

On The Growth Of An E -Valued Meromorphic Function And Its Derivative*

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Abstract

In this article, the relative growth of an E -valued meromorphic function and its derivative is studied and we obtain the bound for $\frac{T(r, f')}{T(r, f)}$ for an E -valued meromorphic function of finite order. We also extend the related results of S. K. Singh and H. S. Gopalakrishna [4] to E -valued meromorphic function. Our results are significant and much stronger than the result obtained by Z. Wu and Y. Chen [5].

1 Introduction

In 1982, H. J. W. Ziegler [6] successively extended the classical Nevanlinna theory of meromorphic functions to vector valued meromorphic functions in finite dimensional spaces. Later in 1996, C. G. Hu and C. C. Yang [3] established the Nevanlinna's theory in an infinite dimensional Hilbert space. C. G. Hu [2] assumed, E is an infinite dimensional Banach space with a Schauder basis $\{e_j\}$, $j = 1, 2, \dots$ and was able to present the statement of first and second fundamental theorem of Nevanlinna and Nevanlinna's deficiency relation in E . In 2006, C. G. Hu and Q. J. Hu [1] successively proved the generalized Poisson-Jensen-Nevanlinna formula, first and second fundamental theorem of Nevanlinna for E -valued meromorphic functions.

2 Basic Notions of Nevanlinna Theory in Infinite Dimensional Banach Space

Assume that E is a infinite dimensional complex Banach space with a Schauder basis $\{e_j\}_{j=1}^{\infty}$ and $\mathbb{C}$ is a complex plane. Let $D = C_r = \{z : |z| < r\}$. An E -valued meromorphic function $f(z)$ in a domain $D \subset \mathbb{C}$ can be written as

$$f(z) = \sum_{j=1}^{\infty} f_j(z)e_j = (f_1(z), f_2(z), \dots, f_j(z), \dots),$$

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where each $f_j(z)$ is a complex-valued meromorphic functions in D . We now introduce the generalized quantities of the Nevanlinna theory (see [1]): For any $a \in E \cup \{\infty\}$, $n(r, a, f) = n(r, a)$ denotes the number of a -points of f in $|z| \leq r$, counted with multiplicities and $n(r, \infty, f) = n(r, f)$ denote the number of poles of f in $|z| \leq r$. Then we have the counting function of finite or infinite a -points as

$$N(r, a) \equiv N(r, a, f) = n(0, a) \log r + \int_0^r \frac{n(t, a) - n(0, a)}{t} dt,$$

$$N(r, f) \equiv N(r, \infty, f) = n(0, f) \log r + \int_0^r \frac{n(t, f) - n(0, f)}{t} dt,$$

$$m(r, f) \equiv m(r, \infty, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \|f(re^{i\phi})\| d\phi,$$

$$m(r, a) \equiv m(r, a, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\|f(re^{i\phi}) - a\|} d\phi, (a \neq \infty),$$

and

$$T(r, f) = m(r, f) + N(r, f),$$

where $\log^+ x = \max\{\log x, 0\}$. The volume function associated with E -valued meromorphic function f is given by

$$V(r, a, f) = \frac{1}{2\pi} \int_{C_r} \log \left| \frac{r}{\xi} \right| \Delta \log \|f(\xi) - a\| d\sigma \wedge d\tau, \quad a \in E$$

and the curvature function is given by

$$V(r, 0, f') = G(r, f) = \int_0^r \frac{dt}{2\pi t} \int_{C_t} \Delta \log \|f'(\xi)\| d\sigma \wedge d\tau.$$

The order ρ of an E -valued meromorphic function f is defined by

$$\rho = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}$$

and the lower order λ of f is defined by

$$\lambda = \liminf_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}.$$

We now define the following deficiencies as in [2]: For any $a \in E \cup \{\infty\}$, the number

$$\delta(a) = \delta(a, f) = \liminf_{r \rightarrow +\infty} \frac{m(r, a)}{T(r, f)} = 1 - \limsup_{r \rightarrow +\infty} \frac{V(r, a) + N(r, a)}{T(r, f)}$$

is called the deficiency of the point a , a point a with $\delta(a) > 0$ is called deficient.

The quantity

$$\theta(a) = \theta(a, f) = \liminf_{r \rightarrow +\infty} \frac{N(r, a) - \overline{N}(r, a)}{T(r, f)}$$

is called the index of multiplicity of a , and

$$\begin{aligned}\Theta(a) = \Theta(a, f) &= \liminf_{r \rightarrow +\infty} \frac{m(r, a) + N(r, a) - \bar{N}(r, a)}{T(r, f)} \\ &= 1 - \limsup_{r \rightarrow +\infty} \frac{V(r, a) + \bar{N}(r, a)}{T(r, f)}.\end{aligned}$$

In particular, we have

$$\begin{aligned}\delta(\infty) &= \liminf_{r \rightarrow +\infty} \frac{m(r, f)}{T(r, f)} = 1 - \limsup_{r \rightarrow +\infty} \frac{N(r, f)}{T(r, f)} \quad \text{since } V(r, \infty) = 0, \\ \theta(\infty) &= \liminf_{r \rightarrow +\infty} \frac{N(r, f) - \bar{N}(r, f)}{T(r, f)}, \\ \Theta(\infty) &= 1 - \limsup_{r \rightarrow +\infty} \frac{\bar{N}(r, f)}{T(r, f)}.\end{aligned}$$

The quantity

$$\delta_G = \delta_G(f) = \liminf_{r \rightarrow +\infty} \frac{G(r, f)}{T(r, f)}$$

is called the Ricci Index of $f(z)$.

The function f is called admissible if $\frac{S(r_\gamma)}{T(r_\gamma, f)} \rightarrow 0$ for a sequence $r_\gamma \rightarrow +\infty$.

THEOREM 1 ([1]). (E -valued Nevanlinna's first fundamental theorem) Let $f(z)$ be an E -valued meromorphic mapping in C_R . Then for $0 < r < R$, $a \in E$, $f(z) \neq a$,

$$T(r, f) = V(r, a) + N(r, a) + m(r, a) + \log \|c_q(a)\| + \epsilon(r, a).$$

Here $\epsilon(r, a)$ is a function such that $|\epsilon(r, a)| \leq \log^+ \|a\| + \log 2$, $\epsilon(r, 0) \equiv 0$, and $c_q(a) \in E$ is the co-efficient of the first term in the Laurent series at the point a .

THEOREM 2 ([1]). (E -valued Nevanlinna's second fundamental theorem) Let $f(z)$ be a non-constant E -valued meromorphic mapping of compact projection in C_R and $a^{[k]} \in E \cup \{\infty\}$ ($k = 1, 2, \dots, q$) be $q \geq 3$ distinct finite or infinite points. Then

$$\sum_{k=1}^q m(r, a^{[k]}) + G(r, f) \leq T(r, f) - N_1(r) + S(r),$$

where $N_1(r) = N(r, 0, f') + 2N(r, f) - N(r, f')$ and

$$G(r, f) = \int_0^r \frac{dt}{2\pi t} \int_{C_t} \Delta \log \|f'(\xi)\| d\sigma \wedge d\tau.$$

If $R = +\infty$, then $S(r)$ satisfies $S(r) = O\{\log T(r, f)\} + O(\log r)$ as $r \rightarrow +\infty$ without exception if $f(z)$ has finite order and otherwise as $r \rightarrow +\infty$ outside a set J of exceptional intervals of finite measure $\int_J dr < +\infty$. If $0 < R < +\infty$, then

$$S(r) = O\{\log^+ T(r, f)\} + O\left\{\log \frac{1}{R-r}\right\}$$

holds as $r \rightarrow R$ without exception if f has finite order

$$\rho = \limsup_{r \rightarrow R} \frac{\log T(r, f)}{\log(1/R - r)},$$

and otherwise as $r \rightarrow R$ outside of a set J exceptional intervals such that $\int_J d\frac{1}{R-r} < +\infty$. In all cases, the exceptional set J is independent of the choice of the finite points $a^{[k]} \in E$ and of their number.

THEOREM 3 ([2]). (*E*-valued Nevanlinna deficiency relation) Let $f(z)$ be an *E*-valued meromorphic function and admissible with the property of compact projection. Then the set $\{a \in E \cup \{\infty\} : \Theta(a) > 0\}$ is at most countable and summing over all such points

$$\sum_a [\delta(a) + \theta(a)] + \delta_G \leq \sum_a \Theta(a) + \delta_G \leq 2.$$

THEOREM 4 (Lemma 3.1(A) of [1]) Let $f(z)$ be an *E*-valued meromorphic function with the property of compact projection, and let

$$\begin{aligned} S_1(r) = & \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{\|f'(re^{i\phi})\|}{\|f(re^{i\phi})\|} d\phi + \frac{1}{2\pi} \int_0^{2\pi} \log^+ [F(re^{i\phi}) \|f(re^{i\phi})\|] d\phi \\ & + p \log^+ \frac{2p}{\delta} - \log \|c'_{l'}\|. \end{aligned}$$

Then

$$G(r) + \sum_{k=1}^{p+1} m(r, a^{[k]}) + N_1(r) \leq 2T(r, f) + S_1(r),$$

where $N_1(r) = N(r, 0, f') + 2N(r, f) - N(r, f')$ is the generalized counting function of multiple points, $a^{[\nu]} = (a_1^{[\nu]}, \dots, a_j^{[\nu]}, \dots)$ ($p \geq 2$) $\in E$ are distinct finite points, and

$$F(z) = \sum_{\nu=1}^p \frac{1}{\|f(z) - a^{[\nu]}\|}.$$

3 Main Results

S. K. Singh and H. S. Gopalkrishna [4] proved the following result:

THEOREM 5. If f is a non-constant meromorphic function of order ρ then

$$\liminf_{r \rightarrow \infty} \frac{T(r, f')}{T(r, f)} \geq \sum_{a \in \mathbb{C}} \Theta(a, f)$$

where $r \rightarrow \infty$ without restriction if ρ is finite and $r \rightarrow \infty$ outside an exceptional set of finite measure if $\rho = +\infty$.

In [5], Z. Wu and Y. Chen proved the following result.

THEOREM 6. Let $f(z)$ be an admissible E -valued meromorphic function of compact projection in $\mathbb{C}$ of finite order and assume $\sum_a \delta(a) = 2$. Then

$$\lim_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \delta(\infty).$$

Now in this article, we obtain a **THEOREM 5** for E -valued meromorphic function $f(z)$ in modified form and also extend the related results of S. K. Singh and H. S. Gopalakrishna [4]. **THEOREM 6** is also proved as a consequence of our main result. We prove the following main results.

THEOREM 7. Let $f(z)$ be an admissible and non-constant E -valued meromorphic function of finite order ρ with compact projection then

$$\sum_{a \in E} \Theta(a, f) + \delta_G \leq \liminf_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)},$$

where $r \rightarrow +\infty$ without restriction if ρ is finite and $r \rightarrow +\infty$ outside an exceptional set of finite measure if $\rho = +\infty$.

To prove **THEOREM 7**, we first prove the following Lemma, which plays an prominent role in the proof of the **THEOREM 7**.

LEMMA 1. Let $f(z)$ be a non-constant E -valued meromorphic function with the property of compact projection in C_r and

$$a^{[\gamma]} = (a_1^{[\gamma]}, a_2^{[\gamma]}, \dots, a_j^{[\gamma]}, \dots) \quad (p \geq 2) \in E$$

are finite or infinite distinct points then

$$\sum_{\mu=1}^p m(r, a^{[\mu]}, f) + N\left(r, \frac{1}{f'}\right) + G(r, f) \leq T(r, f') + S(r, f),$$

where

$$S(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \{F(re^{i\phi}) \|f'^{i\phi}\|\} d\phi - \log \|c'_p\| + p \log^+ \frac{2p}{\delta}$$

and

$$F(z) = \sum_{\nu=1}^p \frac{1}{\|f(z) - a^{[\nu]}\|}.$$

PROOF. Following the proof of Lemma 3.1(A) in [1], we obtain the required result.

PROOF OF THEOREM 7. Let $\{a^{[\mu]}\}$, $\mu = 1, 2, \dots, \infty$ be an infinite sequence of distinct elements of E , which includes every $a \in E$ for which $\Theta(a, f) > 0$. Then

$$\sum_{\mu=1}^{\infty} \Theta(a^{[\mu]}, f) = \sum_{a \in E} \Theta(a, f). \quad (1)$$

We have

$$\sum_{\mu=1}^p m(r, a^{[\mu]}, f) + G(r, f) \leq T(r, f') - N\left(r, \frac{1}{f'}\right) + S(r, f).$$

Adding $\sum_{\mu=1}^p N(r, a^{[\mu]}, f)$ to both sides, we obtain

$$\begin{aligned} \sum_{\mu=1}^p T(r, a^{[\mu]}, f) + G(r, f) &\leq T(r, f') + \sum_{\mu=1}^p N(r, a^{[\mu]}, f) - N\left(r, \frac{1}{f'}\right) + S(r, f) \\ &= T(r, f') + \sum_{\mu=1}^p \bar{N}(r, a^{[\mu]}, f) - N_0\left(r, \frac{1}{f'}\right) + S(r, f), \end{aligned}$$

where $N_0\left(r, \frac{1}{f'}\right)$ is formed with the zeros of f' which are not zeros of any of $f - a^{[\mu]}$ ($\mu = 1, 2, \dots, p$). Since $N_0\left(r, \frac{1}{f'}\right) \geq 0$, we have

$$\sum_{\mu=1}^p T(r, a^{[\mu]}, f) \leq T(r, f') + \sum_{\mu=1}^p \bar{N}(r, a^{[\mu]}, f) - G(r, f) + S(r, f).$$

By an E -valued Nevanlinna's first fundamental theorem, we have

$$T(r, a, f) = T(r, f) - V(r, a, f) + O(1).$$

Using this in the above equation, we obtain

$$\sum_{\mu=1}^p \left[T(r, f) - V(r, a^{[\mu]}, f) + O(1) \right] \leq T(r, f') + \sum_{\mu=1}^p \bar{N}(r, a^{[\mu]}, f) - G(r, f) + S(r, f).$$

We further obtain

$$pT(r, f) \leq T(r, f') + \sum_{\mu=1}^p \left[\bar{N}(r, a^{[\mu]}, f) + V(r, a^{[\mu]}, f) \right] - G(r, f) + S(r, f).$$

Then

$$\begin{aligned} p &\leq \liminf_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} + \sum_{\mu=1}^p \limsup_{r \rightarrow +\infty} \frac{\bar{N}(r, a^{[\mu]}, f) + V(r, a^{[\mu]}, f)}{T(r, f)} - \liminf_{r \rightarrow +\infty} \frac{G(r, f)}{T(r, f)} \\ &\quad + \limsup_{r \rightarrow +\infty} \frac{S(r, f)}{T(r, f)}. \end{aligned}$$

It follows that

$$p \leq \liminf_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} + \sum_{\mu=1}^p \left[1 - \Theta(a^{[\mu]}, f) \right] - \delta_G(f).$$

So

$$\sum_{\mu=1}^p \Theta(a^{[\mu]}, f) + \delta_G(f) \leq \liminf_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)}.$$

Letting $p \rightarrow \infty$ and using (1), we get

$$\sum_{a \in E} \Theta(a, f) + \delta_G(f) \leq \liminf_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)}. \quad (2)$$

COROLLARY 1. Let $f(z)$ be a admissible E -valued meromorphic function of finite order ρ with the property of compact projection such that

$$\sum_{a \in \overline{E}} \Theta(a, f) + \delta_G = 2, \quad \overline{E} = E \cup \{\infty\}.$$

Then

(i)

$$\lim_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \Theta(\infty, f).$$

(ii)

$$\begin{aligned} 1 - \Theta(a, f) + \delta_G &\leq \liminf_{r \rightarrow +\infty} \frac{V(r, a) + \overline{N}(r, a)}{T(r, f)} \leq \limsup_{r \rightarrow +\infty} \frac{V(r, a) + \overline{N}(r, a)}{T(r, f)} \\ &= 1 - \Theta(a, f). \end{aligned}$$

PROOF. Given that

$$\sum_{a \in \overline{E}} \Theta(a, f) + \delta_G = 2,$$

we have

$$\sum_{a \in E} \Theta(a, f) + \Theta(\infty, f) + \delta_G = 2.$$

It follows that

$$\sum_{a \in E} \Theta(a, f) + \delta_G = 2 - \Theta(\infty, f).$$

Using (2), we write

$$\liminf_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} \geq \sum_{a \in E} \Theta(a, f) + \delta_G = 2 - \Theta(\infty, f).$$

On the other hand, we know that

$$\begin{aligned} T(r, f') &= m(r, f') + N(r, f') = m(r, \frac{f'}{f}) + m(r, f) + N(r, f') \\ &\leq T(r, f) + \overline{N}(r, f) + S(r, f) \end{aligned}$$

and

$$\limsup_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} \leq 1 + \limsup_{r \rightarrow +\infty} \frac{\overline{N}(r, f)}{T(r, f)}.$$

So

$$\limsup_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} \leq 2 - \Theta(\infty, f).$$

Thus

$$\lim_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \Theta(\infty, f).$$

(ii) Let $a \in E \cup \{\infty\}$ and $\{a^{[k]}\}, k = 1, 2, \dots, \infty$ be an infinite sequence of distinct elements of $E \cup \{\infty\}$ which includes every $b \in E \cup \{\infty\}$ such that $b \neq a$ and $\Theta(b, f) \neq 0$. Then

$$\sum_{k=1}^{\infty} \Theta(a^{[k]}, f) = \sum_{b \in E, b \neq a} \Theta(b, f) = 2 - \Theta(a, f). \quad (3)$$

By *E*-valued Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} (q-2)T(r, f) + G(r, f) &\leq \sum_{k=1}^{q-1} [V(r, a^{[k]}, f) + \overline{N}(r, a^{[k]}, f)] \\ &\quad + [V(r, a, f) + \overline{N}(r, a, f)] + S(r, f), \end{aligned}$$

$$\begin{aligned} (q-2)T(r, f) &\leq \sum_{k=1}^{q-1} [V(r, a^{[k]}, f) + \overline{N}(r, a^{[k]}, f)] + [V(r, a, f) + \overline{N}(r, a, f)] \\ &\quad - G(r, f) + S(r, f), \end{aligned}$$

$$\begin{aligned} (q-2)T(r, f) &\leq \sum_{k=1}^{q-1} \frac{[V(r, a^{[k]}, f) + \overline{N}(r, a^{[k]}, f)]}{T(r, f)} + \frac{[V(r, a, f) + \overline{N}(r, a, f)]}{T(r, f)} \\ &\quad - \frac{G(r, f)}{T(r, f)} + \frac{S(r, f)}{T(r, f)}, \end{aligned}$$

$$\begin{aligned} (q-2) &\leq \sum_{k=1}^{q-1} \limsup_{r \rightarrow +\infty} \frac{[V(r, a^{[k]}, f) + \overline{N}(r, a^{[k]}, f)]}{T(r, f)} \\ &\quad + \liminf_{r \rightarrow +\infty} \frac{[V(r, a, f) + \overline{N}(r, a, f)]}{T(r, f)} - \liminf_{r \rightarrow +\infty} \frac{G(r, f)}{T(r, f)} + \limsup_{r \rightarrow +\infty} \frac{S(r, f)}{T(r, f)}, \end{aligned}$$

$$\begin{aligned}
(q-2) &\leq \liminf_{r \rightarrow +\infty} \frac{[V(r, a, f) + \overline{N}(r, a, f)]}{T(r, f)} + \sum_{k=1}^{q-1} [1 - \Theta(a^{[k]}, f)] - \delta_G, \\
(q-2) + \delta_G &\leq \liminf_{r \rightarrow +\infty} \frac{[V(r, a, f) + \overline{N}(r, a, f)]}{T(r, f)} + (q-1) - \sum_{k=1}^{q-1} \Theta(a^{[k]}, f), \\
\delta_G - 1 &\leq \liminf_{r \rightarrow +\infty} \frac{[V(r, a, f) + \overline{N}(r, a, f)]}{T(r, f)} - \sum_{k=1}^{q-1} \Theta(a^{[k]}, f).
\end{aligned}$$

So

$$\liminf_{r \rightarrow +\infty} \frac{[V(r, a, f) + \overline{N}(r, a, f)]}{T(r, f)} \geq \sum_{k=1}^{q-1} \Theta(a^{[k]}, f) + \delta_G - 1.$$

Let $q \rightarrow \infty$ and using (3), we get

$$\begin{aligned}
\liminf_{r \rightarrow +\infty} \frac{[V(r, a, f) + \overline{N}(r, a, f)]}{T(r, f)} &\geq \sum_{k=1}^{\infty} \Theta(a^{[k]}, f) + \delta_G - 1 \\
&= 2 - \Theta(a, f) + \delta_G - 1 = 1 - \Theta(a, f) + \delta_G.
\end{aligned}$$

On the other hand, by the definition of $\Theta(a, f)$, we have

$$\limsup_{r \rightarrow +\infty} \frac{[V(r, a, f) + \overline{N}(r, a, f)]}{T(r, f)} = 1 - \Theta(a, f).$$

Thus

$$\begin{aligned}
1 - \Theta(a, f) + \delta_G &\leq \liminf_{r \rightarrow +\infty} \frac{[V(r, a, f) + \overline{N}(r, a, f)]}{T(r, f)} \\
&\leq \limsup_{r \rightarrow +\infty} \frac{[V(r, a, f) + \overline{N}(r, a, f)]}{T(r, f)} = 1 - \Theta(a, f).
\end{aligned}$$

COROLLARY 2 Let $f(z)$ be an admissible E -valued meromorphic function of finite order ρ with the property of compact projection such that

$$\sum_{a \in \overline{E}} \delta(a, f) + \delta_G = 2.$$

Then

$$\lim_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \delta(\infty, f).$$

PROOF. We know that $\delta(a, f) \leq \Theta(a, f)$, $\forall a \in E \cup \{\infty\} = \overline{E}$ and

$$\sum \delta(a, f) + \delta_G \leq \sum \Theta(a, f) + \delta_G \leq 2.$$

Given $\sum \delta(a, f) + \delta_G = 2$. Then $\sum \Theta(a, f) + \delta_G = 2$. We observe that

$$\sum \delta(a, f) + \delta_G = \sum \Theta(a, f) + \delta_G = 2.$$

Then

$$\sum_{a \in \overline{E}} \delta(a, f) = \sum_{a \in \overline{E}} \Theta(a, f).$$

So

$$\delta(a, f) = \Theta(a, f) \quad \forall a \in \overline{E}.$$

By using Corollary 1(i), we have

$$\lim_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \Theta(\infty, f) = 2 - \delta(\infty, f).$$

So

$$\lim_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \delta(\infty, f)$$

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