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ON A SOLUTION AND ITS LONG TIME STABILITY OF A NONSTATIONARY VON KÁRMÁN SYSTEM FOR VISCOELASTIC PLATES *

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Abstract.
We shall deal with the system of quasistationary von Kármán equations describing great deflections of thin viscoelastic plates. We shall concentrate on a long memory material modelled by a quasistationary system with a linear integro-differential main part and a nonlinear integro-differential term. The existence of a solution as well as the convergence of a semidiscrete approximation are verified. The behaviour of a solution for long-time values is studied.

Key words. von Kármán system, viscoelastic plate, integro-differential equations, Rothe method

AMS subject classifications. 74D10, 74K20, 45K05

1. Introduction. Theodor von Kármán (1910) stated the nonlinear system of partial differential equations for great deflections and the Airy stress function of a thin elastic plate. This system has been treated systematically starting in the sixties by Berger and Fife [2] who have proved the existence of buckled states for a plate subjected only to compressive forces. Ciarlet [7] has justified the von Kármán system as the plate model derived from the equations for a 3-dimensional body. The von Kármán system for viscoelastic plates was derived by J.Brilla [4], who considered the linearized stability problem for the generalized n-th order viscoelasticity. We have dealt in [3] with the short memory anisotropic case, where the pseudoparabolic canonical equation with a nonlinear integral term has been derived and solved. The integral term has in the anisotropic case rather complicated form defined by the matrix exponential function. We shall deal here with the isotropic case, where the nonlinear system and the corresponding canonical equation can be derived in a similar way as the traditional elastic von Kármán system.

The dynamic viscoelastic von Kármán systems are studied nowadays mainly in the framework of controllability problems. Muñoz Rivera and Perla Menzala [11]) have considered the memory term only in the linear part of the system.

2. Formulation of the problem. We assume a thin isotropic plate occupying the domain

$$Q = \{(x,z) \in R^3;\ x = (x_1,x_2) \in \Omega,\ -h/2 < z < h/2\},$$

where Ω is a bounded simply connected domain in R^2 with a Lipschitz boundary Γ. The plate is clamped on its boundary and subjected to the perpendicular load $f(t,x),\ t > 0,\ x \in \Omega$.

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Considering the great deflections we have the nonlinear strain-displacement relations

$$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i w \partial_j w) - z\partial_{ij} w, \quad i,j = 1,2;\ \varepsilon_{13} = \varepsilon_{23} = 0.$$

Let $\{\sigma^{ij}\}$ be the stress tensor fulfilling the condition $\sigma^{33} = 0$. The principle of virtual displacements holds in the form

$$\int_\Omega \left(\int_{-h/2}^{h/2} \sigma^{ij} \delta\varepsilon_{ij} dz \right) dx = \int_\Omega f(t,x) v(x) dx \text{ for all } (\omega_1, \omega_2, v) \in U \times U \times V, \tag{1}$$

where v and ω_i are virtual displacements in the directions z and x_i $(i = 1,2)$ respectively and $U = H_0^1(\Omega)$, $V = H_0^2(\Omega)$ are the spaces of admissible displacements. The virtual strains are of the form

$$\delta\varepsilon_{ij} = \frac{1}{2}(\partial_i \omega_j + \partial_j \omega_i + \partial_i w \partial_j v) - z\partial_{ij} v,\ i,j = 1,2. \tag{2}$$

The principle of virtual displacements implies that the stress resultants

$$N_{ij} = \int_{-h/2}^{h/2} \sigma^{ij} dz \tag{3}$$

fulfil the homogeneous equations $\partial_j N_{ij} = 0, \quad i,j = 1,2.$

Then there exists the Airy stress function $\Phi : \Omega \to R$ defined by the equations

$$N_{11} = \partial_{22}\Phi, \quad N_{22} = \partial_{11}\Phi, \quad N_{12} = -\partial_{12}\Phi. \tag{4}$$

The stress-strain relations for the isotropic viscoelastic long memory material of the Boltzmann type are of the form

$$\sigma^{ij} = \frac{E(0)}{1-\mu^2}[(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}] + \frac{E'}{1-\mu^2} * [(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}](t), \tag{5}$$
$$i,j \in \{1,2\}, \quad \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22}, \quad \sigma^{33} = 0$$

with a Poisson ratio $\mu \in (0, \frac{1}{2})$, a positive decreasing relaxation function $E \in C^1(R^+)$ and a convolution product $f * g(t) = \int_0^t f(t-s)g(s)ds$.

Let us set

$$[v,w] = \partial_{11} v \partial_{22} w + \partial_{22} v \partial_{11} w - 2\partial_{12} v \partial_{12} w, \quad v,\ w \in H^2(\Omega). \tag{6}$$

We recall that in the elastic case the well known von Kármán system for the deflection w and the Airy stress function Φ : has the form ([8])

$$D_0 \Delta^2 w - [\Phi, w] = f(x) \text{ in } \Omega, \quad w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma \tag{7}$$

$$\Delta^2 \Phi = -\frac{E_0 h}{2}[w,w] \text{ in } \Omega, \quad \Phi = \frac{\partial \Phi}{\partial \nu} = 0 \text{ on } \Gamma \tag{8}$$

$$D_0 = \frac{h^3 E_0}{12(1-\mu^2)}.$$

In order to convert a system (7), (8) into one equation for a deflection we introduce the bilinear operator $B : V \times V \to V$ defined by the uniquelly solved equation

$$((B(u,v),\varphi)) \;=\; \int_\Omega [u,v]\varphi dx \text{ for all } \varphi \in V, \tag{9}$$

where the scalar product and the norm in the Sobolev space $V = H_0^2(\Omega)$ are

$$((u,v)) = \int_\Omega \Delta u \Delta v dx, \quad \|u\| = ((u,u))^{1/2}.$$

Expressing a weak solution of the boundary value problem (8) by (9) and inserting it into the equation (7) we obtain the nonlinear boundary value problem

$$D_0\Delta^2 w + \frac{hE_0}{2}[B(w,w),w] = f(x) \text{ in } \Omega, \quad w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma. \tag{10}$$

Let us define the element $q \in V$ uniquely defined as a solution of the identity

$$((q,\varphi)) = \frac{1}{D_0}\int_\Omega f\varphi dx \text{ for all } \varphi \in V$$

and the operator

$$C : V \to V, \quad C(v) = \alpha B(B(v,v),v), \quad \alpha = \frac{hE_0}{2D_0} = \frac{6(1-\mu^2)}{h^2}. \tag{11}$$

We formulate the operator equation in the space V

$$w + C(w) = q, \quad w \in V. \tag{12}$$

Definition 2.1. *The equation (12) is called the canonical equation for the boundary value problem (7), (8).*

The operator $C : V \to V$ is compact and not negative. It holds

$$((C(v),v)) = \alpha\|B(v,v)\|^2, \quad v \in V. \tag{13}$$

Moreover C fulfils ([8]) the inequality

$$((C(u)-C(v),u-v)) \le \alpha\|B\|^2_{L(V\times V;V)}\max\{\|u\|^2,\|v\|^2\}\|u-v\|^2, \tag{14}$$

which is very important in continuity and uniqueness considerations. Using a theory of operator equations with compact operators there can be verified a following existence and uniqueness theorem.

Theorem 2.2. *For every $q \in V$ there exists a solution $w \in V$ of the canonical equation (12). There exists $M > 0$ such that a solution of (12) is unique for every q fulfilling the condition $\|q\| \le M$.*

Let us define the material function $D(t) = \frac{h^3}{12(1-\mu^2)}E(t)$. Applying the principle of virtual displacements and the viscoelastic stress-strain relations the following integro-differential von Kármán system for the deflection $w(t,x)$ and the Airy stress function $\Phi(t,x),\ t \ge 0,\ x \in \Omega$; can be derived:

$$D(0)\Delta^2 w + D' * \Delta^2 w - [\Phi, w] = f(t,x), \quad w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma, \tag{15}$$

$$\Delta^2\Phi = -\frac{h}{2}(E(0)[w,w] + E' * [w,w]), \quad \Phi = \frac{\partial \Phi}{\partial \nu} = 0 \text{ on } \Gamma. \tag{16}$$

Expressing the Airy stress function Φ from (16) and inserting into (15) we obtain the integro-differential equation for the deflection w :

$$D(0)\Delta^2 w + D' * \Delta^2 w + \frac{h}{2}[E(0)B(w,w) + E' * B(w,w), w] = f(t,x), \tag{17}$$

with the boundary conditions

$$w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma, \tag{18}$$

A weak (canonical) formulation of the problem (17), (18) is

$$w(t) + g * w + \alpha B(B(w,w) + g * B(w,w), w) = q(t) \ \in V, \ t \geq 0 \tag{19}$$

where $g(t) = \frac{E'(t)}{E(0)}$ and a function $q \in C(R^+, V)$ is defined by the identity

$$((q(t), v)) = \frac{1}{D(0)} \int_\Omega f(t,x) v(x) dx \text{ for all } v \in V. \tag{20}$$

3. On the Exponential Stability of a Solution. We shall suppose the time differentiability of the right-hand side and of a solution. The time regularity of a solution to the equation (19) can be verified in the case of sufficiently small right-hand side q and its derivative $q'(t)$ in a similar way as in [11]. Simultaneously we impose stronger assumptions on the kernel function g. The questions of the exponential stability for a solution of the linear parabolic integro-differential equation were investigated in [1]

THEOREM 3.1. *Let $q \in W^{2,1}(0,T;V)$ and $w \in W^{2,1}(0,T;V)$ be a solution of the canonical equation (19) for every $T > 0$. Let the kernel $g \in C(0,\infty;R)$ be such that*

$$g(t) < 0, \quad g'(t) \geq 0 \ \text{ for every } t \geq 0; \quad 1 + \int_0^\infty g(t)dt = \gamma > 0. \tag{21}$$

Then there exist $\beta > 0$, $K > 0$ such that

$$\begin{aligned} \|w(t)\| &\leq K e^{-\beta t}[\|w(0)\| + \|B(w(0), w(0))\|] \\ &+ K[\max_{s\in[0,t]} \|q(s)\| + \int_0^t e^{-\beta(t-s)} \|\beta q(s) + q'(s)\| ds], \ t > 0. \end{aligned} \tag{22}$$

Proof. We shall use the following modified relation concerning the kernel function g derived in [11]:

$$\begin{aligned} \left(\left(\int_0^t g(t-s)v(s)ds, v'(t)\right)\right) &= -\frac{1}{2}g(t)\|v\|^2 + \frac{1}{2}g' \otimes v \\ &- \frac{1}{2}\frac{d}{dt}\left[g \otimes v - \left(\int_0^t g ds\right)\|v\|^2\right], \ v \in V, \end{aligned} \tag{23}$$

where

$$g \otimes v = \int_0^t g(t-s)\|v(t) - v(s)\|^2 ds.$$

Let us set $u(t) = e^{\beta t}w(t),\ g_\beta(t) = e^{\beta t}g(t),\ g_{2\beta}(t) = e^{2\beta t}g(t),\ \beta > 0$. The equation (19) has then the form

$$u(t) + \int_0^t g_\beta(t-s)u(s)ds + \alpha e^{-2\beta t}B(B(u,u),u)(t)\ +$$

$$\alpha e^{-2\beta t}\left(\left(\int_0^t g^{2\beta}(t-s)B(u,u)(s)ds, u(t)\right)\right) = e^{\beta t}q(t),\ t\geq 0 \tag{24}$$

After multiplying the last expression with $u'(t)$ and integrating we obtain using the relation (23) the identity

$$\begin{aligned}
&\|u(t)\|^2 + \frac{1}{2}\alpha e^{-2\beta t}\|B(u,u)(t)\|^2 + \alpha\beta\int_0^t e^{-2\beta s}\|B(u,u)(s)\|^2 ds \\
&- \int_0^t [g_\beta(s)\|u(s)\|^2 + (g_\beta - \frac{d}{ds}g_\beta)\otimes v](s)ds + \left(\int_0^t g_\beta ds\right)\|u(t)\|^2 \\
&+ \frac{1}{2}\alpha\left\{e^{-2\beta t}\left[\int_0^t g_{2\beta}ds\|B(u,u)(t)\|^2 - g_{2\beta}\otimes B(u,u)(t)\right]\right. \\
&\left. - \int_0^t g(s)\|B(u,u)(s)\|^2 ds\right\} \\
&+ \int_0^t e^{-2\beta s}[\frac{d}{ds}g_{2\beta} - \beta(s)g_{2\beta}]\otimes B(u,u)(s)ds\} = \\
&\|u(0)\|^2 + \frac{1}{2}\alpha\|B(u,u)(0)\|^2\ + \\
&2e^{\beta t}((\ q(t), u(t)\ )) - 2\int_0^t e^{\beta s}((\ \beta q(s) + q'(s), u(s)\ ))ds, \quad t > 0.
\end{aligned} \tag{25}$$

Applying the assumption (21) and restricting the value $\beta \in (0, \frac{1}{2})$ we arrive at the inequality

$$\begin{aligned}
&\gamma\left(\|u(t)\|^2 + \frac{1}{2}\alpha e^{-2\beta t}\|B(u,u)(t)\|^2\right) \leq\ \|u(0)\|^2 + \frac{1}{2}\alpha\|B(u,u)(0)\|^2 + \\
&2e^{\beta t}\|q(t)\|\ \|u(t)\| + 2\int_0^t e^{\beta s}[\|\beta q(s) + q'(s)\|]ds\ \max_{s\in[0,t]}\|u(s)\|, \quad t > 0.
\end{aligned} \tag{26}$$

The estimate (22) follows setting $K = \gamma^{-1}\max\{2, (\frac{\alpha}{2})^{-1/2}\}$.

4. Approximation by the Rothe Method. In order to solve the above problem numerically we first convert the quasistationary system (15), (16) into the system of nonlinear stationary equations similar to the classical von Kármán system. We shall use the Rothe method in a similar way as by Kačur [9] or Slodička [12] in the case of parabolic integro-differential equations.

We assume that the right hand side f belongs to the space $W^{2,1}(0,T; L_2(\Omega))$ for every $T > 0$ and that the assumptions (21) abouth the kernel function g from the previous section hold. Moreover we assume the exponential behaviour of the kernel function g:

$$0 < -g(t) \leq Ke^{-\beta t},\ t\geq 0,\ 0 < K < \beta. \tag{27}$$

For each $n \in N$ we set

$$\tau = \frac{T}{n},\ t_i = i\tau,\ w_i = w(t_i),\ i = 0, 1, ..., n;$$
$$\delta w_j = \frac{1}{\tau}(w_j - w_{j-1}),\ j = 1, ..., n.$$

We substitute the system (15), (16) by

$$D_0\Delta^2 w_i + \tau D_0 \sum_{j=0}^{i-1} g_{i-j}\Delta^2 w_j - [\Phi_i, w_i] = f_i, \quad w_i = \frac{\partial w_i}{\partial \nu} = 0 \text{ on } \Gamma, \tag{28}$$

$$\Delta^2\Phi_i = -\frac{h}{2}E_0\left([w_i, w_i] + \tau\sum_{j=0}^{i-1} g_{i-j}[w_j, w_j]\right), \quad \Phi_i = \frac{\partial \Phi_i}{\partial \nu} = 0 \text{ on } \Gamma. \tag{29}$$

The equation (29) is uniquelly solved and we obtain its weak solution $\Phi_i \in V$ in a form

$$\Phi_i = -\frac{h}{2}E_0\left(B(w_i, w_i) + \tau\sum_{j=0}^{i-1} g_{i-j}B(w_j, w_j)\right). \tag{30}$$

After inserting Φ_i into the equation (28) we arrive at the stationary canonical equation in the Hilbert space V:

$$w_i + \tau\sum_{j=0}^{i-1} g_{i-j}w_j + \alpha B\left(B(w_i, w_i) + \tau\sum_{j=0}^{i-1} g_{i-j}B(w_j, w_j), w_i\right) = q_i, \quad i = 1, ..., n; \tag{31}$$

$$w_0 + \alpha B(B(w_0, w_0), w_0) = q_0. \tag{32}$$

The last equation has a solution due to the theory of stationary von Kármán equations. The equation (31) is the Euler equation for the functional

$$J_i(v) = \frac{1}{2}\|v\|^2 + \frac{\alpha}{4}\|B(v, v)\|^2 + \left(\left(\tau\sum_{j=0}^{i-1} g_{i-j}w_j, v\right)\right) + \alpha\left(\left(\tau\sum_{j=0}^{i-1} g_{i-j}B(w_j, w_j), B(v, v)\right)\right) - ((q_i, v)), v \in V,\ i = 1, ..., n. \tag{33}$$

The functional J_i is weakly lower semicontinuous and coercive. Then there exists an element $w_i \in V$ fulfilling the minimum condition

$$J_i(w_i) = \min_{v \in V} J_i(v), \tag{34}$$

and solving the discrete canonical equation (31). A couple $\{w_i, \Phi_i\} \in V \times V$ is then a weak solution of the system (28), (29).

Let us further define the following functions determined by values $w_i,\ \delta w_i$:

$$w_n : [0,T] \to V,\ w_n(t) = w_{i-1} + (t - t_i)\delta w_i,\ t_{i-1} \le t \le t_i,\ i = 1,...,n;$$
$$\bar{w}_n : [0,T] \to V,\ \bar{w}_n(0) = w_0,\ \bar{w}_n(t) = w_i,\ t_{i-1} < t \le t_i,\ i = 1,...,n;$$
$$\tilde{w}_n : [0,T] \to V,\ \tilde{w}_n(0) = 0,\ \tilde{w}_n(t) = w_i,\ t_i < t \le t_{i+1},\ i = 1,...,n-1.$$

We proceed with a priori estimates.

We set $i = j$ in (31), multiply it with τw_j in V and add for $j = 1,...,i$. After denoting

$$\omega_j = \|w_j\|^2 + \alpha\|(Bw_j, w_j)\|^2 \tag{35}$$

we obtain the identity

$$\sum_{j=1}^{i}\tau\omega_j + \tau^2\left[\left(\left(\sum_{k=0}^{j-1} g_{j-k}w_k, w_j\right)\right) + \alpha\left(\left(\sum_{k=0}^{j-1} g_{j-k}B(w_k,w_k), B(w_j,w_j)\right)\right)\right]$$
$$= \tau\sum_{j=1}^{i}((q_j, w_j)).$$

and the inequality

$$(1-\epsilon)\tau\sum_{j=1}^{i}\omega_j \le$$
$$\tau^3\left(\sum_{j=1}^{i}\|\sum_{k=0}^{j-1} g_{j-k}w_k\|^2 + \alpha\|\sum_{k=0}^{j-1} g_{j-k}B(w_k,w_k)\|^2\right) + \frac{\tau}{\epsilon}\sum_{j=1}^{i}\|q_j\|^2,\ 0 < \epsilon < 1.$$

Applying the assumption (27) and the convexity of the function $\|.\|^2 : V \to R$ we obtain the estimates

$$\tau\sum_{j=1}^{i}\omega_j \le \frac{K^2}{1-\epsilon}\tau^3\sum_{j=1}^{i}e^{-2\beta j\tau}\sum_{k=0}^{j-1}e^{\beta k\tau}\sum_{k=0}^{j-1}e^{\beta k\tau}\omega_k + \frac{\tau}{\epsilon(1-\epsilon)}\sum_{j=1}^{i}\|q_j\|^2$$
$$\le \frac{K^2}{(1-\epsilon)\beta}\ \tau^2\sum_{j=1}^{i}(e^{-\beta j\tau} - e^{-2\beta j\tau})\sum_{k=0}^{j-1}e^{\beta k\tau}\omega_k + \frac{\tau}{\epsilon(1-\epsilon)}\sum_{j=1}^{i}\|q_j\|^2, \tag{36}$$

where we have used the relations

$$\sum_{k=0}^{j-1}e^{\beta k\tau} = \frac{e^{\beta j\tau}-1}{e^{\beta\tau}-1} \le \frac{1}{\beta\tau}(e^{\beta j\tau}-1).$$

We continue substituting the sums by integrals and using the integration by parts in a following way:

$$\tau^2\sum_{j=1}^{i}(e^{-\beta j\tau} - e^{-2\beta j\tau})\sum_{k=0}^{j-1}e^{\beta k\tau}\omega_k \le \tau\sum_{j=1}^{i}(e^{-\beta j\tau} - e^{-2\beta j\tau})\int_0^{j\tau} e^{\beta\sigma}\tilde{\omega}_n(\sigma)d\sigma$$
$$\le \int_0^{t_i}(e^{-\beta s} - e^{-2\beta s})\int_0^{s} e^{\beta\sigma}\tilde{\omega}_n(\sigma)d\sigma + \sum_{j=1}^{i}\int_{(j-1)\tau}^{j\tau}(e^{-\beta s} - e^{-2\beta s})\int_s^{j\tau} e^{\beta\sigma}d\sigma\ \tilde{\omega}_{j-1}d\sigma$$

$$= \frac{1}{\beta}\int_0^{t_i}(1-\frac{1}{2}e^{-\beta s})\tilde{\omega}_n(s)ds - \frac{1}{\beta}(e^{-\beta t_i}-\frac{1}{2}e^{-2\beta t_i})\int_0^{t_i}e^{\beta s}\tilde{\omega}_n(s)ds$$

$$+\frac{1}{\beta}\sum_{j=1}^{i}\int_{(j-1)\tau}^{j\tau}[e^{\beta(j\tau-s)}-1-\frac{1}{2}e^{\beta(j\tau-2s)}+\frac{1}{2}e^{-\beta s}]ds\ \omega_{j-1}$$

$$\le \frac{1}{\beta}\int_0^{t_i}\tilde{\omega}_n(s)ds+\frac{1}{\beta^2}\sum_{j=1}^{i}[e^{\beta\tau}-1-\beta\tau-\frac{1}{4}e^{-\beta j\tau}+\frac{1}{2}e^{-\beta(j-1)\tau}-\frac{1}{4}e^{-\beta(j-2)\tau}]\ \omega_{j-1}$$

$$\le \frac{1}{\beta}\int_0^{t_i}\tilde{\omega}_n(s)ds+\frac{1}{2}\tau^2e^{\beta\tau}\sum_{j=1}^{i}\omega_{j-1}\ \le\ (\frac{1}{\beta}+\frac{1}{2}\tau e^{\beta\tau})\ \tau\sum_{j=0}^{i}\omega_j.$$

Setting $\epsilon = 1-(\frac{K}{\beta})^{1/2}$ and $\tau_0>0$ such that $\frac{1}{2}\beta\tau_0e^{\beta\tau_0}<(\frac{\beta}{K})^{1/2}-1$ we obtain comparing with (32), (35), (36) the a priori estimate

$$\tau\sum_{j=1}^{i}\|w_j\|^2+\alpha\|(Bw_j,w_j)\|^2 \le C_1(\beta,K)\|q_0\|^2+C_2(\beta,K)\tau\sum_{j=1}^{i}\|q_j\|^2$$

$$\le C_1(\beta,K)\|q_0\|^2+C_2(\beta,K)\int_0^T[(1+\tau_0)\|q(t)\|^2+\tau_0\|q'(t)\|^2]dt, \tag{37}$$
$$i=1,...,n,\ \tau\le\tau_0;$$

where

$$C_1(\beta,K)=\frac{2\sqrt{K}}{\beta(\sqrt{\beta}+\sqrt{K})},\ C_2(\beta,K)=\frac{\beta^2}{\sqrt{K}(\sqrt{\beta}-\sqrt{K})(\beta-K)}. \tag{38}$$

We continue with uniform a priori estimates. The equations (31), (35) imply the identity

$$\omega_i=-\left(\left(\tau\sum_{j=0}^{i-1}g_{i-j}w_j,w_i\right)\right)-\alpha\left(\left(\tau\sum_{j=0}^{i-1}g_{i-j}B(w_j,w_j),B(w_i,w_i)\right)\right)+((q_i,w_i))$$

and the inequality

$$\omega_i\le 2\tau^2\left(\left\|\sum_{j=0}^{i-1}g_{i-j}w_j\right\|^2+\alpha\left\|\sum_{j=0}^{i-1}g_{i-j}B(w_j,w_j)\right\|^2\right)+2\|q_i\|^2.$$

Again using the convexity of $\|.\|^2$ and the properties of exponential functions we arrive at the inequality

$$\omega_i\ \le\ \frac{2K^2}{\beta}\tau\sum_{j=0}^{i-1}\omega_j\ +2\|q_i\|^2$$

and applying the estimate (36) we obtain

$$\|w_i\|^2+\alpha\|(Bw_i,w_i)\|^2\le \tag{39}$$
$$C_3(\beta,K)\|q_0\|^2+C_4(\beta,K)\int_0^T[(1+\tau_0)\|q(t)\|^2+\tau_0\|q'(t)\|^2]dt+2\|q\|^2_{C([0,T],V)}$$
$$i=1,...,n,\ \tau\le\tau_0;$$

where

$$C_3(\beta, K) = \frac{K^2}{\beta}[C_1(\beta, K) + \tau_0],\ C_4(\beta, K) = \frac{K^2}{\beta}C_2(\beta, K). \tag{40}$$

In order to obtain the convergence of the scheme we need the estimates of the derivatives δw_i.

After setting $i = j,\ i = j - 1$ in (31) and substracting we have the identities

$$\delta w_1 + g_1 w_0 + \alpha\delta B(B(w_1, w_1), w_1) + \alpha B(g_1 B(w_0, w_0), w_1) = \delta q_1,$$

$$\begin{aligned} &\delta w_j + \alpha\delta B(B(w_j, w_j), w_j) + g_j w_0 + \tau\sum_{k=1}^{j-1} g_{j-k}\delta w_k + \\ &\alpha B(\tau\sum_{k=0}^{j-2}\delta g_{j-k}B(w_k, w_k) + g_1 B(w_{j-1}, w_{j-1}), w_j) + \\ &\tau B(\sum_{k=0}^{j-2} g_{j-1-k}B(w_k, w_k), \delta w_j) = \delta q_j,\ j = 2, ..., i. \end{aligned}$$

After multiplying the last identities in the space V with $\tau\delta w_j, j = 1, ..., i$; and adding we arrive at

$$\begin{aligned} &\tau\sum_{j=1}^{i}\|\delta w_j\|^2 + \alpha\sum_{j=1}^{i}((\ B(B(w_j, w_j), w_j) - B(B(w_{j-1}, w_{j-1}), w_{j-1})\ ,\ \delta w_j\ )) \\ &\quad + \tau\sum_{j=1}^{i}(\ (g_j w_0 + \alpha B(g_0 B(w_{j-1}, w_{j-1}, w_j)\ ,\ \delta w_j)\ )) \\ &\quad + \tau^2\sum_{j=2}^{i}\left(\left(\sum_{k=0}^{j-2} g_{j-k}\delta w_k + \alpha B\left(\sum_{k=0}^{j-1}\delta g_{j-k}B(w_k, w_k), w_j\right)\ ,\ \delta w_j\right)\right) \\ &\quad + \alpha\left(\left(\sum_{k=0}^{j-2} g_{j-1-k}B(w_k, w_k), B(\delta w_j, \delta w_j)\right)\right)\ =\ \tau\sum_{j=1}^{i}((\delta q_j, \delta w_j)). \end{aligned}$$

Using the property (14) of the operator $C : V \to V,\ C(v) = B(B(v, v), v)$ and the properties of the kernel function g we obtain the inequality

$$\begin{aligned} &\tau\sum_{j=1}^{i}\|\delta w_j\|^2\ \le\ 4\alpha\|B\|^2\max_{j\in\{0,\ldots,i\}}\|w_j\|^2\tau\sum_{j=1}^{i}\|\delta w_j\|^2 \\ &+ 2\tau\sum_{j=1}^{i}\left\|g_j w_0 + \alpha B(g_0 B(w_{j-1}, w_{j-1}) + \tau\sum_{k=0}^{j-1}\delta g_{j-k}B(w_k, w_k)\ ,\ w_j) + \delta q_j\right\|^2 \\ &+ 2\tau^3\sum_{j=2}^{i}\left\|\sum_{k=0}^{j-2} g_{j-k}\delta w_k\right\|^2. \end{aligned} \tag{41}$$

Let us assume that

$$4\alpha\|B\|^2\|w_j\|^2 \le 1 - \theta,\ \theta \in (0, 1),\ j = 1, ..., n. \tag{42}$$

Comparing with the a priori estimate (39) we can see that the condition

$$C_3(\beta,K)\|q_0\|^2 + C_4(\beta,K)\int_0^T [(1+\tau_0)\|q(t)\|^2 + \tau_0\|q'(t)\|^2]dt + 2\|q\|^2_{C([0,T],V)} \le \frac{1-\theta}{4\alpha\|B\|^2} \tag{43}$$

is sufficient for fulfilling the estimate (42). Applying the assumption (42), the a priori estimate (39) and the properties of the function g we obtain the inequality

$$\tau\sum_{j=1}^{i}\|\delta w_j\|^2 \le C_5 + C_6\tau^2\sum_{j=2}^{i}\sum_{k=0}^{j-2}\|\delta w_k\|^2. \tag{44}$$

The discrete Gronwall lemma then implies the a priori estimate

$$\tau\sum_{j=1}^{i}\|\delta w_j\|^2 \le C_7,\ i=1,...,n;\ 0<\tau\le\tau_0. \tag{45}$$

The sequence of functions $\{w_n\}$ defined by their discrete values is then bounded in the space $W^{2,1}(0,T;V)$:

$$\|w_n\|_{W^{2,1}(0,T;V)} \le C_8, n\in N. \tag{46}$$

Then there exists its subsequence (again denoted by $\{w_n\}$) and a function $w \in W^{2,1}(0,T;V)$ such that

$$w_n \rightharpoonup w \quad \text{in} \quad W^{2,1}(0,T;V), \tag{47}$$

$$w_n(t) \rightharpoonup w(t),\ \bar{w}_n(t) \rightharpoonup w(t) \text{ in } V \text{ for every } t\in[0,T], \tag{48}$$

$$w_n \rightharpoonup^* w,\ \bar{w}_n \rightharpoonup^* w \text{ in } L^\infty(0,T;V), \tag{49}$$

$$w_n \to w,\ \bar{w}_n \to w \text{ in } L^p(0,T;W^{r,1}(\Omega)),\ p>1,\ r>1. \tag{50}$$

The sequence $\bar{\Phi}_n$ of step functions obtained from the equation (30) is bounded in the space $L^\infty(0,T;V)$ by the estimate

$$\|\bar{\Phi}_n\|_{L^\infty(0,T;V)} \le \frac{hE_0(1-\theta)}{4\alpha\|B\|},\ n=1,2,... \tag{51}$$

We receive the last estimate directly from the expression (30) and the estimate (42) applying the exponential growth of the kernel function g. Then there exists a subsequence (again denoted by $\bar{\Phi}_n$) and a function $\Phi\in L^\infty(0,T;V)$ such that

$$\bar{\Phi}_n \rightharpoonup^* \Phi \text{ in } L^\infty(0,T;V) \tag{52}$$

We shall verify that a function Φ is defined by the expression

$$\Phi = -\frac{h}{2}E_0[B(w,w) + g*B(w,w)]. \tag{53}$$

Let us set

$$B(w,w)=U,\ B(w_n,w_n)=U_n,\ n=1,2,...$$

We can express the functions $\bar{\Phi}_n$ in a following way:

$$\begin{aligned}(54)\qquad \bar{\Phi}_n(t) = -\frac{h}{2}E_0\left[\bar{U}_n(t)+\int_0^t g(t-s)\tilde{U}_n(s)ds-\int_t^{t_i} g(t-s)\tilde{U}_n(s)ds\right]\\ -\frac{h}{2}E_0\left[\int_0^{t_i}[\bar{g}(t_i-s)-g(t-s)]\tilde{U}_n(s)ds\right],\ t_{i-1}<t\le t_i,\ i=1,...,n.\end{aligned}$$

Let $v\in L^q(0,T;V),\ q>1$ be arbitrary. Using the property ([8])

$$(55)\qquad ([u,v],w)=(u,[v,w]),\ u,v,w\in V$$

and the definition of B we have the relations

$$((\bar{U}_n(t),v))=([\bar{w}_n(t),v(t)],\bar{w}_n(t)]),\ t\in[0,T],\ n=1,2...$$

and

$$\left|\int_0^T([\bar{w}_n,v],\bar{w}_n)-([w,v],w)(t)dt\right|\le$$
$$\left|\int_0^T([\bar{w}_n-w,v],\bar{w}_n)dt\right|+\left|\int_0^T([w,v],\bar{w}_n-w)(t)dt\right|.$$

Applying the convergence (50) and the boundedness of $\{\bar{w}_n\}$ in $L^\infty(0,T;V)$ and hence also in $L^\infty(0,T;L^2(\Omega))$ we obtain the converegence

$$(56)\qquad \bar{U}_n \rightharpoonup^* U \text{ in } L^\infty(0,T;V).$$

Applying the relation

$$\int_0^T((\bar{U}_n(t)-\tilde{U}_n(t),v(t)))dt=\sum_{i=1}^n\int_{(i-1)\tau}^{i\tau}([w_i,w_i]-[w_{i-1},w_{i-1}],v(t))dt=$$
$$\tau\sum_{i=1}^n\int_{(i-1)\tau}^{i\tau}([w_i+w_{i-1},v],\delta w_i)$$

we obtain using the a priori estimate (46) the convergence

$$\lim_{n\to\infty}\int_0^T((\bar{U}_n(t)-\tilde{U}_n(t),v(t)))dt=0$$

and comparing with (56)

$$(57)\qquad \tilde{U}_n\rightharpoonup U \text{ in } L^p(0,T;V),\ \frac{1}{p}+\frac{1}{q}=1.$$

The operator $G: L^p(0,T;V)\to L^p(0,T;V)$ defined by

$$(Gu)(t)=\int_0^t g(t-s)u(s)ds,\ u\in L^p(0,T;V);$$

is linear and continuous and the convergence

$$(58)\qquad G\tilde{U}_n\rightharpoonup GU \text{ in } L^p(0,T;V)$$

follows. The function defined by the sum of the second and the third integral in (54) converges weakly to 0 in $L^p(0,T;V)$ as a consequence of previous a priori estimates and properties of the function g. Then we obtain using (56), (58) the relations (52), (53). Using the strong convergence (50), property (55) and the estimate ([8])

$$|([u,v],w)| \le C\|u\| \ \|v\|_{W^{1,4}(\Omega)} \ \|w\|_{W^{1,4}(\Omega)}, \ u,v,w \in V \tag{59}$$

we obtain after limiting process in a weak form of the equation (28) that a function w is a solution of the canonical equation (19) and the couple $\{w,\Phi\} \in W^{2,1}(0,T;V) \times L^\infty(0,T;V)$ is a weak solution of the von Kármán system (15), (16) in a form

$$\begin{aligned} &\int_0^T \left(\left(D(0)w + \int_0^t D'(t-s)w(s)ds - B(\Phi(t),w(t)),v(t)\right)\right) dt \\ &\qquad = D(0)\int_0^T ((\ q(t),v(t)\ ))dt \ \forall v \in L^2(0,T;V), \end{aligned} \tag{60}$$

$$\begin{aligned} &\int_0^T ((\ \Phi(t),\Psi(t)\ ))dt = \\ &-\frac{h}{2}\int_0^T ((\ E(0)B(w,w)(t) + \int_0^t E'(t-s)B(w,w)(s)ds, \Psi(t)\ ))dt \ \forall \Psi \in L^2(0,T;V). \end{aligned} \tag{61}$$

The function $w:[0,T]\to V$ is $C^{0,1/2}$ Hölder continuous due to continuous imbedding (see [5], Corollary 1.4.38)

$$W^{2,1}(0,T;V) \subset C^{0,1/2}([0,T],V).$$

The function $B(w,w):[0,T]\to V$ is continuous due to the relation

$$B(u,u) - B(v,v) = B(u+v,u-v), \ u,v \in V.$$

The functions $\Phi:[0,T]\to V, B(\Phi,w):[0,T]\to V$ are then continuous too. We can then substitute the integral form of the equations (60), (61). The result is a following theorem.

Theorem 4.1. *Let $\{\bar w_n, \bar\Phi_n\}$ is a sequence of step functions achieved from the discrete values defined by the system (40), (41), (42). There its exists a subsequence (with the same notion) such that the convergence (51)-(56) holds and a couple $\{w,\Phi\} \in W^{2,1}(0,T;V) \times C([0,T;V)$ is a weak solution of the von Kármán system (15), (16) in a form*

$$\begin{aligned} &((D(0)w(t) + (D' * w)(t), v)) = \int_\Omega ([\Phi,w] + f(t,x))v(x)dx \\ &\qquad \text{for all } v \in V, \ t \in [0,T]; \end{aligned} \tag{62}$$

$$\begin{aligned} &((\Phi,v)) = -\frac{h}{2}\int_\Omega (\ E(0)[w,w](t) + (E' * [w,w])(t)\ )\Psi(x)dx \\ &\qquad \text{for all } v \in V, \ t \in [0,T]. \end{aligned} \tag{63}$$

Remark. We have verified the convergence of the method under the restrictive assumption (43) on the right-hand side q. This condition is closely connected with questions of continuity and uniqueness of a weak solution $\{w,\Phi\}$. It is possible to improve this condition, but one cannot expect the convergence of the Rothe method for an arbitrary right-hand side.

The exponential decreasing of the kernel function g corresponds to the most of viscoelastic materials (see [6] for the details).

Combining the Rothe method with finit elements with respect to space coordinates we obtain the nonlinear algebraic system for every time line $t \equiv t_i$. The mixed formulation of the problem due to Miyoshi [10] converting the weak formulation of the problem (28), (29) into the problem involving 8 unknown functions with at most 2-nd order derivatives is more suitable to full numerical approximation. The scheme is based on the fact that a solution pairs $\{w_i, \Phi_i\}$ achieve $H^3(\Omega)$ regularity in the case of the bounded convex region Ω with a Lipschitz boundary. After applying the Green formula the system (28), (29) can be substituted by a following weak formulation in the space $\mathcal{H} = H_0^1(\Omega) \times [H^1(\Omega)]^3$:

$$D_0\mathcal{L}(W_i, U) + D_0\tau \sum_{j=0}^{i-1} g_{i-j}\mathcal{L}(W_j, U) - ([W_i, \Psi_i], u) = (f_i, u), \tag{64}$$

$$\mathcal{L}(\Psi_i, \tilde{U}) = -\frac{h}{2}E_0 \left([W_i, W_i] + \tau \sum_{j=0}^{i} g_{i-j}([W_j, W_j]\ ,\ \tilde{u} \right) \tag{65}$$

$$\text{for all } \{U, \tilde{U}\} \in \mathcal{H}^2,\ U = (u, U_{11}, U_{12}, U_{22}),\ \tilde{U} = (\tilde{u}, \tilde{U}_{11}, \tilde{U}_{12}, \tilde{U}_{22}),$$
$$U_{12} = U_{21},\ \tilde{U}_{12} = \tilde{U}_{21},$$
$$\mathcal{L}(W, U) = \int_\Omega \left(\sum_{\alpha,\beta} \partial_\alpha W_{\alpha\beta}\partial_\beta u + \sum_{\alpha\le\beta} \partial_\beta w \partial_\alpha U_{\alpha\beta} + W_{\alpha\beta}U_{\alpha\beta} \right) dx,$$
$$[U, W] = U_{11}V_{22} + U_{22}V_{11} - 2U_{12}V_{12}.$$

Piecewise linear finite elements on the regularly triangulated domain Ω can be used in order to solve the system (64), (65). A special choise of basic functions near the boundary enables to satisfy the vanishing of the normal derivatives of the functions w_i, Φ_i on the boundary. These boundary conditions have the character of natural (mechanical) boundary conditions in the system (64), (65). The number of the resulting equations can be reduced essentially using certain types of piecewise constant functions (see [10] for the details where also the linearization process is proposed).

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