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Wave Equation With Distributional Propagation Speed And Mass Term: Numerical Simulations*

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Abstract

In this paper we explore numerically the theoretical results of the paper [10], and report on the numerical study of the Cauchy–Dirichlet problem for the 1D wave equation with distributional propagation speed and mass term. The analysis of this note shows that the notion of very weak solutions introduced in [3] is very well adapted for numerical simulations. Moreover, by the recently constructed theory of very weak solutions we can talk about uniqueness of numerical solutions to differential equations with δ–like coefficients in some appropriate sense.

1 Introduction

In this paper we slightly modify theoretical results of the paper [10] to adapt them to the numerical setting. As in [10] let $\mathcal{L}$ be a densely defined linear operator on a separable Hilbert space $\mathcal{H}$ with the discrete spectrum $\{\lambda_\xi \geq 0 : \xi \in \mathbb{N}\}$. We assume that the system of eigenfunctions $\{e_\xi : \xi \in \mathbb{N}\}$ is a Riesz basis in $\mathcal{H}$.

Here, for non-negative functions $a = a(t) \geq 0$, $q = q(t) \geq 0$ and for the source term $f = f(t) \in \mathcal{H}$, we are interested in the well–posedness of the Cauchy problem for the operator $\mathcal{L}$ with the propagation speed a and with a time-dependent mass term q:

$$\begin{cases} \partial_t^2 u(t) + a(t)\mathcal{L}u(t) + q(t)u(t) = f(t), \ t \in [0,T], \\ u(0) = u_0 \in \mathcal{H}, \\ \partial_t u(0) = u_1 \in \mathcal{H}. \end{cases} \tag{1}$$

We allow a and q to be of distributional type, in particular, we analyse δ and/or δ' cases.

In general, in equations with distributional coefficients we face a mathematical difficulty caused by the general impossibility to multiply distributions, going back to the famous Schwartz impossibility result [6]. To study equations with distributional

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coefficients and initial data the notion of very weak solutions has been introduced in [3] to analyse partial differential equations of hyperbolic type. In [9, 11, 12] this notion was applied to prove the (very weak) well-posedness for the Landau Hamiltonian wave equations in distributional electro-magnetic fields. Also, in [5] an acoustic problem was studied from the viewpoint of very weak solutions. For more background information on very weak solutions, we refer to [3, 9] and references therein.

Below, concerning the Cauchy problem (1), we obtain the well-posedness results in Sobolev spaces $H^s_{\mathcal{L}}$ associated to $\mathcal{L}$ for any $s \in \mathcal{R}$. We use the Sobolev type space adapted to $\mathcal{L}$,

$$H^s_{\mathcal{L}} := \left\{ f \in H^{-\infty}_{\mathcal{L}} : \mathcal{L}^{s/2} f \in \mathcal{H} \right\},$$

with the norm $\|f\|_{H^s_{\mathcal{L}}} := \|\mathcal{L}^{s/2} f\|_{\mathcal{H}}$. The global space of $\mathcal{L}$-distributions $H^{-\infty}_{\mathcal{L}}$ is defined in Appendix 4. Anticipating the material of Appendix 4, by using Plancherel's identity (7), we can express the Sobolev norm as

$$\|f\|_{H^s_{\mathcal{L}}} = \left(\sum_{\xi \in \mathbb{N}} |\lambda_\xi|^s (f, e^*_\xi)(e_\xi, f) \right)^{1/2}, \tag{2}$$

for any $s \in \mathbb{R}$, where $(\cdot, \cdot)$ is the inner product of the space $\mathcal{H}$.

Also, it is convenient to use the notation $H^\infty_{\mathcal{L}}$ for the space of test functions, defined by $H^\infty_{\mathcal{L}} := \bigcap_{s \geq 0} H^s_{\mathcal{L}}$. We refer to Appendix 4 for all the appearing notation.

2 Main Results

We now repeat discussions of [3, 9, 10] about the very weak solutions and formulate results for distributions $a, q \in \mathcal{D}'([0,T])$ and $f \in \mathcal{D}'([0,T]) \bar{\otimes} H^{-\infty}_{\mathcal{L}}$. We start by regularising the distributional coefficients a, q and the source term f by convolution with a suitable mollifier ψ. We obtain families of smooth functions $(a_\varepsilon)_\varepsilon$, $(q_\varepsilon)_\varepsilon$ and $(f_\varepsilon)_\varepsilon$, namely

$$a_\varepsilon = a * \psi_{\omega(\varepsilon)}, \;\; q_\varepsilon = q * \psi_{\omega(\varepsilon)}, \;\; f_\varepsilon = f(\cdot) * \psi_{\omega(\varepsilon)},$$

where

$$\psi_{\omega(\varepsilon)}(t) = \omega(\varepsilon)^{-1} \psi(t/\omega(\varepsilon))$$

and $\omega(\varepsilon)$ is a positive function converging to 0 as $\varepsilon \to 0$ (to be chosen later). Here ψ is a Friedrichs–mollifier: $\psi \in C^\infty_0(\mathbb{R})$, $\psi \geq 0$ and $\int \psi = 1$. It turns out that the net $(a_\varepsilon)_\varepsilon$ is C^∞-*moderate*, in the sense that its C^∞-seminorms can be estimated by a negative power of ε. More precisely, we will make use of the following notions of moderateness.

In what follows "$K \Subset \mathbb{R}$" means that K is a compact set in $\mathbb{R}$.

DEFINITION 1. (i) We say that a net of functions $(f_\varepsilon)_\varepsilon \in C^\infty(\mathbb{R})^{(0,1]}$ is C^∞-moderate if for arbitrary $K \Subset \mathbb{R}$ and for any $\alpha \in \mathbb{N}_0$ there exist $N \in \mathbb{N}_0$ and $c > 0$ such that

$$\sup_{t \in K} |\partial^\alpha f_\varepsilon(t)| \leq c\varepsilon^{-N-\alpha},$$

for all $\varepsilon \in (0,1]$.

(ii) We say that a net of functions $(u_\varepsilon)_\varepsilon \in C^\infty([0,T]; H^s_{\mathcal{L}})^{(0,1]}$ is $C^\infty([0,T]; H^s_{\mathcal{L}})$-moderate if there are $N \in \mathbb{N}_0$ and $c_k > 0$ for any $k \in \mathbb{N}_0$ such that

$$\|\partial_t^k u_\varepsilon(t,\cdot)\|_{H^s_{\mathcal{L}}} \leq c_k \varepsilon^{-N-k},$$

for all $\varepsilon \in (0,1]$ and $t \in [0,T]$.

Note that these notions of moderateness are natural in the sense that regularisations of distributions are moderate. For more discussions on it, we refer to [3]. Now we are in a position to introduce a notion of a 'very weak solution' for the Cauchy problem

$$\begin{cases} \partial_t^2 u(t) + a(t)\mathcal{L}u(t) + q(t)u(t) = f(t), \ t \in [0,T], \\ u(0) = u_0 \in \mathcal{H}, \\ \partial_t u(0) = u_1 \in \mathcal{H}. \end{cases} \tag{3}$$

DEFINITION 2. Let s be a real number. We say that the net $(u_\varepsilon)_\varepsilon \subset C^\infty([0,T]; H^s_{\mathcal{L}})$ is *a very weak solution of s-type* of the Cauchy problem (3) if there are

$$C^\infty\text{-moderate regularisation } a_\varepsilon,\ q_\varepsilon \text{of coefficients } a \text{ and } q$$

and

$$C^\infty([0,T]; H^s_{\mathcal{L}})\text{-moderate regularisation } f_\varepsilon \text{ of } f,$$

such that $(u_\varepsilon)_\varepsilon$ solves the regularised problem

$$\begin{cases} \partial_t^2 u_\varepsilon(t) + a_\varepsilon(t)\mathcal{L}u_\varepsilon(t) + q_\varepsilon(t)u_\varepsilon(t) = f_\varepsilon(t), \ t \in [0,T], \\ u_\varepsilon(0) = u_0 \in \mathcal{H}, \\ \partial_t u_\varepsilon(0) = u_1 \in \mathcal{H}, \end{cases}$$

for arbitrary $\varepsilon \in (0,1]$, and is $C^\infty([0,T]; H^s_{\mathcal{L}})$-moderate.

In what follows we make assumptions that a is a nonnegative or a strictly positive distribution and q is a nonnegative distribution. That is, a is a strictly positive distribution if there exists a constant $a_0 > 0$ such that $a - a_0 (\geq 0)$ is a positive distribution. Here $a - a_0 \geq 0$ means that $\langle a - a_0, \psi \rangle \geq 0$ for arbitrary $\psi \in C_0^\infty(\mathbb{R})$, $\psi \geq 0$.

The main results of the paper are existence, uniqueness and consistency Theorems 1, 2 and 3.

THEOREM 1 [Existence]. Let $s \in \mathbb{R}$. Let the coefficients a and q of the Cauchy problem (3) be positive distributions with compact support included in $[0,T]$, such that $a \geq a_0$ and $q \geq q_0$ for some constants $a_0 > 0$ and $q_0 > 0$. Let $u_0, u_1 \in H^s_{\mathcal{L}}$ and $f \in \mathcal{D}'([0,T]) \bar{\otimes} H^s_{\mathcal{L}}$. Then the Cauchy problem (3) has a very weak solution of s-type.

Now, we show that the very weak solution is unique in an appropriate sense. To formulate the uniqueness result we will use the Colombeau algebras' language:

DEFINITION 3. We say that $(u_\varepsilon)_\varepsilon$ is C^∞*-negligible* if for arbitrary $K \Subset \mathbb{R}$, for all $\alpha \in \mathbb{N}$ and for any $\ell \in \mathbb{N}$ there is a constant $C > 0$ such that

$$\sup_{t\in K} |\partial^\alpha u_\varepsilon(t)| \le C\,\varepsilon^\ell$$

holds for all $\varepsilon \in (0,1]$.

Indeed, it is enough to put $K = [0,T]$ in this paper since we have only time-dependent distributions supported in the interval $[0,T]$.

We introduce the Colombeau algebra as the quotient

$$\mathcal{G}(\mathbb{R}) = \frac{C^\infty - \text{moderate nets}}{C^\infty - \text{negligible nets}}.$$

For more general information about $\mathcal{G}(\mathbb{R})$ we refer to e.g. Oberguggenberger [13].

THEOREM 2 [Uniqueness]. Let a and q be positive distributions with compact support included in $[0,T]$, such that $a \ge a_0$ and $q \ge q_0$ for some constants $a_0 > 0$ and $q_0 > 0$. Let $u_0 \in H^{s+1}_{\mathcal{L}}$, $u_1 \in H^s_{\mathcal{L}}$ and $f \in \mathcal{G}([0,T]; H^s_{\mathcal{L}})$ for some $s \in \mathbb{R}$. Then there are embeddings of the coefficients a and q into $\mathcal{G}([0,T])$, such that the Cauchy problem (3), that is

$$\begin{cases} \partial_t^2 u(t) + a(t)\mathcal{L}u(t) + q(t)u(t) = f(t),\ t \in [0,T], \\ u(0) = u_0 \in \mathcal{H}, \\ \partial_t u(0) = u_1 \in \mathcal{H}, \end{cases}$$

has a unique solution $u \in \mathcal{G}([0,T]; H^s_{\mathcal{L}})$.

Denote by $L^\infty_1([0,T])$ the space of bounded L^∞–functions on the interval $[0,T]$ with the derivative also in L^∞.

THEOREM 3 [Consistency]. Assume that $a \in L^\infty_1([0,T])$ and $q \in L^\infty([0,T])$ are such that $a(t) \ge a_0 > 0$ and $q(t) \ge a_0 > 0$. Let $s \in \mathbb{R}$ and consider the Cauchy problem

$$\begin{cases} \partial_t^2 u(t) + a(t)\mathcal{L}u(t) + q(t)u(t) = f(t),\ t \in [0,T], \\ u(0) = u_0 \in \mathcal{H}, \\ \partial_t u(0) = u_1 \in \mathcal{H}, \end{cases} \tag{4}$$

with $u_0 \in H^{1+s}_{\mathcal{L}}$, $u_1 \in H^s_{\mathcal{L}}$ and $f \in C([0,T]; H^s_{\mathcal{L}})$. Let u be a very weak solution of s-type of (4). Then for any regularising families a_ε, q_ε, and f_ε in Definition 2, any representative $(u_\varepsilon)_\varepsilon$ of u converges in $C([0,T]; H^{1+s}_{\mathcal{L}}) \cap C^1([0,T]; H^s_{\mathcal{L}})$ as $\varepsilon \to 0$ to the unique classical solution in $C([0,T]; H^{1+s}_{\mathcal{L}}) \cap C^1([0,T]; H^s_{\mathcal{L}})$ of the Cauchy problem (4) if the latter exists.

3 Numerical Experiments

As a most commonly encountered model, for the subsequent numerical analysis, let us put

$$\mathcal{L} := -\frac{\partial^2}{\partial x^2}$$

on the segment $\Omega := [0, 10]$ with the Dirichlet boundary conditions. Thus, we consider

$$\begin{cases} \partial_t^2 u(t,x) - a(t)\partial_x^2 u(t,x) + q(t)u(t,x) = 0, \ (t,x) \in [0,T] \times [0,10], \\ u(t,0) = 0, \ t \in [0,T], \\ u(t,10) = 0, \ t \in [0,T], \\ u(0,x) = u_0(x), \ x \in [0,10], \\ \partial_t u(0,x) = u_1(x), \ x \in [0,10]. \end{cases} \tag{5}$$

In this work we consider several particular cases of the coefficient $a(t)$ and the mass term $q(t)$. Here we allow them to be distributional, in particular, to have δ–like singularities.

As it was theoretically outlined in [9] and [11], we start to analyse our problem by regularising distributions $a(t)$ and $q(t)$ by a parameter ε, that is, we set

$$a_\varepsilon(t) := (a * \varphi_\varepsilon)(t), \quad q_\varepsilon(t) := (q * \varphi_\varepsilon)(t),$$

as the convolution with the mollifier

$$\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi(t/\varepsilon),$$

with

$$\varphi(t) = \begin{cases} \frac{1}{c} e^{1/(t^2+1)}, |t| \le 1, \\ 0, |t| > 1, \end{cases}$$

where $c \simeq 0.443994$ to get $\int_{-1}^{1} \varphi(t)dt = 1$. Then, instead of (5) we consider the regularised problem

$$\begin{cases} \partial_{tt}^2 u_\varepsilon(t,x) - a_\varepsilon(t)\partial_{xx}^2 u_\varepsilon(t,x) + q_\varepsilon(t)u_\varepsilon(t,x) = 0, \ (t,x) \in [0,T] \times [0,10], \\ u_\varepsilon(t,0) = 0, \ t \in [0,T], \\ u_\varepsilon(t,10) = 0, \ t \in [0,T], \\ u_\varepsilon(0,x) = u_0(x), \ x \in [0,10], \\ \partial_t u_\varepsilon(0,x) = 0, \ x \in [0,10]. \end{cases} \tag{6}$$

Here, we put $u_1(x) \equiv 0$ and

$$u_0(x) = \begin{cases} e^{1/((x-4.1)^2-0.1)}, \ |x-4.1| < 0.1, \\ 0, \ |x-4.1| \ge 0.1. \end{cases}$$

Note that $\operatorname{supp} u_0 \subset [4, 4.2]$.

For a and q we consider the following combinations of the possible cases, with δ denoting the standard Dirac's delta-distribution:

(A1) $a = 1$, $q = 0$;

(A2) $a = 1 + 5\delta(t-3)$, $q = 0$;

(A3) $a = 1 + 5\delta(t-3)$, $q = 1$;

(B4) $a = 1 + 5\delta(t-3)$, $q = 10\delta(t-7)$;

(B5) $a = 1 + 5\delta(t-3)$, $q = 1 + 10\delta(t-7)$;

(A6) $a = 1 + 5\delta(t-3)$, $q = 1 + 5\delta'(t-1.5)$;

In Figure 1, we compare solutions of the problem (6) in different cases. In the upper-left plot, we compare the behaviours of the solution corresponding to the cases (A1) and (A2) coloured by blue and red, respectively, at $t = 3.2$ for $\varepsilon = 0.1$. In the upper-right plot, we compare the behaviours of the solution corresponding to the cases (A1) and (A3) coloured by blue and red, respectively, at $t = 3.2$ for $\varepsilon = 0.1$. In the bottom plot, we compare the behaviour of the solutions of the problem (6) corresponding to the cases (A1) and (A6) coloured by blue and red, respectively, at $t = 3.2$ for $\varepsilon = 0.1$. Here when the mass term q is positive, we see that the wave level is lower than when it is absent. But when $q = 1 + 5\delta'(t-1.5)$, it can be interpreted as a quickly changeable mass (not only volume but also its sign), and we get less stable waves, as it is shown in the plot.

In Figure 2 in the left plot, from up to bottom we see the solution of the problem (6) coloured by blue in the case (A1) and by red in the case (A2) at $t = 3, 3.2, 3.4$. In the right plot, from up to bottom we can see the solution of the problem (6) coloured by blue in the case (A2) and by red in the case (B4) at $t = 7, 7.2, 7.4$, for $\varepsilon = 0.1$.

In Figure 3, we compare the "kinetic energy"

$$E(t) = \int_0^{10} |\partial_t u_\varepsilon(x,t)|^2 dx$$

of the system corresponding to the problem (6) in the case (B5) for different values of $\varepsilon = 0.03, 0.05, 0.08, 0.1$. In Figure 4 in the left plot, we compare the kinetic energy of the system corresponding to the problem (6) in the cases (A1), coloured by blue, and (A2), which is coloured by red, for $\varepsilon = 0.1$. In the right plot, we compare the kinetic energy of the system corresponding to the problem (6) in the cases (A3), coloured by blue, and (B5), which is coloured by red, for $\varepsilon = 0.1$. The analysis shows that the kinetic energy of the singular problems is higher than the problems without δ–like terms. However, in the both cases the kinetic energy decays in time. In the left plot, we analyse the behaviour of the kinetic energy, in particular, how it depends on the parameter ε. Even if the energy function shows impulses at the shocked moments, in general, it decays in time.

All numerical computations are made in C++ by using the sweep method. In above numerical simulations, we used the Matlab R2017b. For all simulations we take $\Delta t = 0.01$, $\Delta x = 0.1$.

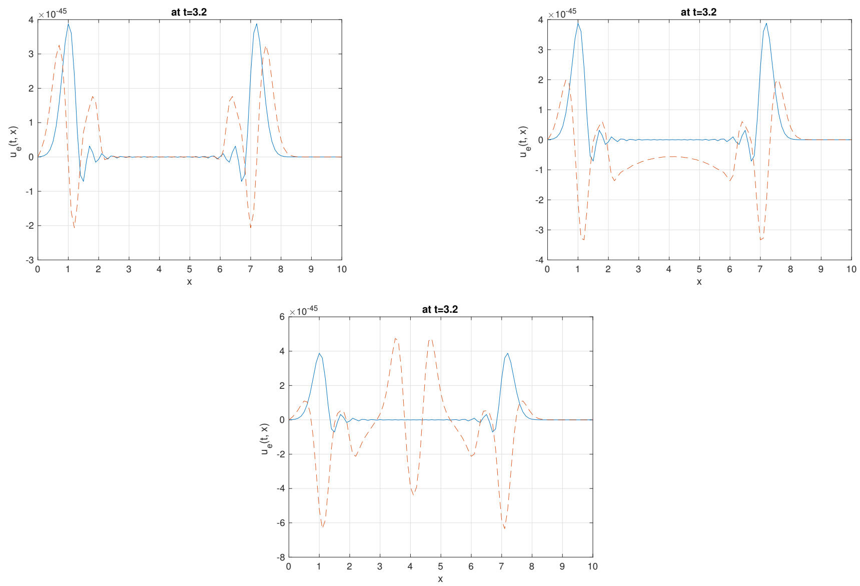

Figure 1: In all plots, from right to left we see the solution of the problem (6) in the case (A1) coloured by blue at $t = 3.2$ and $\varepsilon = 0.1$. Red lines: in the upper-left plot, we see the solution of the problem (6) in the case (A2); in the upper-right plot, we see the solution of the problem (6) in the case (A3); in the bottom plot, we see the solution of the problem (6) in the case (A6) at $t = 3.2$, for $\varepsilon = 0.1$.

4 Appendix: $\mathcal{L}$-Fourier Analysis

Here, we recall some necessary elements of the global Fourier analysis developed in [2, 7, 8]. We call $H^\infty_{\mathcal{L}} := \mathrm{Dom}(\mathcal{L}^\infty)$ the space of test functions for $\mathcal{L}$. Define

$$\mathrm{Dom}(\mathcal{L}^\infty) := \bigcap_{k=1}^{\infty} \mathrm{Dom}(\mathcal{L}^k),$$

where $\mathrm{Dom}(\mathcal{L}^k)$ is the domain of the iterated operator $\mathcal{L}^k$, defined as

$$\mathrm{Dom}(\mathcal{L}^k) := \{f \in \mathcal{H} : \ \mathcal{L}^j f \in \mathrm{Dom}(\mathcal{L}), \ \ j = 0, 1, 2, \ldots, k-1\}.$$

The Fréchet topology of $H^\infty_{\mathcal{L}}$ is given by the semi-norms

$$\|\varphi\|_{H^k_{\mathcal{L}}} := \max_{j \le k} \|\mathcal{L}^j \varphi\|_{\mathcal{H}}, \quad k \in \mathbb{N}_0, \ \varphi \in H^\infty_{\mathcal{L}}.$$

We call the space of linear continuous functionals $H^{-\infty}_{\mathcal{L}} := \mathcal{L}(H^\infty_{\mathcal{L}^*}, \mathbb{C})$ the space of $\mathcal{L}$-distributions. We write $w(\varphi) = \langle w, \varphi \rangle$ for $w \in H^{-\infty}_{\mathcal{L}}$ and $\varphi \in H^\infty_{\mathcal{L}^*}$,

The system $\{e_\xi : \ \xi \in \mathbb{N}\}$ of eigenfunctions of $\mathcal{L}$ is a Riesz basis in the Hilbert space $\mathcal{H}$ and its biorthogonal system $\{e^*_\xi : \ \xi \in \mathbb{N}\}$ is also a Riesz basis in the space $\mathcal{H}$ (see

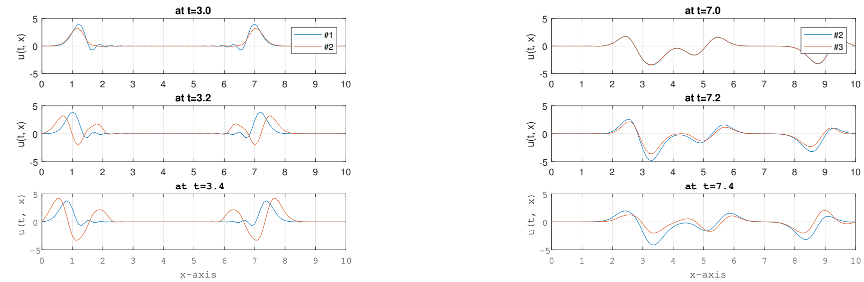

Figure 2: In the left plot, from up to bottom we see the solution of the problem (6) coloured by blue in the case (A1) and by red in the case (A2) at $t = 3, 3.2, 3.4$. In the right plot, from up to bottom we can see the solution of the problem (6) coloured by blue in the case (A2) and by red in the case (B4) at $t = 7, 7.2, 7.4$, for $\varepsilon = 0.1$.

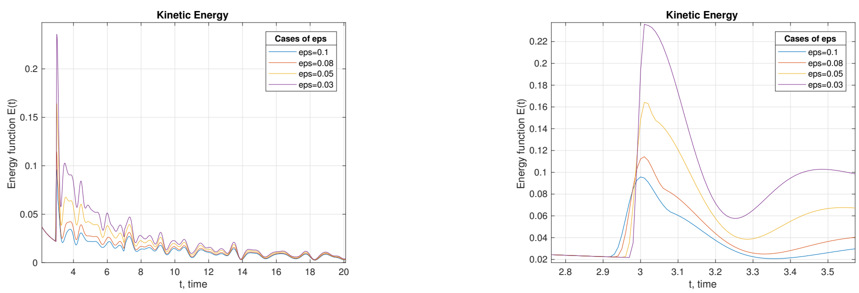

Figure 3: In the both plots, we compare the kinetic energy of the system corresponding to the problem (6) in the case (B5) for different values of $\varepsilon = 0.03, 0.05, 0.08, 0.1$. In the right plot we focus near $t = 3$ and can see an influence to the energy of the singular propagation speed: the propagation speed effects comparatively more than the mass term.

[1, 4]). We note that e^*_ξ is an eigenfunction of the conjugate operator $\mathcal{L}^*$ corresponding to the eigenvalue $\overline{\lambda}_\xi$ for all $\xi \in \mathbb{N}$. For them we have the orthogonality relations

$$(e_\xi, e^*_\eta) = \delta_{\xi\eta},$$

where $\delta_{\xi\eta}$ is the Kronecker delta.

Denote $\langle\xi\rangle := (1 + |\lambda_\xi|)^{1/2}$. We say that the function $\varphi : \mathbb{N} \to \mathbb{C}$ belongs to the space of rapidly decaying functions $\mathcal{S}(\mathbb{N})$ if for arbitrary $m < \infty$ there is a constant $C_{\varphi,m}$ such that

$$|\varphi(\xi)| \leq C_{\varphi,m}\langle\xi\rangle^{-m}$$

holds for all $\xi \in \mathbb{N}$.

We give the topology on $\mathcal{S}(\mathbb{N})$ by the seminorms p_k, where $k \in \mathbb{N}_0$ and

$$p_k(\varphi) := \sup_{\xi\in\mathbb{N}} \langle\xi\rangle^k |\varphi(\xi)|.$$

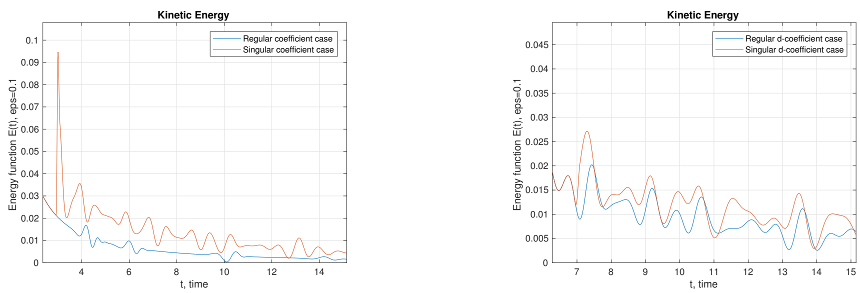

Figure 4: In the left plot, we compare the kinetic energy of the system corresponding to the problem (6) in the cases (A1), coloured by blue, and (A2), which is coloured by red, for $\varepsilon = 0.1$. In the right plot, we compare the kinetic energy of the system corresponding to the problem (6) in the cases (A3), coloured by blue, and (B5), which is coloured by red, for $\varepsilon = 0.1$.

Let us define the $\mathcal{L}$-Fourier transform on $H^\infty_{\mathcal{L}}$ as

$$(\mathcal{F}_{\mathcal{L}} f)(\xi) = (f \mapsto \widehat{f}) : H^\infty_{\mathcal{L}} \to \mathcal{S}(\mathbb{N})$$

by

$$\widehat{f}(\xi) := (\mathcal{F}_{\mathcal{L}} f)(\xi) = (f, e^*_\xi),$$

and introduce the $\mathcal{L}^*$-Fourier transform on $H^\infty_{\mathcal{L}^*}$ as

$$(\mathcal{F}_{\mathcal{L}^*} g)(\xi) = (g \mapsto \widehat{g}_*) : H^\infty_{\mathcal{L}^*} \to \mathcal{S}(\mathbb{N})$$

by

$$\widehat{g}_*(\xi) := (\mathcal{F}_{\mathcal{L}^*} g)(\xi) = (g, e_\xi).$$

The transform $\mathcal{F}_{\mathcal{L}} : H^\infty_{\mathcal{L}} \to \mathcal{S}(\mathbb{N})$ is a bijective homeomorphism. For the inverse

$$\mathcal{F}^{-1}_{\mathcal{L}} : \mathcal{S}(\mathbb{N}) \to H^\infty_{\mathcal{L}}$$

we have

$$(\mathcal{F}^{-1}_{\mathcal{L}} h) = \sum_{\xi \in \mathbb{N}} h(\xi) e_\xi, \quad h \in \mathcal{S}(\mathbb{N}),$$

so that the inversion formula becomes

$$f = \sum_{\xi \in \mathbb{N}} \widehat{f}(\xi) e_\xi$$

for any $f \in H^\infty_{\mathcal{L}}$.

The Plancherel identity has the form

$$\|f\|_{\mathcal{H}} = \left(\sum_{\xi \in \mathbb{N}} \widehat{f}(\xi) \overline{\widehat{f}_*(\xi)} \right)^{1/2}. \tag{7}$$

Let us introduce $\mathcal{H}$–norms of functions as

$$\|f\|_{\mathcal{H}} := \left(\sum_{\xi \in \mathbb{N}} |\widehat{f}(\xi)|^2 \right)^{1/2}.$$

Roughly speaking, for any linear continuous operator $L : H^{\infty}_{\mathcal{L}} \to H^{\infty}_{\mathcal{L}}$ (or for more general operator $L : H^{\infty}_{\mathcal{L}} \to H^{-\infty}_{\mathcal{L}}$), requiring that e_ξ does not have zeros, its symbol $\sigma_L(\xi)$ can be defined by formula

$$e_\xi \sigma_L(\xi) := L e_\xi.$$

Then we have

$$Lf = \sum_{\xi \in \mathbb{N}} \sigma_L(\xi)\, \widehat{f}(\xi)\, e_\xi.$$

The correspondence between linear operators and symbols is one-to-one.

For arbitrary $s \in \mathbb{R}$ define Sobolev spaces $H^s_{\mathcal{L}}$ associated to $\mathcal{L}$:

$$H^s_{\mathcal{L}} := \left\{ f \in H^{-\infty}_{\mathcal{L}} : \mathcal{L}^{s/2} f \in \mathcal{H} \right\},$$

with the norm $\|f\|_{H^s_{\mathcal{L}}} := \|\mathcal{L}^{s/2} f\|_{\mathcal{H}}$. By taking into account the providing arguments, we may also understand it as

$$\|f\|_{H^s_{\mathcal{L}}} := \|\mathcal{L}^{s/2} f\|_{\mathcal{H}} := \left(\sum_{\xi \in \mathbb{N}} |\sigma_{\mathcal{L}}(\xi)|^s |\widehat{f}(\xi)|^2 \right)^{1/2},$$

justifying the expression (2) since $\sigma_{\mathcal{L}}(\xi) = \lambda_\xi$.

5 Conclusion

The analysis carried out in this paper showed that the numerical methods work well in the situations where a rigorous mathematical formulation of the problem is difficult within the classical theory of distributions. The concept of very weak solutions eliminates this difficulty, giving the well-posedness results for equations with δ–like coefficients. Numerical experiments showed that a notion of very weak solutions introduced in [3] is very well adapted for numerical simulations. Moreover, by the recently constructed theory of very weak solutions we can talk about uniqueness of the numerical solutions to differential equations with δ–like coefficients in a suitable appropriate sense.

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