Applied Mathematics E-Notes, 20(2020), 39-45 ©
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/ ISSN 1607-2510

Mapping Properties Of Certain Subclasses Of Analytic Functions Associated With Generalized Distribution Series*

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Received 25 Feburary 2019

Abstract

The purpose of the present paper is to obtain some necessary and sufficient conditions for generalized distribution series to be in certain special subclasses.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and univalent in $\mathbb{U}$. As usual, we denote by $\mathcal{T}$ [22] the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad n = 2, 3, \ldots. \tag{1.2}$$

A function $f(z)$ of the form (1.1) is said to be starlike of order α $(0 \le \alpha < 1)$, if it satisfies the following condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad z \in \mathbb{U}.$$

A function $f(z)$ of the form (1.1) is said to be convex of order α $(0 \le \alpha < 1)$, if it satisfies the following condition

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad z \in \mathbb{U}.$$

The classes of all starlike and convex functions of order α are denoted by $S^*(\alpha)$ and $K(\alpha)$ were introduced and studied by Robertson [19] and Silverman [22]. We also write $S^*(0) = S^*$ and $K(0) = K$ are the well-known classes of starlike and convex functions.

A function $f(z)$ of the form (1.1) is said to be in the class $\mathcal{G}(\lambda, \alpha)$, if it satisfies the following condition

$$\Re\left\{\frac{zf'(z) + \lambda z^2 f''(z)}{f(z)}\right\} > \alpha, \qquad z \in \mathbb{U},$$

where $0 \le \alpha < 1$ and $0 \le \lambda < 1$.

A function $f(z)$ of the form (1.1) is said to be in the class $\mathcal{K}(\lambda, \alpha)$, if it satisfies the following condition

$$\Re\left\{\frac{z\left[zf'(z) + \lambda z^2 f''(z)\right]'}{zf'(z)}\right\} > \alpha, \qquad z \in \mathbb{U},$$

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where $0 \leq \alpha < 1$ and $0 \leq \lambda < 1$.

Also we write $\mathcal{TG}(\lambda, \alpha) \equiv \mathcal{G}(\lambda, \alpha) \cap \mathcal{T}$ and $\mathcal{TK}(\lambda, \alpha) \equiv \mathcal{K}(\lambda, \alpha) \cap \mathcal{T}$.

Remark 1. *It is easy to verify that for $\lambda = 0$, we have $\mathcal{G}(\lambda, \alpha) \equiv \mathcal{S}^*(\alpha)$ and $\mathcal{K}(\lambda, \alpha) \equiv \mathcal{K}(\alpha)$.*

A function $f \in \mathcal{A}$ is said to be in the class $f \in \Re^\tau(A, B)$ ($\tau \in \mathbb{C}\backslash\{0\}$, $-1 \leq B < A \leq 1$), if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1, \quad (z \in \mathbb{U}).$$

The class $\Re^\tau(A, B)$ was introduced earlier by Dixit and Pal [6].

The applications of hypergeometric function ([5], [9], [12], [23], [24]), confluent hypergeometric functions ([4], [7]), Wright's function [18], generalized Bessel functions ([3], [8], [15]) are interesting topics of research in Geometric Function Theory. In 2014, Porwal [13] introduced Poisson distribution series and give a nice application on analytic univalent functions and co-relates probability density function with univalent function. After the appearance of this paper several researchers introduced hypergeometric distribution series [1], hypergeometric distribution type series [16], confluent hypergeometric distribution series [17], Binomial distribution series [11], Mittag-Leffler type distribution series [2] and obtain sufficient conditions and inclusion relations for various classes of univalent functions. Very recently, Porwal [14] introduced generalized distribution and studied its geometric properties associated with univalent functions. Now we recall the definition of generalized distribution. The probability mass function of the generalized distribution is given as

$$p(n) = \frac{t_n}{S}, \quad n = 0, 1, 2, \ldots,$$

where $t_n \geq 0$ and the series $\sum_{n=0}^{\infty} t_n$ is convergent and

$$S = \sum_{n=0}^{\infty} t_n. \tag{1.3}$$

Further, we introduce the series

$$\phi(x) = \sum_{n=0}^{\infty} t_n x^n. \tag{1.4}$$

From (1.3) it is easy to see that the series given by (1.4) is convergent for $|x| < 1$ and for $x = 1$, it is also convergent. Porwal [14] introduce generalized distribution series as

$$K_\phi(z) = z + \sum_{n=2}^{\infty} \frac{t_{n-1}}{S} z^n. \tag{1.5}$$

Now, we define

$$TK_\phi(z) = 2z - K_\phi(z) = z - \sum_{n=2}^{\infty} \frac{t_{n-1}}{S} z^n. \tag{1.6}$$

We define the convolution (or Hadamard product) of two functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in \mathbb{U}). \tag{1.7}$$

Next, we introduce the convolution operator $K_\phi(f, z)$ for functions f of the form (1.1) as follows

$$K_\phi(f, z) = K_\phi(z) * f(z),$$

$$K_\phi(f,z) = z + \sum_{n=2}^{\infty} \frac{a_n t_{n-1}}{S} z^n. \tag{1.8}$$

Motivated by results on connections between various subclasses of analytic univalent functions by using generalized Bessel functions [8, 15], hypergeometric functions by [4, 5, 7, 9, 12, 20, 21, 23, 24], Wright's hypergeometric functions [18], Poisson distribution series [10], Hypergeometric distribution series [1, 16, 17], Binomial distribution series [11], we obtain necessary and sufficient condition for functions $T_\phi(z)$ in $\mathcal{TG}(\lambda,\alpha)$ and $\mathcal{TK}(\lambda,\alpha)$. Finally, we estimate certain inclusion relations between the classes $\Re^\tau(A,B)$ and $\mathcal{K}(\lambda,\alpha)$.

2. Main Results

To prove the main results, we need the following Lemmas.

Lemma 1 ([6]). *A function $f \in \Re^\tau(A,B)$ is of form (1.1), then*

$$|a_n| \le (A-B)\frac{|\tau|}{n}, \quad n \in \mathbb{N}\backslash\{1\}. \tag{2.1}$$

The bound given in (2.1) is sharp.

Lemma 2 ([25]). *A function $f \in \mathcal{A}$ belongs to the class $\mathcal{G}(\lambda,\alpha)$ if*

$$\sum_{n=2}^{\infty}(n+\lambda n(n-1)-\alpha)|a_n| \le 1-\alpha. \tag{2.2}$$

Lemma 3 ([25]). *A function $f \in \mathcal{A}$ belongs to the class $\mathcal{K}(\lambda,\alpha)$ if*

$$\sum_{n=2}^{\infty} n(n+\lambda n(n-1)-\alpha)|a_n| \le 1-\alpha. \tag{2.3}$$

Further we can easily prove that the conditions are also necessary if $f \in \mathcal{T}$.

Lemma 4. *A function $f \in \mathcal{T}$ belongs to the class $\mathcal{TG}(\lambda,\alpha)$ if and only if (2.2) is satisfied.*

Lemma 5. *A function $f \in \mathcal{T}$ belongs to the class $\mathcal{TK}(\lambda,\alpha)$ if and only if (2.3) is satisfied.*

Theorem 6. *Let $TK_\phi(z)$ be of the form (1.6) is in the class $\mathcal{TG}(\lambda,\alpha)$ if and only if*

$$\lambda\phi''(1) + (1+2\lambda)\phi'(1) + (1-\alpha)\phi(1) \le (1-\alpha)(S+\phi(0)). \tag{2.4}$$

Proof. To prove that $TK_\phi(z) \in \mathcal{TG}(\lambda,\alpha)$, from Lemma 4 we have to show that

$$\sum_{n=2}^{\infty}(n+\lambda n(n-1)-\alpha)\frac{t_{n-1}}{S} \le 1-\alpha.$$

By the given hypothesis, we see that

$$\begin{aligned}
& \sum_{n=2}^{\infty}(n+\lambda n(n-1)-\alpha)\frac{t_{n-1}}{S} \\
= \ & \sum_{n=2}^{\infty}[\lambda(n-1)(n-2)+(1+2\lambda)(n-1)+(1-\alpha)]\times\frac{t_{n-1}}{S} \\
= \ & \frac{1}{S}\left[\sum_{n=2}^{\infty}[\lambda(n-1)(n-2)t_{n-1}+(1+2\lambda)(n-1)t_{n-1}+(1-\alpha)t_{n-1}]\right] \\
= \ & \frac{1}{S}\left[\sum_{n=1}^{\infty}[\lambda n(n-1)t_n+(1+2\lambda)nt_n+(1-\alpha)t_n]\right] \\
= \ & \frac{1}{S}\left[\lambda\phi''(1)+(1+2\lambda)\phi'(1)+(1-\alpha)(\phi(1)-\phi(0))\right] \\
\leq \ & 1-\alpha.
\end{aligned}$$

This completes the proof of Theorem 6. ■

Theorem 7. *Let $TK_\phi(z)$ be of the form (1.6) is in the class $\mathcal{TK}(\lambda,\alpha)$, if and only if it satisfies the condition*

$$\lambda\phi'''(1)+(1+5\lambda)\phi''(1)+(3+4\lambda-\alpha)\phi'(1)+(1-\alpha)\phi(1)\leq(1-\alpha)(S+\phi(0)). \tag{2.5}$$

Proof. To prove that $TK_\phi(z)\in\mathcal{TK}(\lambda,\alpha)$, from Lemma 5 we have to show that

$$\sum_{n=2}^{\infty}n(n+\lambda n(n-1)-\alpha)\frac{t_{n-1}}{S}\leq 1-\alpha.$$

By the given hypothesis, we see that

$$\begin{aligned}
& \sum_{n=2}^{\infty}n(n+\lambda n(n-1)-\alpha)\frac{t_{n-1}}{S} \\
= \ & \sum_{n=2}^{\infty}\Big[\lambda(n-1)(n-2)(n-3)+(1+5\lambda)(n-1)(n-2) \\
& +(3+4\lambda-\alpha)(n-1)+(1-\alpha)\Big]\frac{t_{n-1}}{S} \\
= \ & \frac{1}{S}\sum_{n=2}^{\infty}\Big[\lambda(n-1)(n-2)(n-3)t_{n-1}+(1+5\lambda)(n-1)(n-2)t_{n-1} \\
& +(3+4\lambda-\alpha)(n-1)t_{n-1}+(1-\alpha)t_{n-1}\Big] \\
= \ & \frac{1}{S}\left[\sum_{n=1}^{\infty}[\lambda n(n-1)(n-2)t_n+(1+5\lambda)n(n-1)t_n+(3+4\lambda-\alpha)nt_n+(1-\alpha)t_n]\right] \\
= \ & \frac{1}{S}\left[\lambda\phi'''(1)+(1+5\lambda)\phi''(1)+(3+4\lambda-\alpha)\phi'(1)+(1-\alpha)(\phi(1)-\phi(0))\right] \\
\leq \ & 1-\alpha.
\end{aligned}$$

This establishes the proof of Theorem 7. ■

Theorem 8. *If $f\in\Re^\tau(A,B)(\tau\in\mathbb{C}\backslash\{0\},-1\leq B<A\leq 1)$ and the inequality*

$$\frac{(A-B)|\tau|}{S}\left[\lambda\phi''(1)+(1+2\lambda)\phi'(1)+(1-\alpha)(\phi(1)-\phi(0))\right]\leq 1-\alpha.$$

is satisfied, then $\mathcal{K}_\phi(f,z) \in \mathcal{K}(\lambda,\alpha)$.

Proof. Let f be of the form (1.1) belong to the class $\Re^\tau(A,B)$. Then by virtue of Lemma 3, it suffices to show that

$$\sum_{n=2}^{\infty} n(n^2\lambda + n(1-\lambda) - \alpha)\frac{t_{n-1}}{S}|a_n| \leq 1-\alpha.$$

By Lemma 1 and given hypothesis, we see that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n+\lambda n(n-1)-\alpha)\frac{t_{n-1}}{S}|a_n| \\
&= \frac{(A-B)|\tau|}{S}\left[\sum_{n=2}^{\infty}\left[\lambda(n-1)(n-2)+(1+2\lambda)(n-1)+(1-\alpha)\right]t_{n-1}\right] \\
&= \frac{(A-B)|\tau|}{S}\left[\sum_{n=2}^{\infty}\left[\lambda(n-1)(n-2)t_{n-1}+(1+2\lambda)(n-1)t_{n-1}+(1-\alpha)t_{n-1}\right]\right] \\
&= \frac{(A-B)|\tau|}{S}\left[\sum_{n=1}^{\infty}\left[\lambda n(n-1)t_n+(1+2\lambda)nt_n+(1-\alpha)t_n\right]\right] \\
&= \frac{(A-B)|\tau|}{S}\left[\lambda\phi''(1)+(1+2\lambda)\phi'(1)+(1-\alpha)(\phi(1)-\phi(0))\right] \\
&\leq 1-\alpha.
\end{aligned}$$

This completes the proof of Theorem 8. ■

Theorem 9. *Let*

$$TG_\phi(f,z) = \int_0^z \frac{TG_\phi(f,t)}{t})dt$$

is in $\mathcal{TK}(\lambda,\alpha)$ *if and only if* (2.4) *is satisfied.*

Proof. Since

$$TG_\phi(f,z) = z - \sum_{n=2}^{\infty}\frac{t_{n-1}}{S}\frac{z^n}{n}$$

by Lemma 5, we need only to show that

$$\sum_{n=2}^{\infty} n(n^2\lambda + n(1-\lambda) - \alpha)\frac{t_{n-1}}{nS} \leq 1-\alpha.$$

By the given hypothesis, we see that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n+\lambda n(n-1)-\alpha)\frac{t_{n-1}}{nS} \\
&= \sum_{n=2}^{\infty}\left[\lambda(n-1)(n-2)+(1+2\lambda)(n-1)+(1-\alpha)\right]\frac{t_{n-1}}{S} \\
&= \frac{1}{S}\left[\sum_{n=2}^{\infty}\left[\lambda(n-1)(n-2)t_{n-1}+(1+2\lambda)(n-1)t_{n-1}+(1-\alpha)t_{n-1}\right]\right] \\
&= \frac{1}{S}\left[\sum_{n=1}^{\infty}\left[\lambda n(n-1)t_n+(1+2\lambda)nt_n+(1-\alpha)t_n\right]\right] \\
&= \frac{1}{S}\left[\lambda\phi''(1)+(1+2\lambda)\phi'(1)+(1-\alpha)(\phi(1)-\phi(0))\right] \\
&\leq 1-\alpha.
\end{aligned}$$

This completes the proof of Theorem 9. ■

Remark 2. *If we take $t_n = \frac{m^n}{n!}$, then we obtain the corresponding results of Murugusundaramoorthy et al. [10].*

Acknowledgement. The author is thankful to the referee for his/her valuable comments and observations which helped in improving the paper.

References

[1] M. S. Ahmad, Q. Mehmood, W. Nazeer and A. U. Haq, An application of a Hypergeometric distribution series on certain analytic functions, Sci. Int., 27(2015), 2989–2992.

[2] Divya Bajpai, A Study of Univalent Functions Associated with Distribution Series and q-Calculus, M. Phil. Dissertation, CSJM University, Kanpur, India, 2016.

[3] A. Baricz, Generalized Bessel Functions of the First Kind, Lecture Notes in Math., Vol. 1994 , Springer-Verlag, 2010.

[4] N. Bohra and V. Ravichandran, On confluent hypergeometric functions and generalized Bessel functions, Anal. Math., 43(2017), 1–13.

[5] B. C. Carleson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, SIAM J. Math. Anal., 15(1984), 737–745.

[6] K. K. Dixit and S. K. Pal, On a class of univalent functions related to complex order, Indian J. Pure. Appl. Math., 26(1995), 889–896.

[7] S. Miller and P. T. Mocanu, Univalence of Gaussian and confluent hypergeometric functions, Proc. Amer. Math. Soc., 110(1990), 333–342.

[8] S. R. Mondal and A. Swaminathan, Geometric properties of Generalized Bessel functions, Bull. Malaysian Math. Sci. Soc., 35(2012), 179–194.

[9] A. O. Mostafa, A study on starlike and convex properties for hypergeometric functions, J. Inequal. Pure Appl. Math., 10(2009), Art. 87, 1–16.

[10] G. Murugusundaramoorthy, K. Vijaya and Saurabh Porwal, Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series, Hacettepe J. Math. Stat., 45(2016), 1101–1107.

[11] W. Nazeer, Q. Mehmood, S. M. Kang and A. U. Haq, An application of a Binomial distribution series on certain analytic functions, J. Comput. Anal. Appl., 26(2019), 11–17.

[12] S. Ponnusamy and F. Rønning, Starlikeness properties for convolution involving hypergeometric series, Ann. Univ. Mariae Curie-Sklodowska L.H.1, 16(1998), 141–155.

[13] S. Porwal, An application of a Poisson distribution series on certain analytic functions, J. Complex Anal., Vol.(2014), Art. ID 984135, 1–3.

[14] S. Porwal, Generalized distribution and its geometric properties associated with univalent functions, J. Complex Anal., Vol.(2018), Art. ID 8654506, 1–5.

[15] Saurabh Porwal and Moin Ahmad, Some sufficient condition for generalized Bessel functions associated with conic regions, Vietnam J. Math., 43(2015), 163–172.

[16] Saurabh Porwal and Ankita Gupta, Some properties of convolution for hypergeometric distribution type series on certain analytic univalent functions, Acta Univ. Apul., 56(2018), 69–80.

[17] Saurabh Porwal and Shivam Kumar, Confluent hypergeometric distribution and its applications on certain classes of univalent functions, Afr. Mat., 28(2017), 1–8.

[18] R. K. Raina, On univalent and starlike Wright's hypergeometric functions, Rend. Sem. Mat. Univ. Padova, 95(1996), 11–22.

[19] M. S. Robertson, On the theory of univalent functions, Ann. Math., 37(1936), 374–408.

[20] S. Ruscheweyh and V. Singh, On the order of starlikeness of hypergeometric functions, J. Math. Anal. Appl., 113(1986), 1–11.

[21] H. Silverman, Starlike and convexity properties for hypergeometric functions, J. Math. Anal. Appl., 172(1993), 574–581.

[22] H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51(1975), 109–116.

[23] H. M. Srivastava, G. Murugusundaramoorthy and S. Sivasubramanian, Hypergeometric functions in the parabolic starlike and uniformly convex domains, Integr. Transf. Spec. Func., 18(2007), 511–520.

[24] A. Swaminathan, Certain Sufficiency conditions on Gaussian hypergeometric functions, J. Ineq. Pure Appl. Math., 5(2004), Art. 83, 1–10.

[25] T. Thulasiram, K. Suchithra, T. V. Sudharsan and G. Murugusundaramoorthy, Some inclusion results associated with certain subclass of analytic functions involving Hohlov operator, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math., 108(2014), 711–720.