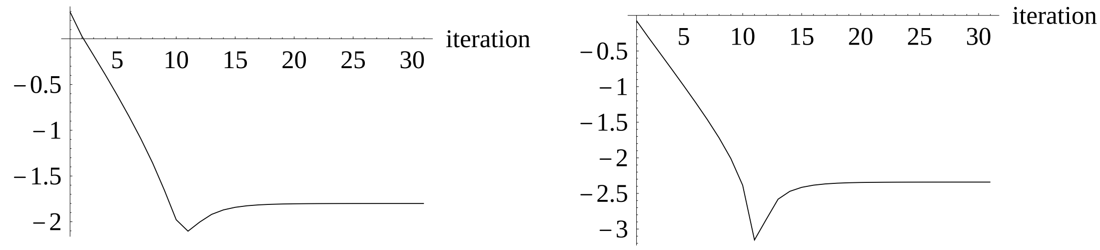
Consider a regular triangulation $\mathcal{T}_h$ (h denotes the mesh diameter) of the domain Ω. On each element $\mathcal{T}\in\mathcal{T}_h$ we define three linear basis functions associated with the edges of $\mathcal{T}$, i.e., a basis function has the value 1 at the midpoint of one edge and vanishes at the midpoints of the other edges of the triangle. Further, we define a bubble function on $\mathcal{T}$, which is a polynomial function of third order vanishing on the boundary $\partial\mathcal{T}$, such that its integral average value on $\mathcal{T}$ is 1. In this way we have enriched the standard linear non-conforming space by bubbles. We solve a linear elliptic problem in this space replacing the velocity field $\mathbf{q}$ by its projection on the Raviart-Thomas space RT_0. For more details see [1].

We have chosen a fixed uniform mesh consisting of 5 000 triangles, which corresponds to $\Delta x=\Delta y=0.02$. The logarithms of absolute and relative $L_2(\Omega)$-errors of u_k versus the iteration number $k=1,\ldots,30$ are plotted in Fig. 4.2. The logarithm of the $L_2(\Omega)$-error for flux the $\boldsymbol{q}_k=-\nabla u_k-\boldsymbol{a}_{con}u_k$ and the logarithm of the α_k-error are depicted in Fig. 4.3. Note that one can see the convergence rate on the y-axes in each picture. The behavior of all graphs is similar. First, we observe a monotonically decreasing part of the curve for $k=1,\ldots,11$. Then the curve turns up and later becomes more and less constant. This can be easily explained. The resulting error consists of two parts: the linearization and the discretization error. At the beginning



$L_2(\Omega)$-error   Relative $L_2(\Omega)$-error

FIG. 4.2. *Logarithms of errors for u_k versus iterations*



$L_2(\Omega)$-error of flux   Error of α

FIG. 4.3. *Logarithms of flux $\boldsymbol{q}_k$- and α_k-errors versus iterations*

of the iteration process (for $k = 1, \ldots, 11$), the linearization error is dominant. Therefore, it makes no sense to iterate after the turning point (in our case it is $k = 11$). Let us note that the discretization error can be diminished by taking a smaller mesh diameter h.

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