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Computing The Upper Bounds For The Metric Dimension Of Cellulose Network*

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Abstract

Let $G = (V, E)$ be a connected graph and $d(x, y)$ be the distance between the vertices x and y in G. A set of vertices W resolves a graph G if every vertex is uniquely determined by its vector of distances to the vertices in W. A metric dimension of G is the minimum cardinality of a resolving set of G and is denoted by $dim(G)$. In this paper we study three dimensional chemical structure of cellulose network and then we converted it into planar chemical structure, consequently we obtained cellulose network graphs denoted by CL_n^k. We prove that $dim(CL_n^k) \leq 4$ in certain cases.

1 Introduction

In this paper we constructed a planar chemical structure from three dimensional chemical structure of cellulose, consequently we obtained cellulose network graphs and calculated its metric dimension. We did this due to the importance of networks and metric dimension in daily life. We have treated the structure of cellulose as an example of multiprocessor networks. To the extent of our knowledge, no such work has been carried out previously.

These days much attention has been directed towards Multiprocessor interconnection networks, which are often required to connect thousands of homogeneously replicated processor-memory pairs [6, 7], each of which is called a processing node. Instead of using a shared memory, all synchronization and communication between processing nodes for program execution is often done via message passing. Multiprocessor interconnection networks have recently become significant due to the availability of inexpensive, powerful microprocessors and memory chips [5, 16, 20, 21].

Metric dimension is a parameter that has appeared in various applications of graph theory, as diverse as, pharmaceutical chemistry [3, 4], robot navigation [12] and combinatorial optimization [14]. A chemical compound can be represented by more than one

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suggested structures but only one of them, which expresses the physical and chemical properties of compound, is acceptable. The chemists require mathematical representation for a set of chemical compounds in a way that gives distinct representations to distinct compounds. The structure of a chemical compound can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Thus, a graph theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. This is the subject of the papers [1, 10].

In a connected graph $G(V, E)$ where V is the set of vertices and E is the set of edges, the distance $d(u, v)$ between two vertices $u, v \in V$ is the length of shortest path between them. Let $W = \{w_1, w_2, ..., w_k\}$ be an order set of vertices of G and let v be a vertex of G. The representation $r(v/W)$ of v with respect to W is the k-tuple $(d(v, w_1), d(v, w_2), d(v, w_3), ..., d(v, w_k))$ where W is called a resolving set [4] or locating set [17] if every vertex of G is uniquely identified by its distances from the vertices of W, or equivalently, if distinct vertices of G have distinct representations with respect to W. A resolving set of minimum cardinality is called a basis for G and cardinality is the metric dimension of G, denoted by $dim(G)$ [8]. The concept of resolving set and metric basis have previously appeared in the literature [9, 10, 11].

For a given ordered set of vertices $W = \{w_1, w_2, ..., w_k\}$ of a graph G, the ith component of $r(v/W)$ is 0 if and only if $V = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x/W) \neq r(y/W)$ for each pair of distinct vertices $x, y \in V(G) \setminus W$.

Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension was introduced by slater in [15, 17] and studied independently by Harary and Melter in [11]. Application of this invariant to the navigation of robots in networks are discussed in [12] and application to chemistry in [4] while application to the problem of pattern recognition and image processing, some of which involve the use of hierarchical data structures, are given in [10].

Let F be a family of connected graphs $G_n : F = (G_n)_{n\geq 1}$ depending on n and $|V(G)| = \varphi(n)$, $\lim_{n\to\infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $dim(G_n) \leq C$ for every $n \geq 1$ then we shall say that F has bounded metric dimension ; otherwise F has unbounded metric dimension.

If all graphs in F have the same metric dimension (which does not depend on n), F is called a family with constant metric dimension [18]. A connected graph G has $dim(G) = 1$ if and only if G is a path [4]; cycle C_n have metric dimension 2 for every $n \geq 3$ also honeycomb networks [13] have metric dimension 3.

Other families of graphs with unbounded metric dimension are regular bipartite graphs [9], wheel graph and jahangir graph [19].

2 Cellulose Networks

Cellulose is among the most abundant organic compounds found in nature. It belongs to polysaccharide class of carbohydrates. It is a major component of tough cell walls that surround plant cells thus making plant stems, leaves, and branches rigid and strong. The bonds between cellulose molecules are very strong, which makes cellulose very hard to break down. The rigid structure of cellulose allow plants to stand upright, and without the strength of cellulose, we wouldn't have lumber, paper, or cotton fabric. Chemically it is the most abundant organic compound found on the earth with the general formula $(C_6H_{10}O_5)_n$ consisting of over $3,000$ D-glucose units which are joined by $\beta(1 \longrightarrow 4)$ glycosidic bonds. Cellulose is a straight chain polymer: unlike starch, no coiling or branching occurs, and the molecule adopts an extended and rather stiff rod-like conformation, aided by the equatorial conformation of the glucose residues.

The multiple hydroxyl groups on the glucose from one chain form hydrogen bonds with oxygen atoms on the same or on a neighbor chain, holding the chains firmly together side-by-side and forming microfibrils with high tensile strength.

Figure 1 represents one unit of cellulose molecule and figure 2 represents three dimensional cellulose network. We converted this three dimensional network into equivalent planar network in which represents oxygen atom, HO or OH represent hydroxyl groups and their bonding is expressed in figure 3. Taking atoms as vertices and bonds as edges we can draw cellulose network graph represented as CL_n^k in particular CL_9^{16} is expressed in figure 4, suppose we have p hexagons in the hexagonal chain and q hexagonal chains in the network then $n = p+1$ and $k = 6q-2$, $p, q \in N$ see figure 3.

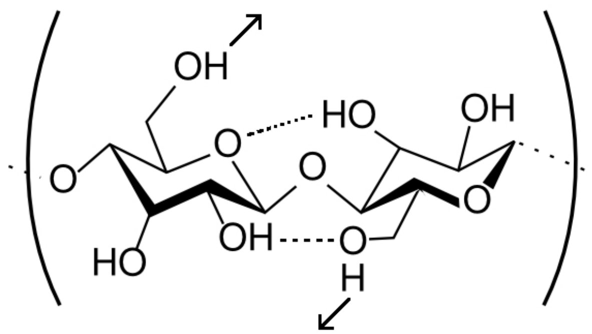

Figure 1: Cellulose.

3 Main Results

First, we have the following definition.

DEFINITION. Cellulose network graphs is the combination of hexagons, octagons and decagons. Take a hexagon (CL_2^4) and adding two hexagons to the two opposite boundary edges of hexagon then we shall get a chain of three hexagons (CL_4^4). Similarly we can extend this chain upto any number of hexagons (CL_n^4)in which every hexagon is sharing two opposite edges and four vertices with two neighboring hexagons, we will call remaining two vertices as free vertices.

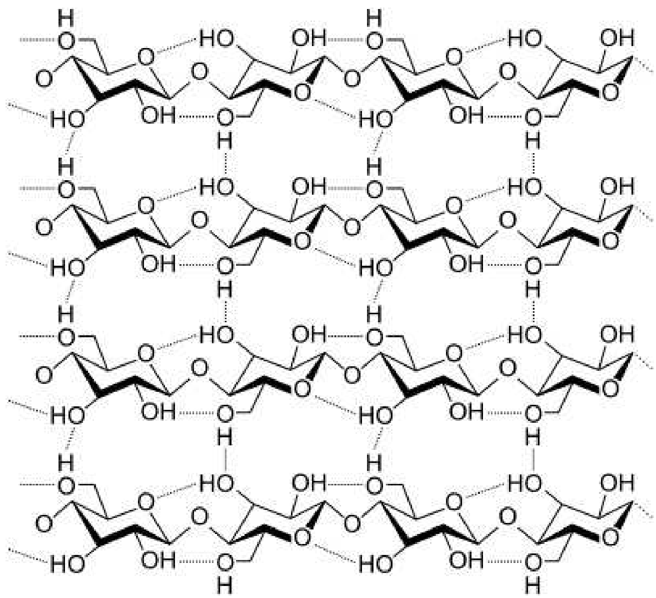

Figure 2: 3 Dimensional Cellulose Network.

Suppose our chain of hexagons is fixed from left and moving toward right. We shall construct the graph by adding a pendant to each free vertex on one side of the chain except the vertices at $4m-3, m \in N$ position and we shall call it as upper side of hexagonal chain, we shall also add a pendant to each free vertex on second side of the hexagonal chain except the vertices at $4m-1, m \in N$ position and we shall call it as a as lower side of hexagonal chain consequently this hexagonal chain is called chain with pendant. Pendants at upper and lower sides of hexagonal chain are called upper and lower pendants respectively. Pendant at i^{th} vertex will be called i^{th} pendant $1 \leq i \leq n$. Take a hexagonal chain with pendant and neglect all its lower pendants call it first chain put a second chain with pendant parallel to it having equal number of hexagons.

Join first lower pendant of second chain with second upper pendant of first chain (note that pendant chains has no i^{st} upper pendant and in general having no $(4m-1)th$ upper pendants as mention above) then joining every $2(2m-1)th$ and $4mth, m \in N$ lower pendants of 2^{nd} chain with $(4m-1)th, m \in N$ upper pendant of i^{st} chain and joining every $4mth$ and $2(2m+1)th$ upper pendants of i^{st} chain with $(4m+1)th, m \in N$ lower pendant of 2^{nd} chain. Now take a third hexagonal chain with pendant and put it parallel to second pendant chain, repeat the process on second and third pendant chain, then third and fourth pendant chain and so on. Finally we shall not add the upper pendant of last hexagonal pendant chain and hence we have the cellulose network graph CL_n^k. Figure 4 represents the graph CL_9^{16}.

Cellulose network graphs are an important class of graphs, which can be used in

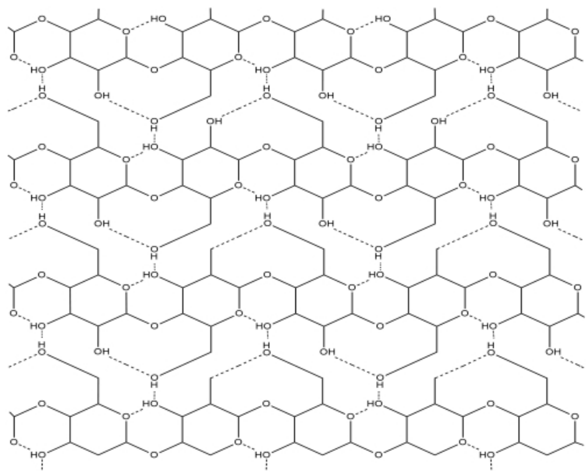

Figure 3: Cellulose planar network.

the design of local area networks [20]. We will represent vertices of cellulose network graph CL_n^k by v_i^j where $1 \le i \le n$ and $1 \le j \le k$.
It has $\frac{11nk-8k+4n+8}{12}$ vertices and $\frac{15nk-16k+16}{12}$ edges when n is even, $\frac{11nk-9k+4n+12}{12}$ vertices and $\frac{15nk-17k+20}{12}$ edges when n is odd. CL_n^k contains vertices of degree two or three so it is a bipartite graph. Now we present our main results on metric dimension of cellulose network graph CL_n^k. In certain cases we prove that $dim(G) \le 4$.

@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@ THEOREM 1. For $G \cong$

CL_n^k where $k = 2n-2$ when $n = 12p+3, p \in N$ and $k = 2n-4$ when $n = 12p+7, p \in N$, we have $dim(G) \le 4$.

PROOF. Let $W = \{V_1^1, V_{n-1}^1, V_1^{k-7}, V_{n-1}^k\}$ be the resolving set of G. Then

$$r(V_i^j/W) = (a_i^j, b_i^j, c_i^j, d_i^j), \quad 1 \le i \le n \text{ and } 1 \le j \le k,$$

where
$a_i^j = 2i + j - 3, 1 \le i \le 3, 1 \le j \le k, j = 6r - 5, r \in N,$

$$a_i^j = \begin{cases} \frac{1}{3}(j + 6i - 7), & i = 4t, t \in \{1, 2\}, 1 \le j \le \frac{3i}{2} + 1, j = 6r - 5, r \in N, \\ j + \frac{i}{2} - 1, & i = 4t, t \in \{1, 2\}, \frac{3i}{2} + 7 \le j \le k, j = 6r + \frac{3i}{2} + 1, r \in N, \end{cases}$$

$$a_i^j = \begin{cases} 2i + \frac{1}{3}(j - 7), & 5 \le i \le 7, 1 \le j \le 7, j = 6r - 5, r \in N, \\ 2i + j - 9, & 5 \le i \le 7, 13 \le j \le k, j = 6r + 7, r \in N, \end{cases}$$

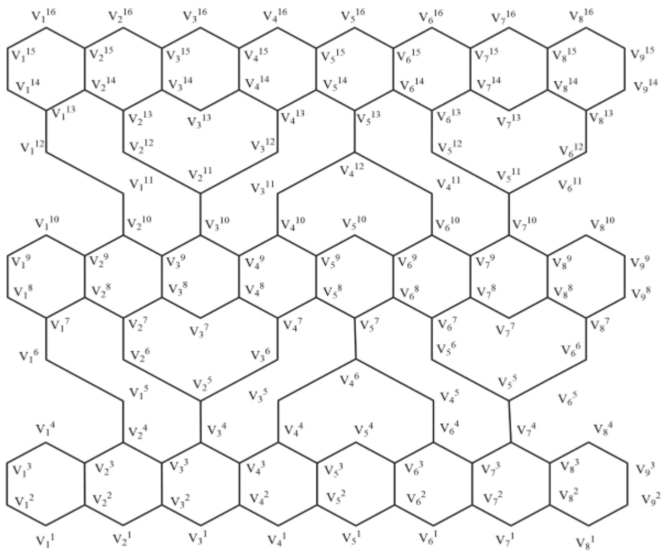

Figure 4: CL_9^{16}.

$$a_i^j = \begin{cases} 2i + \frac{1}{3}(j-7), & m-2 \le i \le m, m = 7+4t, t \in N, 1 \le j \le \frac{3m-19}{2}, \\ & j = 6r-5, r \in N, \\ 2i + \frac{2}{3}(j - \frac{3m+7}{4}), & m-2 \le i \le m, m = 7+4t, t \in N, \\ & \frac{3m-7}{2} \le j \le 3m-14, j = 6r + \frac{3m-19}{2}, r \in N, \\ 2i + j - \frac{3}{2}(m-1), & m-2 \le i \le m, m = 7+4t, t \in N, 3m-8 \le j \le k, \\ & j = 6r + 3m - 14, r \in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{1}{3}(j+6i-7), & i = 8+4t, t \in N, 1 \le j \le \frac{3i}{2} - 5, \\ & j = 6r-5, r \in N, \\ \frac{2}{3}(j + \frac{9i}{4} - 4), & i = 8+4t, t \in N, \\ & \frac{3i}{2} \le j \le 3i-11, j = 6r + \frac{3i}{2} - 5, r \in N, \\ j + \frac{i}{2} - 1, & i = 8+4t, t \in N, 3i-5 \le j \le k, \\ & j = 6r + 3i - 11, r \in N, \end{cases}$$

$$a_i^j = j-1, i \in \{1,2\}, 2 \le j \le k, j = 6r-4, r \in N,$$

$$a_i^j = j+1, i = 3, 2 \le j \le k, j = 6r-4, r \in N,$$

$$a_i^j = \begin{cases} \frac{1}{3}(2j+11), & i = 4, 2 \le j \le 8, j = 6r-4, r \in N, \\ j+1, & i = 4, 14 \le j \le k, j = 6r+8, r \in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{1}{3}(j+19), & i = 5, 2 \le j \le 8, j = 6r-4, r \in N, \\ j+1, & i = 5, 14 \le j \le k, j = 6r+8, r \in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{1}{3}(j+25), & i = 6, 2 \le j \le 8, j = 6r-4, r \in N, \\ j+1, & i = 6, 14 \le j \le k, j = 6r+8, r \in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{1}{3}(j+31), & i=7, 2 \le j \le 8, j=6r-4, r \in N, \\ j+3, & i=7, 14 \le j \le k, j=6r+8, r \in N, \end{cases}$$

$$a_i^j = j-1, i \in \{1,2\}, 3 \le j \le k, j=6r-3, r \in N,$$

$$a_i^j = j+1, i=3, 3 \le j \le k, j=6r-3, r \in N,$$

$$a_i^j = \begin{cases} \frac{2}{3}(j+6), & i=4, 3 \le j \le 9, j=6r-3, r \in N, \\ j+1, & i=4, 15 \le j \le k, j=6r+9, r \in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{1}{3}(j+21), & i=5, 3 \le j \le 9, j=6r-3, r \in N, \\ j+1, & i=5, 15 \le j \le k, j=6r+9, r \in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{1}{3}(j+27), & i=6, 3 \le j \le 9, j=6r-3, r \in N, \\ j+1, & i=5, 15 \le j \le k, j=6r+9, r \in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{1}{3}(j+33), & i=7, 3 \le j \le 9, j=6r-3, r \in N, \\ j+3, & i=7, 15 \le j \le k, j=6r+9, r \in N, \end{cases}$$

$$a_i^j = j-1, i \in \{1,2\}, 4 \le j \le k, j=6r-2, r \in N,$$

$$a_i^j = j+1, i=3, 4 \le j \le k, j=6r-2, r \in N,$$

$$a_i^j = \begin{cases} \frac{1}{3}(2j+13), & i=4, 4 \le j \le 10, j=6r-2, r \in N, \\ j+1, & i=4, 16 \le j \le k, j=6r+10, r \in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{1}{3}(j+35), & i=7, 4 \le j \le 10, j=6r-2, r \in N, \\ j+3, & i=7, 16 \le j \le k, j=6r+10, r \in N, \end{cases}$$

$$a_i^j = j-1, i=1, j=6r-1, r \in N,$$

$$a_i^j = j+1, i=2, j=6r-1, r \in N,$$

$$a_i^j = \begin{cases} \frac{1}{3}(2j+14), & i=3, 5 \le j \le 11, j=6r-1, r \in N, \\ j+1, & i=3, 17 \le j \le k, j=6r+11, r \in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{2}{3}(j+10), & i=4, 5 \le j \le 17, j=6r-1, r \in N, \\ j+1, & i=4, 23 \le j \le k, j=6r+17, r \in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{1}{3}(j+37), & i=5, 5 \le j \le 11, j=6r-1, r \in N, \\ j+3, & i=5, 17 \le j \le k, j=6r+11, r \in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{1}{3}(j+8i-3), & i=5+3t, t \in N, 5 \le j \le 2i+1, \\ & j=6r-1, r \in N, \\ \frac{2}{3}(3i+j-2), & i=5+3t, t \in N, \\ & 2i+7 \le j \le 4i-3, j=6r+2i+1, r \in N, \\ j+\frac{2i}{3}-\frac{1}{3}, & i=5+3t, t \in N, 4i+3 \le j \le k, \\ & j=6r+4i-3, r \in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{1}{3}(j+8i-5), & i=3+3t, t\in N, 5\le j\le 2i-1, \\ & j=6r-1, r\in N, \\ \frac{2}{3}(3i+j-2), & i=3+3t, t\in N, \\ & 2i+5\le j\le 4i-1, j=6r+2i-1, r\in N, \\ j+\frac{2i}{3}-1, & i=3+3t, t\in N, 4i+5\le j\le k, \\ & j=6r+4i-1, r\in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{1}{3}(j+8i-7), & i=4+3t, t\in N, 5\le j\le 2i-3, \\ & j=6r-1, r\in N, \\ \frac{2}{3}(3i+j-2), & i=4+3t, t\in N, \\ & 2i+3\le j\le 4i+1, j=6r+2i-3, r\in N, \\ j+\frac{2i}{3}-\frac{5}{3}, & i=4+3t, t\in N, 4i+7\le j\le k, \\ & j=6r+4i+1, r\in N, \end{cases}$$

$$a_i^j = j-1, i=1, 6\le j\le k, j=6r, r\in N,$$

$$a_i^j = j+1, i\in\{2,3\}, 6\le j\le k, j=6r, r\in N,$$

$$a_i^j = \begin{cases} \frac{2j}{3}+5, & i=4, 6\le j\le 12, j=6r, r\in N, \\ j+1, & i=4, 18\le j\le k, j=6r+12, r\in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{2j}{3}+9, & i=5, 6\le j\le 18, j=6r, r\in N, \\ j+3, & i=5, 24\le j\le k, j=6r+18, r\in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{j}{3}+13, & i=6, 6\le j\le 12, j=6r, r\in N, \\ j+3, & i=6, 18\le j\le k, j=6r+12, r\in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{1}{3}(j+8i-7), & i=5+3t, t\in N, 6\le j\le 2i-4, \\ & j=6r, r\in N, \\ \frac{2j}{3}+2i-1, & i=5+3t, t\in N, \\ & 2i+2\le j\le 4i-2, j=6r+2i-4, r\in N, \\ j+\frac{2i}{3}-\frac{1}{3}, & i=5+3t, t\in N, 4i+4\le j\le k, \\ & j=6r+4i-2, r\in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{1}{3}(2j+8i-11), & i=4+3t, t\in N, 6\le j\le 2i-2, \\ & j=6r, r\in N, \\ \frac{2j}{3}+2i-3, & i=4+3t, t\in N, \\ & 2i+4\le j\le 4i-4, j=6r+2i-2, r\in N, \\ j+\frac{2i}{3}-\frac{5}{3}, & i=4+3t, t\in N, 4i+2\le j\le k, \\ & j=6r+4i-4, r\in N, \end{cases}$$

$$a_i^j = \begin{cases} \frac{1}{3}(j+8i-9), & i=6+3t, t\in N, 6\le j\le 2i, \\ & j=6r, r\in N, \\ \frac{2j}{3}+2i-3, & i=6+3t, t\in N, \\ & 2i+6\le j\le 4i-6, j=6r+2i, r\in N, \\ j+\frac{2i}{3}-1, & i=6+3t, t\in N, 4i\le j\le k, \\ & j=6r+4i-6, r\in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{4j}{3}-\frac{4}{3}, & i=n-1, 1\le j\le 7, j=6r-5, r\in N, \\ j+1, & i=n-1, 13\le j\le k, j=6r+7, r\in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{2j}{3} + \frac{4}{3}, & i = n-2, 1 \leq j \leq 7, j = 6r-5, r \in N, \\ j-1, & i = n-2, 13 \leq j \leq k, j = 6r+7, r \in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{4j}{3} - \frac{5}{3}, & i = n, 2 \leq j \leq 8, j = 6r-4, r \in N, \\ j+1, & i = n, 14 \leq j \leq k, j = 6r+8, r \in N, \end{cases}$$

$$b_i^j = j-1, i = n-1, 2 \leq j \leq k, j = 6r-4, r \in N,$$

$$b_i^j = j-1, i = n-1, 3 \leq j \leq k, j = 6r-3, r \in N,$$

$$b_i^j = \begin{cases} \frac{2j}{3} + 2, & i = n-2, 3 \leq j \leq 9, j = 6r-3, r \in N, \\ j-1, & i = n-2, 15 \leq j \leq k, j = 6r+9, r \in N, \end{cases}$$

$$b_i^j = j-1, i = n-1, n-2, 4 \leq j \leq k, j = 6r-2, r \in N,$$

$$b_i^j = \begin{cases} \frac{2j}{3} + \frac{7}{3}, & i = n-3, 4 \leq j \leq 10, j = 6r-2, r \in N, \\ j-1, & i = n-3, 16 \leq j \leq k, j = 6r+10, r \in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{j}{3} + \frac{29}{3}, & i = n-6, 4 \leq j \leq 10, j = 6r-2, r \in N, \\ j+1, & i = n-6, 16 \leq j \leq k, j = 6r+10, r \in N, \end{cases}$$

$$b_i^j = j-1, i = \frac{3n-5}{4}, 5 \leq j \leq k, j = 6r-1, r \in N,$$

$$b_i^j = \begin{cases} \frac{2j}{3} + \frac{8}{3}, & i = \frac{3n-9}{4}, 5 \leq j \leq 11, j = 6r-1, r \in N, \\ j-1, & i = \frac{3n-9}{4}, 17 \leq j \leq k, j = 6r+11, r \in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{2j}{3} + \frac{26}{3}, & i = \frac{3n-21}{4}, 5 \leq j \leq 23, j = 6r-1, r \in N, \\ j+1, & i = \frac{3n-21}{4}, 29 \leq j \leq k, j = 6r+23, r \in N, \end{cases}$$

$$b_i^j = j-1, i = \frac{3n-5}{4}, 6 \leq j \leq k, j = 6r, r \in N,$$

$$b_i^j = \begin{cases} \frac{2j}{3} + 5, & i = \frac{3n-9}{4}, \frac{3n-13}{4}, 6 \leq j \leq 12, j = 6r, r \in N, \\ j+1, & i = \frac{3n-9}{4}, \frac{3n-13}{4}, 18 \leq j \leq k, j = 6r+12, r \in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{2j}{3} + 7, & i = \frac{3n-17}{4}, 6 \leq j \leq 18, j = 6r, r \in N, \\ j+1, & i = \frac{3n-17}{4}, 24 \leq j \leq k, j = 6r+18, r \in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{2j}{3} + 11, & i = \frac{3n-21}{4}, 6 \leq j \leq 24, j = 6r, r \in N, \\ j+3, & i = \frac{3n-21}{4}, 30 \leq j \leq k, j = 6r+24, r \in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{1}{3}(j - 8i + 6n - 5), & i = -1+3t, t \in N, i \leq \frac{3n-25}{4}, 6 \leq j \leq (\frac{3n}{2} - 2i - \frac{1}{2}), \\ & j = 6r, r \in N, \\ \frac{1}{3}(2j + \frac{9n}{2} - 6i - \frac{9}{2}), & i = -1+3t, t \in N, i \leq \frac{3n-25}{4} \\ & \frac{3n}{2} - 2i + \frac{11}{2} \leq j \leq 3n - 4i - 1, j = 6r + \frac{3n}{2} - 2i - \frac{1}{2}, r \in N, \\ j + \frac{n}{2} - \frac{2i}{3} - \frac{7}{6}, & i = -1+3t, t \in N, 3n - 4i + 5 \leq j \leq k, \\ & j = 6r + 3n - 4i - 1, r \in N, \end{cases}$$

$$b_i^j=\begin{cases}\frac{1}{3}(j-8i+6n-7), & i=-2+3t, t\in N, i\le\frac{3n-25}{4}, 6\le j\le(\frac{3n}{2}-2i-\frac{5}{2}),\\ & j=6r, r\in N,\\ \frac{1}{3}(2j+\frac{9n}{2}-6i-\frac{9}{2}), & i=-2+3t, t\in N, i\le\frac{3n-25}{4}\\ & \frac{3n}{2}-2i+\frac{7}{2}\le j\le 3n-4i+1, j=6r+\frac{3n}{2}-2i-\frac{5}{2}, r\in N,\\ j+\frac{n}{2}-\frac{2i}{3}-\frac{11}{6}, & i=-2+3t, t\in N, 3n-4i+7\le j\le k,\\ & j=6r+3n-4i+1, r\in N,\end{cases}$$

$$b_i^j=\begin{cases}\frac{1}{3}(j-8i+6n-3), & i=3t, t\in N, i\le\frac{3n-25}{4}, 6\le j\le(\frac{3n}{2}-2i-\frac{9}{2}),\\ & j=6r, r\in N,\\ \frac{1}{3}(2j+\frac{9n}{2}-6i+\frac{3}{2}), & i=3t, t\in N, i\le\frac{3n-25}{4},\\ & \frac{3n}{2}-2i+\frac{3}{2}\le j\le 3n-4i+3, j=6r+\frac{3n}{2}-2i-\frac{9}{2}, r\in N,\\ j+\frac{n}{2}-\frac{2i}{3}-\frac{1}{2}, & i=3t, t\in N, 3n-4i+9\le j\le k,\\ & j=6r+3n-4i+3, r\in N,\end{cases}$$

$$c_i^j=\begin{cases}6, & i=1, j=k-3,\\ k-j-7, & i=1, 1\le j\le k-9, j=6r-5, r\in N,\end{cases}$$

$$c_i^j=\begin{cases}8, & i=2, j=k-3,\\ \frac{2k}{3}-\frac{2j}{3}-2, & i=2, k-15\le j\le k-9, j=6r+k-21, r\in N,\\ k-j-7, & i=2, 1\le j\le k-21, j=6r-5, r\in N,\end{cases}$$

$$c_i^j=\begin{cases}10, & i=3, j=k-3,\\ \frac{k}{3}-\frac{j}{3}-\frac{7}{3}, & i=3, k-15\le j\le k-9, j=6r+k-21, r\in N,\\ k-j-7, & i=3, 1\le j\le k-21, j=6r-5, r\in N,\end{cases}$$

$$c_i^j=\begin{cases}8, & i=4, j=k-3,\\ \frac{k}{3}-\frac{j}{3}+5, & i=4, k-15\le j\le k-9, j=6r+k-21, r\in N,\\ k-j-7, & i=4, 1\le j\le k-21, j=6r-5, r\in N,\end{cases}$$

$$c_i^j=\begin{cases}10, & i=5, j=k-3,\\ \frac{k}{3}-\frac{j}{3}+7, & i=5, k-15\le j\le k-9, j=6r+k-21, r\in N,\\ k-j-5, & i=5, 1\le j\le k-21, j=6r-5, r\in N,\end{cases}$$

$$c_i^j=\begin{cases}7, & i=2, j=k-2,\\ \frac{2k}{3}-\frac{2j}{3}-\frac{7}{3}, & i=2, k-14\le j\le k-8, j=6r+k-20, r\in N,\\ k-j-7, & i=2, 2\le j\le k-20, j=6r-4, r\in N,\end{cases}$$

$$c_i^j=\begin{cases}9, & i=3, j=k-2,\\ \frac{k}{3}-\frac{j}{3}+\frac{7}{3}, & i=3, k-14\le j\le k-8, j=6r+k-20, r\in N,\\ k-j-7, & i=3, 2\le j\le k-20, j=6r-4, r\in N,\end{cases}$$

$$c_i^j=\begin{cases}9, & i=4, j=k-2,\\ \frac{k}{3}-\frac{j}{3}+\frac{13}{3}, & i=4, k-14\le j\le k-8, j=6r+k-20, r\in N,\\ k-j-7, & i=4, 2\le j\le k-20, j=6r-4, r\in N,\end{cases}$$

$$c_i^j=\begin{cases}9, & i=5, j=k-2,\\ \frac{k}{3}-\frac{j}{3}+\frac{19}{3}, & i=5, k-14\le j\le k-8, j=6r+k-20, r\in N,\\ k-j-5, & i=4, 2\le j\le k-20, j=6r-4, r\in N,\end{cases}$$

$$c_i^j = \begin{cases} 10, & i=4, j=k-1, \\ \frac{k}{3}-\frac{j}{3}+\frac{11}{3}, & i=4, k-13 \le j \le k-7, j=6r+k-19, r \in N, \\ k-j-7, & i=4, 3 \le j \le k-19, j=6r-3, r \in N, \end{cases}$$

$$c_i^j = \begin{cases} 9, & i=1, j=k, \\ k-j-5, & i=1, 4 \le j \le k-6, j=6r-2, r \in N, \end{cases}$$

$$c_i^j = \begin{cases} 9, & i=2, j=k, \\ \frac{k}{3}-\frac{j}{3}+1, & i=2, k-12 \le j \le k-6, j=6r+k-18, r \in N, \\ k-j-7, & i=2, 4 \le j \le k-18, j=6r-2, r \in N, \end{cases}$$

$$c_i^j = \begin{cases} 11, & i=3, j=k, \\ \frac{k}{3}-\frac{j}{3}+3, & i=3, k-12 \le j \le k-6, j=6r+k-18, r \in N, \\ k-j-7, & i=3, 4 \le j \le k-18, j=6r-2, r \in N, \end{cases}$$

$$c_i^j = \begin{cases} 11, & i=5, j=k, \\ \frac{k}{3}-\frac{j}{3}+7, & i=5, k-12 \le j \le k-6, j=6r+k-18, r \in N, \\ k-j-3, & i=2, 4 \le j \le k-18, j=6r-2, r \in N, \end{cases}$$

$$c_i^j = \begin{cases} 13, & i=6, j=k, \\ \frac{k}{3}-\frac{j}{3}+9, & i=6, k-18 \le j \le k-6, j=6r+k-24, r \in N, \\ k-j-5, & i=6, 4 \le j \le k-24, j=6r-2, r \in N, \end{cases}$$

$$c_i^j = \begin{cases} 4, & i=1, j=k-5, \\ k-j-7, & i=1, 5 \le j \le k-11, j=6r-1, r \in N, \end{cases}$$

$$c_i^j = \begin{cases} 6, & i=2, j=k-5, \\ \frac{2k}{3}-\frac{2j}{3}-\frac{4}{3}, & i=2, k-17 \le j \le k-11, j=6r+k-23, r \in N, \\ k-j-7, & i=2, 5 \le j \le k-23, j=6r-1, r \in N, \end{cases}$$

$$c_i^j = \begin{cases} 8, & i=3, j=k-5, \\ \frac{2k}{3}-\frac{2j}{3}+\frac{8}{3}, & i=3, k-17 \le j \le k-11, j=6r+k-23, r \in N, \\ k-j-5, & i=3, 5 \le j \le k-23, j=6r-1, r \in N, \end{cases}$$

$$c_i^j = \begin{cases} 10, & i=4, j=k-5, \\ \frac{k}{3}-\frac{j}{3}+\frac{25}{3}, & i=4, k-17 \le j \le k-11, j=6r+k-23, r \in N, \\ k-j-5, & i=4, 5 \le j \le k-23, j=6r-1, r \in N, \end{cases}$$

$$c_i^j = \begin{cases} 5, & i=1, j=k-4, \\ k-j-7, & i=1, 6 \le j \le k-10, j=6r, r \in N, \end{cases}$$

$$c_i^j = \begin{cases} 7, & i=2, j=k-4, \\ \frac{2k}{3}-\frac{2j}{3}-\frac{5}{3}, & i=2, k-16 \le j \le k-10, j=6r+k-22, r \in N, \\ k-j-7, & i=2, 6 \le j \le k-22, j=6r, r \in N, \end{cases}$$

$$c_i^j = \begin{cases} 7, & i=3, j=k-4, \\ \frac{2k}{3}-\frac{2j}{3}+\frac{1}{3}, & i=3, k-16 \le j \le k-10, j=6r+k-22, r \in N, \\ k-j-7, & i=3, 6 \le j \le k-22, j=6r, r \in N, \end{cases}$$

$$c_i^j = \begin{cases} 9, & i=4, j=k-4, \\ \frac{k}{3}-\frac{j}{3}+\frac{23}{3}, & i=4, k-16 \le j \le k-10, j=6r+k-22, r\in N, \\ k-j-5, & i=4, 6\le j\le k-22, j=6r, r\in N, \end{cases}$$

$$d_i^j = k-j, i=n-1, n-2, 1\le j\le k-3, j=6r-5, r\in N,$$

$$d_i^j = \begin{cases} \frac{2k}{3}-\frac{2j}{3}+3, & i=n-3, k-9\le j\le k-3, j=6r+k-15, r\in N, \\ k-j, & i=n-3, 1\le j\le k-15, j=6r-5, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+6, & i=n-4, k-9\le j\le k-3, j=6r+k-15, r\in N, \\ k-j, & i=n-4, 1\le j\le k-15, j=6r-5, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+8, & i=n-5, k-9\le j\le k-3, j=6r+k-15, r\in N, \\ k-j, & i=n-5, 1\le j\le k-15, j=6r-5, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+10, & i=n-6, k-9\le j\le k-3, j=6r+k-15, r\in N, \\ k-j+2, & i=n-6, 1\le j\le k-15, j=6r-5, r\in N, \end{cases}$$

$$d_i^j = k-j, i=n, n-1, 2\le j\le k-2, j=6r-4, r\in N,$$

$$d_i^j = \begin{cases} \frac{2k}{3}-\frac{2j}{3}+\frac{8}{3}, & i=n-2, k-8\le j\le k-2, j=6r+k-14, r\in N, \\ k-j, & i=n-2, 2\le j\le k-14, j=6r-4, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+\frac{16}{3}, & i=n-3, k-8\le j\le k-2, j=6r+k-14, r\in N, \\ k-j, & i=n-3, 2\le j\le k-14, j=6r-4, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+\frac{22}{3}, & i=n-4, k-8\le j\le k-2, j=6r+k-14, r\in N, \\ k-j, & i=n-4, 1\le j\le k-14, j=6r-4, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+\frac{28}{3}, & i=n-5, k-8\le j\le k-2, j=6r+k-14, r\in N, \\ k-j+2, & i=n-5, 2\le j\le k-14, j=6r-4, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{1}{3}(k-j+6n-6i-2), & i=4t, t\in N, k-\frac{3n}{2}+\frac{3i}{2}+\frac{5}{2}\le j\le k-2, \\ & j=6r+k-\frac{3n}{2}+\frac{3i}{2}-\frac{7}{2}, r\in N, \\ \frac{1}{3}(2k-2j+\frac{9n}{2}-\frac{9i}{2}-\frac{11}{2}), & i=4t, t\in N, k-3n+3i+1\le j\le k-\frac{3n}{2}+\frac{3i}{2}-\frac{7}{2}, \\ & j=6r+k-3n+3i-5, r\in N, \\ k-j+\frac{n}{2}-\frac{i}{2}-\frac{3}{2}, & i=4t, t\in N, 2\le j\le k-3n+3i-5, \\ & j=6r-4, r\in N, \end{cases}$$

$$d_i^j = k-j, i=n, n-1, 3\le j\le k-2, j=6r-3, r\in N,$$

$$d_i^j = \begin{cases} \frac{2k}{3}-\frac{2j}{3}+\frac{7}{3}, & i=n-2, k-7\le j\le k-1, j=6r+k-13, r\in N, \\ k-j, & i=n-2, 3\le j\le k-13, j=6r-3, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+\frac{14}{3}, & i=n-3, k-7\le j\le k-1, j=6r+k-13, r\in N, \\ k-j, & i=n-3, 3\le j\le k-13, j=6r-3, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+\frac{20}{3}, & i=n-4, k-7\le j\le k-1, j=6r+k-13, r\in N, \\ k-j, & i=n-4, 3\le j\le k-13, j=6r-3, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+\frac{26}{3}, & i=n-5, k-7\le j\le k-1, j=6r+k-13, r\in N, \\ k-j+2, & i=n-5, 3\le j\le k-13, j=6r-3, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{1}{3}(k-j+6n-6i-4), & i=-3+4t, t\in N, k-\frac{3n}{2}+\frac{3i}{2}+2\le j\le k-1,\\ & j=6r+k-\frac{3n}{2}+\frac{3i}{2}-4, r\in N,\\ \frac{1}{3}(2k-2j+\frac{9n}{2}-\frac{9i}{2}-2), & i=-3+4t, t\in N, k-3n+3i-1\le j\le k-\frac{3n}{2}+\frac{3i}{2}-4,\\ & j=6r+k-3n+3i-7, r\in N,\\ k-j+\frac{n}{2}-\frac{i}{2}-1, & i=-3+4t, t\in N, 3\le j\le k-3n+3i-7,\\ & j=6r-3, r\in N,\end{cases}$$

$$d_i^j = \begin{cases} \frac{1}{3}(k-j+6n-6i-4), & i=-1+4t, t\in N, k-\frac{3n}{2}+\frac{3i}{2}-1\le j\le k-1,\\ & j=6r+k-\frac{3n}{2}+\frac{3i}{2}-7, r\in N,\\ \frac{1}{3}(2k-2j+\frac{9n}{2}-\frac{9i}{2}-5), & i=-1+4t, t\in N, k-3n+3i-1\le j\le k-\frac{3n}{2}+\frac{3i}{2}-7,\\ & j=6r+k-3n+3i-7, r\in N,\\ k-j+\frac{n}{2}-\frac{i}{2}-2, & i=-1+4t, t\in N, 3\le j\le k-3n+3i-7,\\ & j=6r-3, r\in N,\end{cases}$$

$$d_i^j = k-j, i=n-1, 4\le j\le k, j=6r-2, r\in N,$$

$$d_i^j = k-j+2, i=n-2, 4\le j\le k, j=6r-2, r\in N,$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+4, & i=n-3, k-6\le j\le k, j=6r+k-12, r\in N,\\ k-j, & i=n-3, 4\le j\le k-12, j=6r-2, r\in N,\end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+6, & i=n-4, k-6\le j\le k, j=6r+k-12, r\in N,\\ k-j, & i=n-4, 4\le j\le k-12, j=6r-2, r\in N,\end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+8, & i=n-5, k-6\le j\le k, j=6r+k-12, r\in N,\\ k-j+2, & i=n-5, 4\le j\le k-12, j=6r-2, r\in N,\end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+10, & i=n-6, k-6\le j\le k, j=6r+k-12, r\in N,\\ k-j+4, & i=n-6, 4\le j\le k-12, j=6r-2, r\in N,\end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+2n-2i-2, & i=-3+4t, t\in N, k-\frac{3n}{2}+\frac{3i}{2}+9\le j\le k,\\ & j=6r+k-\frac{3n}{2}+\frac{3i}{2}+3, r\in N,\\ \frac{2k}{3}-\frac{2j}{3}+\frac{3n}{2}-\frac{3i}{2}-1, & i=-3+4t, t\in N, k-3n+3i+6\le j\le k-\frac{3n}{2}+\frac{3i}{2}+3,\\ & j=6r+k-3n+3i, r\in N,\\ k-j+\frac{n}{2}-\frac{i}{2}+1, & i=-3+4t, t\in N, 4\le j\le k-3n+3i,\\ & j=6r-2, r\in N,\end{cases}$$

$$d_i^j = k-j, i=\tfrac{3n-5}{4}, \tfrac{3n-9}{4}, 5\le j\le k-5, j=6r-1, r\in N,$$

$$d_i^j = \begin{cases} \frac{2k}{3}-\frac{2j}{3}+\frac{11}{3}, & i=\frac{3n-13}{4}, k-11\le j\le k-5, j=6r+k-17, r\in N,\\ k-j, & i=\frac{3n-13}{4}, 5\le j\le k-17, j=6r-1, r\in N,\end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3}-\frac{2j}{3}+\frac{13}{3}, & i=\frac{3n-17}{4}, k-11\le j\le k-5, j=6r+k-17, r\in N,\\ k-j+2, & i=\frac{3n-17}{4}, 5\le j\le k-17, j=6r-1, r\in N,\end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+\frac{34}{3}, & i=\frac{3n-21}{4}, k-11\le j\le k-5, j=6r+k-17, r\in N,\\ k-j+2, & i=\frac{3n-21}{4}, 5\le j\le k-17, j=6r-1, r\in N,\end{cases}$$

$$d_i^j = k-j, i=\tfrac{3n-5}{4}, 6\le j\le k-4, j=6r, r\in N,$$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + \frac{10}{3}, & i = \frac{3n-9}{4}, k-10 \leq j \leq k-4, j = 6r+k-16, r \in N, \\ k-j, & i = \frac{3n-9}{4}, 6 \leq j \leq k-16, j = 6r, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + \frac{16}{3}, & i = \frac{3n-13}{4}, k-10 \leq j \leq k-4, j = 6r+k-16, r \in N, \\ k-j, & i = \frac{3n-13}{4}, 6 \leq j \leq k-16, j = 6r, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3} - \frac{j}{3} + \frac{32}{3}, & i = \frac{3n-17}{4}, k-10 \leq j \leq k-4, j = 6r+k-16, r \in N, \\ k-j+2, & i = \frac{3n-17}{4}, 6 \leq j \leq k-16, j = 6r, r \in N, \end{cases}$$

This complete the proof.

THEOREM 2. For $G \cong CL_n^k$ where $k = 2n-2$ when $n = 12p-3, p \in N$ and $k = 2n-4$ when $n = 12p+1, p \in N$ we have $dim(G) \leq 4$.

PROOF. Let $W = \{V_1^1, V_{n-1}^1, V_1^{k-7}, V_{n-1}^k\}$ be the resolving set of G. Then

$$r(V_i^j/W) = (a_i^j, b_i^j, c_i^j, d_i^j), 1 \leq i \leq n, 1 \leq j \leq k.$$

where a_i^j and c_i^j contain same values as in Theorem[1] butb_i^j and d_i^j have different values as given below.
$b_i^j = j-1, i = n-1, 1 \leq j \leq k-3, j = 6r-5, r \in N,$
$b_i^j = j+1, i = n-2, 1 \leq j \leq k-3, j = 6r-5, r \in N,$

$$b_i^j = \begin{cases} \frac{j}{3} + \frac{11}{3}, & i = n-3, 1 \leq j \leq 7, j = 6r-5, r \in N, \\ j-1, & i = n-3, 13 \leq j \leq k-3, j = 6r+7, r \in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{j}{3} + \frac{17}{3}, & i = n-4, 1 \leq j \leq 7, j = 6r-5, r \in N, \\ j-1, & i = n-4, 13 \leq j \leq k-3, j = 6r+7, r \in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{j}{3} + \frac{23}{3}, & i = n-5, 1 \leq j \leq 7, j = 6r-5, r \in N, \\ j+1, & i = n-5, 13 \leq j \leq k-3, j = 6r+7, r \in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{1}{3}(j+29), & i = n-6, 1 \leq j \leq 7, j = 6r-5, r \in N, \\ j+3, & i = n-6, 13 \leq j \leq k-3, j = 6r+7, r \in N, \end{cases}$$

$b_i^j = j-1, i = n, n-1, 2 \leq j \leq k-2, j = 6r-4, r \in N,$

$$b_i^j = \begin{cases} \frac{2j}{3} + \frac{5}{3}, & i = n-2, 2 \leq j \leq 8, j = 6r-4, r \in N, \\ j-1, & i = n-2, 14 \leq j \leq k, j = 6r+8, r \in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{j}{3} + \frac{13}{3}, & i = n-3, 2 \leq j \leq 8, j = 6r-4, r \in N, \\ j-1, & i = n-3, 14 \leq j \leq k-2, j = 6r+8, r \in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{j}{3} + \frac{19}{3}, & i = n-4, 2 \leq j \leq 8, j = 6r-4, r \in N, \\ j-1, & i = n-4, 14 \leq j \leq k-2, j = 6r+8, r \in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{j}{3} + \frac{25}{3}, & i = n-5, 2 \leq j \leq 8, j = 6r-4, r \in N, \\ j+1, & i = n-5, 14 \leq j \leq k-2, j = 6r+8, r \in N, \end{cases}$$

$b_i^j = j-1, i = n, n-1, 3 \leq j \leq k-1, j = 6r-3, r \in N,$

$$b_i^j = \begin{cases} \frac{2j}{3}+2, & i=n-2, 3\le j\le 9, j=6r-3, r\in N,\\ j-1, & i=n-2, 15\le j\le k-1, j=6r+9, r\in N,\end{cases}$$

$$b_i^j = \begin{cases} \frac{j}{3}+5, & i=n-3, 3\le j\le 9, j=6r-3, r\in N,\\ j-1, & i=n-3, 15\le j\le k, j=6r+9, r\in N,\end{cases}$$

$$b_i^j = \begin{cases} \frac{j}{3}+7, & i=n-4, 3\le j\le 9, j=6r-3, r\in N,\\ j-1, & i=n-4, 15\le j\le k-3, j=6r+9, r\in N,\end{cases}$$

$$b_i^j = \begin{cases} \frac{j}{3}+9, & i=n-5, 3\le j\le 9, j=6r-3, r\in N,\\ j+1, & i=n-5, 15\le j\le k, j=6r+9, r\in N,\end{cases}$$

$$b_i^j = \begin{cases} \frac{1}{3}(j-6i+6n-3), & i=-3+4t, t\in N, 3\le j\le (\frac{3n}{2}-\frac{3i}{2}-3),\\ & j=6r-3, r\in N,\\ \frac{1}{3}(2j+\frac{9n-9i}{2}-6), & i=-3+4t, t\in N,\\ & \frac{1}{2}(3n-3i)+3\le j\le 3(n-i)+3, j=6r+\frac{1}{2}(3n-3i)-3, r\in N,\\ j+\frac{n-i}{2}-3, & i=-3+4t, t\in N, 3(n-i)+9\le j\le k-1, ,\\ & j=6r+3(n-i)+3, r\in N,\end{cases}$$

$$b_i^j = \begin{cases} \frac{1}{3}(j-6i+6n-3), & i=4t, t\in N, 3\le j\le (\frac{3n}{2}-\frac{3i}{2}+\frac{3}{2}),\\ & j=6r-3, r\in N,\\ \frac{1}{3}(2j+\frac{9n-9i}{2}-\frac{9}{2}), & i=4t, t\in N,\\ & \frac{1}{2}(3n-3i)+\frac{15}{2}\le j\le 3(n-i), j=6r+\frac{1}{2}(3n-3i)+\frac{3}{2}, r\in N,\\ j+\frac{n-i}{2}-\frac{3}{2}, & i=4t, t\in N, 3(n-i)+6\le j\le k-1, ,\\ & j=6r+3(n-i), r\in N,\end{cases}$$

$$b_i^j = j-1, i=n-1, n-2, 4\le j\le k, j=6r-2, r\in N,$$

$$b_i^j = \begin{cases} \frac{2j}{3}+\frac{7}{3}, & i=n-3, 4\le j\le 10, j=6r-2, r\in N,\\ j-1, & i=n-3, 16\le j\le k, j=6r+10, r\in N,\end{cases}$$

$$b_i^j = \begin{cases} \frac{j}{3}+\frac{17}{3}, & i=n-4, 4\le j\le 10, j=6r-2, r\in N,\\ j-1, & i=n-4, 16\le j\le k, j=6r+10, r\in N,\end{cases}$$

$$b_i^j = \begin{cases} \frac{j}{3}+\frac{23}{3}, & i=n-5, 4\le j\le 10, j=6r-2, r\in N,\\ j-1, & i=n-5, 16\le j\le k, j=6r+10, r\in N,\end{cases}$$

$$b_i^j = \begin{cases} \frac{j}{3}+\frac{29}{3}, & i=n-6, 4\le j\le 10, j=6r-2, r\in N,\\ j+1, & i=n-6, 16\le j\le k, j=6r+10, r\in N,\end{cases}$$

$$b_i^j = \begin{cases} \frac{1}{3}(j-6i+6n-7), & i=-2+4t, t\in N, 4\le j\le (\frac{3n}{2}-\frac{3i}{2}-\frac{1}{2}),\\ & j=6r-2, r\in N,\\ \frac{1}{3}(2j+\frac{9n-9i}{2}-\frac{13}{2}), & i=-2+4t, t\in N,\\ & \frac{1}{2}(3n-3i)+\frac{11}{2}\le j\le 3(n-i)+1, j=6r+\frac{1}{2}(3n-3i)-\frac{1}{2}, r\in N,\\ j+\frac{n-i}{2}-\frac{5}{2}, & i=-2+4t, t\in N, 3(n-i)+7\le j\le k, ,\\ & j=6r+3(n-i)+1, r\in N,\end{cases}$$

$$b_i^j = \begin{cases} \frac{1}{3}(j-6i+6n-7), & i=-1+4t, t\in N, 4\le j\le (\frac{3n}{2}-\frac{3i}{2}+1), \\ & j=6r-2, r\in N, \\ \frac{1}{3}(2j+\frac{9n-9i}{2}-8), & i=-1+4t, t\in N, \\ & \frac{1}{2}(3n-3i)+7\le j\le 3(n-i)-2, j=6r+\frac{1}{2}(3n-3i)+1, r\in N, \\ j+\frac{n-i}{2}-2, & i=-3+4t, t\in N, 3(n-i)+4\le j\le k, , \\ & j=6r+3(n-i)-2, r\in N, \end{cases}$$

$$b_i^j = j-1, i=\frac{3n-7}{4}, 5\le j\le k-5, j=6r-1, r\in N,$$

$$b_i^j = \begin{cases} \frac{2j}{3}+\frac{8}{3}, & i=\frac{3n-11}{4}, 5\le j\le 11, j=6r-1, r\in N, \\ j-1, & i=\frac{3n-11}{4}, 17\le j\le k-5, j=6r+11, r\in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{2j}{3}+\frac{14}{3}, & i=\frac{3n-15}{4}, 5\le j\le 11, j=6r-1, r\in N, \\ j-1, & i=\frac{3n-15}{4}, 17\le j\le k-5, j=6r+11, r\in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{j}{3}+\frac{31}{3}, & i=\frac{3n-19}{4}, 5\le j\le 11, j=6r-1, r\in N, \\ j+1, & i=\frac{3n-19}{4}, 17\le j\le k-5, j=6r+11, r\in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{1}{3}(j-8i+6n-7), & i=-1+3t, t\in N, i\le\frac{3n-23}{4}, 5\le j\le(\frac{3n}{2}-2i-\frac{9}{2}), \\ & j=6r-1, r\in N, \\ \frac{1}{3}(2j+\frac{9n}{2}-6i-\frac{17}{2}), & i=-1+3t, t\in N, i\le\frac{3n-23}{4}, \\ & \frac{3n}{2}-2i+\frac{3}{2}\le j\le 3n-4i-2, j=6r+\frac{3n}{2}-2i-\frac{9}{2}, r\in N, \\ j+\frac{n}{2}-\frac{2i}{3}-\frac{13}{6}, & i=-1+3t, t\in N, 3n-4i+4\le j\le k-5, , \\ & j=6r+3n-4i-2, r\in N, \end{cases}$$

$$b_i^j = j-1, i=\frac{3n-3}{4}, \frac{3n-7}{4}, 6\le j\le k-4, j=6r, r\in N,$$

$$b_i^j = \begin{cases} \frac{2j}{3}+3, & i=\frac{3n-11}{4}, 6\le j\le 12, j=6r, r\in N, \\ j-1, & i=\frac{3n-11}{4}, 18\le j\le k-4, j=6r+12, r\in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{2j}{3}+7, & i=\frac{3n-15}{4}, 6\le j\le 18, j=6r, r\in N, \\ j+1, & i=\frac{3n-15}{4}, 24\le j\le k-4, j=6r+18, r\in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{j}{3}+11, & i=\frac{3n-19}{4}, 6\le j\le 12, j=6r, r\in N, \\ j+1, & i=\frac{3n-19}{4}, 18\le j\le k-4, j=6r+12, r\in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{1}{3}(j-8i+6n-7), & i=-2+3t, t\in N, i\le\frac{3n-23}{4}, 6\le j\le(\frac{3n}{2}-2i+\frac{1}{2}), \\ & j=6r, r\in N, \\ \frac{1}{3}(2j+\frac{9n}{2}-6i-\frac{15}{2}), & i=-2+3t, t\in N, i\le\frac{3n-23}{4}, \\ & \frac{3n}{2}-2i+\frac{13}{2}\le j\le 3n-4i+1, j=6r+\frac{3n}{2}-2i+\frac{1}{2}, r\in N, \\ j+\frac{n}{2}-\frac{2i}{3}-\frac{17}{6}, & i=-2+3t, t\in N, 3n-4i+7\le j\le k-4, , \\ & j=6r+3n-4i+1, r\in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{1}{3}(j-8i+6n-3), & i=3t, t\in N, i\le\frac{3n-23}{4}, 6\le j\le(\frac{3n}{2}-2i-\frac{3}{2}), \\ & j=6r, r\in N, \\ \frac{1}{3}(2j+\frac{9n}{2}-6i-\frac{3}{2}), & i=3t, t\in N, i\le\frac{3n-23}{4}, \\ & \frac{3n}{2}-2i+\frac{9}{2}\le j\le 3n-4i+3, j=6r+\frac{3n}{2}-2i-\frac{3}{2}, r\in N, \\ j+\frac{n}{2}-\frac{2i}{3}-\frac{3}{2}, & i=3t, t\in N, 3n-4i+9\le j\le k-4, , \\ & j=6r+3n-4i+3, r\in N, \end{cases}$$

$$b_i^j = \begin{cases} \frac{1}{3}(j-8i+6n-5), & i=-1+3t, t\in N, i\leq \frac{3n-23}{4}, 6\leq j\leq (\frac{3n}{2}-2i-\frac{7}{2}), \\ & j=6r, r\in N, \\ \frac{1}{3}(2j+\frac{9n}{2}-6i-\frac{15}{2}), & i=-1+3t, t\in N, i\leq \frac{3n-23}{4}, \\ & \frac{3n}{2}-2i+\frac{5}{2}\leq j\leq 3n-4i-1, j=6r+\frac{3n}{2}-2i-\frac{7}{2}, r\in N, \\ j+\frac{n}{2}-\frac{2i}{3}-\frac{13}{6}, & i=-1+3t, t\in N, 3n-4i+5\leq j\leq k-4, , \\ & j=6r+3n-4i-1, r\in N, \end{cases}$$

$d_i^j = k-j, i=n-1, n-2, 1\leq j\leq k-3, j=6r-5, r\in N,$

$$d_i^j = \begin{cases} \frac{2k}{3}-\frac{2j}{3}+3, & i=n-3, k-9\leq j\leq k-3, j=6r+k-15, r\in N, \\ k-j, & i=n-3, 1\leq j\leq k-15, j=6r-5, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3}-\frac{2j}{3}+5, & i=n-4, k-9\leq j\leq k-3, j=6r+k-15, r\in N, \\ k-j+2, & i=n-4, 1\leq j\leq k-15, j=6r-5, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3}-\frac{2j}{3}+7, & i=n-5, k-9\leq j\leq k-3, j=6r+k-15, r\in N, \\ k-j+2, & i=n-5, 1\leq j\leq k-15, j=6r-5, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+10, & i=n-6, k-9\leq j\leq k-3, j=6r+k-15, r\in N, \\ k-j+2, & i=n-6, 1\leq j\leq k-15, j=6r-5, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+2n-2i-2, & i=-2+4t, t\in N, k-\frac{3n}{2}+\frac{3i}{2}+\frac{3}{2}\leq j\leq k-3, \\ & j=6r+k-\frac{3n}{2}+\frac{3i}{2}-\frac{9}{2}, r\in N, \\ \frac{2k}{3}-\frac{2j}{3}+\frac{3n}{2}-\frac{3i}{2}-\frac{3}{2}, & i=-2+4t, t\in N, \\ & k-3n+3i\leq j\leq k-\frac{3n}{2}+\frac{3i}{2}-\frac{9}{2}, j=6r+k-3n+3i-6, r\in N, \\ k-j+\frac{n}{2}-\frac{i}{2}-\frac{3}{2}, & i=-2+4t, t\in N, 1\leq j\leq k-3n+3i-6, , \\ & j=6r-5, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+2n-2i-2, & i=-3+4t, t\in N, k-\frac{3n}{2}+\frac{3i}{2}+3\leq j\leq k-3, \\ & j=6r+k-\frac{3n}{2}+\frac{3i}{2}-3, r\in N, \\ \frac{2k}{3}-\frac{2j}{3}+\frac{3n}{2}-\frac{3i}{2}-1, & i=-3+4t, t\in N, \\ & k-3n+3i+3\leq j\leq k-\frac{3n}{2}+\frac{3i}{2}-3, j=6r+k-3n+3i-3, r\in N, \\ k-j+\frac{n}{2}-\frac{i}{2}, & i=-3+4t, t\in N, 1\leq j\leq k-3n+3i-3, , \\ & j=6r-5, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3}-\frac{j}{3}+2n-2i-2, & i=4t, t\in N, k-\frac{3n}{2}+\frac{3i}{2}+\frac{9}{2}\leq j\leq k-3, \\ & j=6r+k-\frac{3n}{2}+\frac{3i}{2}-\frac{3}{2}, r\in N, \\ \frac{2k}{3}-\frac{2j}{3}+\frac{3n}{2}-\frac{3i}{2}-\frac{1}{2}, & i=4t, t\in N, \\ & k-3n+3i\leq j\leq k-\frac{3n}{2}+\frac{3i}{2}-\frac{3}{2}, j=6r+k-3n+3i-6, r\in N, \\ k-j+\frac{n}{2}-\frac{i}{2}-\frac{1}{2}, & i=4t, t\in N, 1\leq j\leq k-3n+3i-6, , \\ & j=6r-5, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{4k}{3}-\frac{4j}{3}-\frac{2}{3}, & i=n, k-8\leq j\leq k-2, j=6r+k-14, r\in N, \\ k-j+2, & i=n, 2\leq j\leq k-14, j=6r-4, r\in N, \end{cases}$$

$d_i^j = k-j, i=n-1, 2\leq j\leq k-2, j=6r-4, r\in N,$

$$d_i^j = \begin{cases} \frac{2k}{3}-\frac{2j}{3}+\frac{8}{3}, & i=n-2, k-8\leq j\leq k-2, j=6r+k-14, r\in N, \\ k-j, & i=n-2, 2\leq j\leq k-14, j=6r-4, r\in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + \frac{14}{3}, & i = n-3, k-8 \le j \le k-2, j = 6r+k-14, r \in N, \\ k-j+2, & i = n-3, 2 \le j \le k-14, j = 6r-4, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + \frac{20}{3}, & i = n-4, k-8 \le j \le k-2, j = 6r+k-14, r \in N, \\ k-j+2, & i = n-4, 2 \le j \le k-14, j = 6r-4, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3} - \frac{j}{3} + \frac{28}{3}, & i = n-5, k-8 \le j \le k-2, j = 6r+k-14, r \in N, \\ k-j+2, & i = n-5, 2 \le j \le k-14, j = 6r-4, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{1}{3}(k-j+6n-6i-2), & i = -3+4t, t \in N, k - \frac{3n}{2} + \frac{3i}{2} + 4 \le j \le k-2, \\ & j = 6r+k-\frac{3n}{2}+\frac{3i}{2}-2, r \in N, \\ \frac{1}{3}(2k-2j+\frac{9n}{2}-\frac{9i}{2}+2), & i = -3+4t, t \in N, k-3n+3i-2 \le j \le k-\frac{3n}{2}+\frac{3i}{2}-2, \\ & j = 6r+k-3n+3i-8, r \in N, \\ k-j+\frac{n}{2}-\frac{i}{2}, & i = -3+4t, t \in N, 2 \le j \le k-3n+3i-8, , \\ & j = 6r-4, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{4k}{3} - \frac{4j}{3} - \frac{1}{3}, & i = n, k-7 \le j \le k-1, j = 6r+k-13, r \in N, \\ k-j, & i = n, 3 \le j \le k-13, j = 6r-3, r \in N, \end{cases}$$

$$d_i^j = k-j, i = n-1, 3 \le j \le k-1, j = 6r-3, r \in N,$$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + \frac{7}{3}, & i = n-2, k-7 \le j \le k-1, j = 6r+k-13, r \in N, \\ k-j, & i = n-2, 3 \le j \le k-13, j = 6r-3, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + \frac{13}{3}, & i = n-3, k-7 \le j \le k-1, j = 6r+k-13, r \in N, \\ k-j+2, & i = n-3, 3 \le j \le k-13, j = 6r-3, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + \frac{19}{3}, & i = n-4, k-7 \le j \le k-1, j = 6r+k-13, r \in N, \\ k-j+2, & i = n-4, 3 \le j \le k-13, j = 6r-3, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3} - \frac{j}{3} + \frac{26}{3}, & i = n-5, k-7 \le j \le k-1, j = 6r+k-13, r \in N, \\ k-j+2, & i = n-5, 3 \le j \le k-13, j = 6r-3, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{1}{3}(k-j+6n-6i-4), & i = -3+4t, t \in N, k - \frac{3n}{2} + \frac{3i}{2} + 5 \le j \le k-1, \\ & j = 6r+k-\frac{3n}{2}+\frac{3i}{2}-1, r \in N, \\ \frac{1}{3}(2k-2j+\frac{9n}{2}-\frac{9i}{2}+1), & i = -3+4t, t \in N, k-3n+3i-1 \le j \le k-\frac{3n}{2}+\frac{3i}{2}-1, \\ & j = 6r+k-3n+3i-7, r \in N, \\ k-j+\frac{n}{2}-\frac{i}{2}, & i = -3+4t, t \in N, 3 \le j \le k-3n+3i-7, , \\ & j = 6r-3, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{4k}{3} - \frac{4j}{3}, & i = n-1, k-6 \le j \le k, j = 6r+k-12, r \in N, \\ k-j+2, & i = n-1, 4 \le j \le k-12, j = 6r-2, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + 2, & i = n-2, k-6 \le j \le k, j = 6r+k-12, r \in N, \\ k-j, & i = n-2, 4 \le j \le k-12, j = 6r-2, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + 4, & i = n-3, k-6 \le j \le k, j = 6r+k-12, r \in N, \\ k-j+2, & i = n-3, 4 \le j \le k-12, j = 6r-2, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + 6, & i = n-4, k-6 \le j \le k, j = 6r+k-12, r \in N, \\ k-j+4, & i = n-4, 4 \le j \le k-12, j = 6r-2, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3} - \frac{j}{3} + 6, & i = n-5, k-6 \le j \le k, j = 6r+k-12, r \in N, \\ k - j + 2, & i = n-5, 4 \le j \le k-12, j = 6r-2, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3} - \frac{j}{3} + 2n - 2i - 2, & i = -1+4t, t \in N, k - \frac{3n}{2} + \frac{3i}{2} + 3 \le j \le k, \\ & j = 6r + k - \frac{3n}{2} + \frac{3i}{2} - 3, r \in N, \\ \frac{2k}{3} - \frac{2j}{3} + \frac{3n}{2} - \frac{3i}{2} - 1, & i = -1+4t, t \in N, k - 3n + 3i + 3 \le j \le k - \frac{3n}{2} + \frac{3i}{2} - 3, \\ & j = 6r + k - 3n + 3i - 6, r \in N, \\ k - j + \frac{n}{2} - \frac{i}{2} - 1, & i = -1+4t, t \in N, 4 \le j \le k - 3n + 3i - 6, , \\ & j = 6r - 2, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{k}{3} - \frac{j}{3} + 2n - 2i - 2, & i = -2+4t, t \in N, k - \frac{3n}{2} + \frac{3i}{2} + \frac{9}{2} \le j \le k, \\ & j = 6r + k - \frac{3n}{2} + \frac{3i}{2} - \frac{3}{2}, r \in N, \\ \frac{2k}{3} - \frac{2j}{3} + \frac{3n}{2} - \frac{3i}{2} - \frac{1}{2}, & i = -2+4t, t \in N, k - 3n + 3i + 3 \le j \le k - \frac{3n}{2} + \frac{3i}{2} - \frac{3}{2}, \\ & j = 6r + k - 3n + 3i - 3, r \in N, \\ k - j + \frac{n}{2} - \frac{i}{2} + \frac{1}{2}, & i = -2+4t, t \in N, 4 \le j \le k - 3n + 3i - 3, , \\ & j = 6r - 2, r \in N, \end{cases}$$

$d_i^j = k - j, i = \frac{3n-7}{4}, 5 \le j \le k-5, j = 6r-1, r \in N,$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + \frac{17}{3}, & i = \frac{3n-11}{4}, \frac{3n-15}{4}, k-11 \le j \le k-5, j = 6r+k-17, r \in N, \\ k - j + 2, & i = \frac{3n-11}{4}, \frac{3n-15}{4}, 5 \le j \le k-17, j = 6r-1, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + \frac{23}{3}, & i = \frac{3n-19}{4}, k-17 \le j \le k-5, j = 6r+k-23, r \in N, \\ k - j + 2, & i = \frac{3n-19}{4}, 5 \le j \le k-23, j = 6r-1, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + \frac{35}{3}, & i = \frac{3n-23}{4}, k-23 \le j \le k-5, j = 6r+k-29, r \in N, \\ k - j + 4, & i = \frac{3n-23}{4}, 5 \le j \le k-29, j = 6r-1, r \in N, \end{cases}$$

$d_i^j = k - j, i = \frac{3n-3}{4}, 6 \le j \le k-4, j = 6r, r \in N,$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + \frac{10}{3}, & i = \frac{3n-7}{4}, k-10 \le j \le k-4, j = 6r+k-16, r \in N, \\ k - j, & i = \frac{3n-7}{4}, 6 \le j \le k-16, j = 6r, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + \frac{16}{3}, & i = \frac{3n-11}{4}, k-10 \le j \le k-4, j = 6r+k-16, r \in N, \\ k - j + 2, & i = \frac{3n-11}{4}, 6 \le j \le k-16, j = 6r, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + \frac{22}{3}, & i = \frac{3n-15}{4}, k-16 \le j \le k-4, j = 6r+k-22, r \in N, \\ k - j + 2, & i = \frac{3n-15}{4}, 6 \le j \le k-22, j = 6r, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{2k}{3} - \frac{2j}{3} + \frac{28}{3}, & i = \frac{3n-19}{4}, k-22 \le j \le k-4, j = 6r+k-28, r \in N, \\ k - j + 2, & i = \frac{3n-19}{4}, 6 \le j \le k-28, j = 6r, r \in N, \end{cases}$$

$$d_i^j = \begin{cases} \frac{1}{3}(k - j + 6n - 8i - 6), & i = -1+3t, t \in N, k - \frac{3n}{2} + 2i + \frac{11}{2} \le j \le k-4, i \le \frac{3n-23}{4}, \\ & j = 6r + k - \frac{3n}{2} + 2i - \frac{1}{2}, r \in N, \\ \frac{1}{3}(2k - 2j + \frac{9n-12i-1}{2}), & i = -1+3t, t \in N, k - 3n + 4i - 3 \le j \le k - \frac{3n}{2} + 2i - \frac{1}{2}, \\ & i \le \frac{3n-23}{4}, j = 6r + k - 3n + 4i - 9, r \in N, \\ k - j + \frac{3n-4i-7}{6}, & i = -1+3t, t \in N, 6 \le j \le k - 3n + 4i - 9, , \\ & i \le \frac{3n-23}{4}, j = 6r, r \in N, \end{cases}$$

This completes the proof. @@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@

4 Conclusion

In the foregoing section we constructed cellulose network graphs CL_n^k from 3 dimensional cellulose chemical structure and then studied its metric dimension and we prove that $dim(CL_n^k) \leq 4$. We close this section by raising the following open problem.
Open Problem 1. Determine the exact value of metric dimension for CL_n^k.

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