Applied Mathematics E-Notes, 12(2012), 136-152 © ISSN 1607-2510
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A Survey On Functions Of Bounded Boundary And Bounded Radius Rotation*

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Received 28 September 2011

Abstract

The article aims at giving a brief survey of functions of bounded boundary and bounded radius rotation. The generalizations of these classes are also discussed along with numerous properties of these generalized classes. We also list some samples which reflect our recent investigations in the geometric function theory.

1 Introduction

A function analytic and locally univalent in a given simply connected domain is said to be of bounded boundary rotation if its range has bounded boundary rotation which is defined as the total variation of the direction angle of the tangent to the boundary curve under a complete circuit.

Let V_k denote the class of analytic functions f defined in the open unit disc $E = \{z : |z| < 1\}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in E. \tag{1}$$

and which maps E conformally onto an image domain of boundary rotation at most $k\pi$.

The concept of functions of bounded boundary rotation originates from Loewner [19] in 1917 but he did not use the present terminology. It was Paatero [86, 87] who systematically developed their properties and made an exhaustive study of the class V_k. Paatero [86] has shown that $f \in V_k$ if and only if

$$f'(z) = \exp\left\{-\int_0^{2\pi} \log(1 - ze^{-it})d\mu(t)\right\}, \tag{2}$$

*Mathematics Subject Classifications: 35A07, 35Q53.

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where $\mu(t)$ is a real-valued function of bounded variation for which

$$\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq k. \tag{3}$$

For a fixed $k \geq 2$, it can also be expressed as

$$\int_0^{2\pi} \left| \mathrm{Re} \frac{(zf'(z))'}{f'(z)} \right| d\theta \leq k\pi, \quad z = re^{i\theta}. \tag{4}$$

Clearly, if $k_1 < k_2$, then $V_{k_1} \subset V_{k_2}$, that is the class V_k obviously expands as k increases. V_2 is simply the class C of convex univalent functions. Paatero [86] showed that $V_4 \subset S$, where S is the class of normalized univalent functions. For more explanations of C and S, see [11]. Later, Pinchuk [89] proved that functions in V_k are close-to-convex in E if $2 \leq k \leq 4$ and hence univalent.

A function f analytic in E is said to be close-to-convex, if there exists a function $g \in C$ such that

$$\mathrm{Re} \frac{f'(z)}{g'(z)} > 0 \quad \text{for all} \quad z \in E. \tag{5}$$

Kirwan [17] showed that the radius of univalence of V_k for $k > 4$ is $\tan(\frac{\pi}{k})$. In [3], Brannan showed that V_k is a subclass of the class $K(\alpha)$ of close-to-convex functions of order $\alpha = \frac{k}{2} - 1$. The Class $K(\alpha)$ for $\alpha \geq 0$ has been introduced by Goodman [11].

Paatero [87] gave the distortion bounds for the functions $f \in V_k$, that is for $|z| = r < 1$,

$$\frac{(1-r)^{\frac{k}{2}-1}}{(1+r)^{\frac{k}{2}+1}} \leq |f'(z)| \leq \frac{(1+r)^{\frac{k}{2}-1}}{(1-r)^{\frac{k}{2}+1}}. \tag{6}$$

Both bounds in (6) are sharp for each r in $(0,1)$ by the function

$$F_k(z) = \frac{1}{k}\left(\frac{1+z}{1-z}\right)^{\frac{k}{2}} - \frac{1}{k} = z + \sum_{n=2}^{\infty} B_n(k) z^n. \tag{7}$$

In particular F_4 is the Koebe function which is $F_4 = \frac{z}{(1-z)^2}$ and F_2 is the half-plane mapping $\frac{z}{1-z}$, the typical extremal function for problems involving convex functions. The sharpness of the function means that it is impossible, under the given conditions to improve the inequality (decrease an upper bound, or increase a lower bound) because there is an admissible function for which the equal sign holds.

The problem of finding the coefficient bounds f in V_k was opened for twenty years till finally solved by Aharonov and Friedland [1] and Brannan [4]. For $f \in V_k$ given by (1), we have

$$|a_n| \leq B_n(k), \tag{8}$$

where $B_n(k)$ is defined by (7). Many well-known mathematicians tried to derive a solution for this problem. There is a long list including particularly Lehto [18], Schiffer and Tammi [95], Lonka and Tammi [20], Coonce [7], Noonan [24] and Brannan et al. [17].

Brannan [3] gave another representation for functions of bounded boundary rotation in terms of starlike functions. For detailed study of these functions, see [11]. That is, $f \in V_k$ if and only if there exist two starlike functions s_1, s_2 in E, such that

$$f'(z) = \frac{\left(\frac{s_1(z)}{z}\right)^{\left(\frac{k}{4}+\frac{1}{2}\right)}}{\left(\frac{s_2(z)}{z}\right)^{\left(\frac{k}{4}-\frac{1}{2}\right)}}. \tag{9}$$

In 1971, Pinchuk [90] introduced and studied the classes P_k and R_k, where R_k generalizes the starlike functions in the same manner as the class V_k generalizes convex functions.

Let R_k denote the class of analytic functions f of the form (1) having the representation

$$f(z) = z \exp\left\{-\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\mu(t)\right\}, \tag{10}$$

where $\mu(t)$ is as given in (3). Pinchuk also showed that Alexander type relation between the classes V_k and R_k exists

$$f \in V_k \text{ if and only if } zf' \in R_k. \tag{11}$$

R_k consists of those functions f which satisfy

$$\int_{-\pi}^{\pi} \left|\operatorname{Re}\left\{re^{i\theta}\frac{f'^{i\theta})}{f(re^{i\theta})}\right\}\right| d\theta \leq k\pi \text{ for } r < 1,\ z = re^{i\theta}. \tag{12}$$

Geometrically, the condition is that the total variation of angle between radius vector $f(re^{i\theta})$ makes with positive real axis is bounded by $k\pi$. Thus R_k is the class of functions of bounded radius rotation bounded by $k\pi$.

P_k denotes the class of functions $p(0) = 1$ analytic in E and having the representation

$$p(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t), \tag{13}$$

where $\mu(t)$ is defined by (3). Clearly $P_2 = P$, where P is the class of analytic functions with positive real part. For more details see [11].

From (13), one can easily find that $p \in P_k$ can also be written as

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \text{ where } p_1,\ p_2 \in P. \tag{14}$$

Pinchuk [90] has shown that the classes V_k and R_k can be defined by using the class P_k as given below.

$$f \in V_k \text{ if and only if } \frac{(zf'(z))'}{f'(z)} \in P_k \tag{15}$$

and

$$f \in R_k \text{ if and only if } \frac{zf'(z)}{f(z)} \in P_k. \tag{16}$$

2 Some Classes Related with V_k and R_k

The concept of order of a function for both V_k and R_k was first given by Noonan [24] in 1972. But the purpose of his article is to extend the results in [14] to V_k and R_k. The actual concept and thorough study of order referring to Roberston [92] was first presented by Padmanabhan and Parvatham [88] in 1975 for the classes V_k and R_k. They introduced the class $P_k(\alpha)$ as follows:

DEFINITION 2.1. Let $P_k(\alpha)$ be the class of functions p analytic in E satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\mathrm{Re}p(z) - \alpha}{1-\alpha} \right| d\theta \leq k\pi, \tag{17}$$

where $z = re^{i\theta}, k \geq 2$ and $0 \leq \alpha < 1$. For $\alpha = 0$, we have $P_k(0) = P_k$.

DEFINITION 2.2. Let f be analytic function given by (1). Then $f \in V_k(\alpha), k \geq 2,$ if and only if

$$1 + \frac{zf''(z)}{f'(z)} \in P_k(\alpha),\ 0 \leq \alpha < 1,\ z \in E. \tag{18}$$

DEFINITION 2.3. Let f be analytic function given by (1). Then $f \in R_k(\alpha)$, $k \geq 2$ if and only if

$$\frac{zf'(z)}{f(z)} \in P_k(\alpha),\ 0 \leq \alpha < 1,\ z \in E. \tag{19}$$

For $k = 2$, we have $V_2(\alpha) = C(\alpha)$ and $R_2(\alpha) = S^*(\alpha)$, where $C(\alpha)$ and $S^*(\alpha)$ are the classes of convex and starlike functions of order α respectively. These classes have been introduced by Roberston [92] in 1936. The classes $V_k(\alpha)$ and $R_k(\alpha)$ are thoroughly investigated by Padmanabhan and Parvatham. Later Noor [35] explored the different properties for these classes. Radius of convexity of order α for class $V_k(\alpha)$ and some integral results for these classes are discussed. Noor [37] proved that each class $P_k(\alpha)$ is a convex set. In the same article, distortion bounds for the class $P_k(\alpha), V_k(\alpha)$, coefficient bound for $V_k(\alpha)$ and radius of convexity for $V_k(\alpha)$ have also been studied. Here, we give the radius of convexity for $V_k(\alpha)$ which is the largest number r_0 such that $f(r_0 z)$ is convex in E for all $f \in V_k(\alpha)$.

THEOREM 2.1. Let $f \in V_k(\alpha), \alpha \neq \frac{1}{2}$. Then f maps $|z| < r_0$ onto a convex domain, where r_0 is given as

$$r_0 = \frac{2}{k(1-\alpha) + \sqrt{k^2(1-\alpha)^2 - 4(1-2\alpha)}}, \alpha \neq \frac{1}{2}. \tag{20}$$

This result is sharp by taking f_0, where

$$f_0'(z) = \frac{(1+\delta_1 z)^{(\frac{k}{2}-1)(1-\alpha)}}{(1-\delta_2 z)^{(\frac{k}{2}+1)(1-\alpha)}}, \;\; |\delta_1| = |\delta_2| = 1. \tag{21}$$

For $\alpha = 0$, we have radius of convexity for the class V_k proved by Pinchuk [90]. Noor et al. [79, 70] and Noor [43, 47, 49] defined these classes by using Noor integral operator, Ruscheweyh derivative operator, generalized Bernardi integral and Jim-Kim-Srivastava operator respectively. Some differential operators are discussed under these classes in [56]. Recently Noor et al. [62] proved the result of Goel [13] for the classes $V_k(\alpha)$ and $R_k(\alpha)$ by using three different techniques.

THEOREM 2.2. Let $f \in V_k(\alpha)$, $0 \le \alpha < 1$. Then $f \in R_k(\beta)$, where

$$\beta = \beta(\alpha) = \begin{cases} \frac{4^\alpha(1-2\alpha)}{4-2^{2\alpha+1}}, & \alpha \neq \frac{1}{2} \\ \frac{1}{2\ln 2}, & \alpha = \frac{1}{2}. \end{cases}$$

For $\alpha = 0$, this result also generalizes the result of Marx [21] and Strohhacker [94]. In 1977, Nasr [23] introduced the class $V_k(b), b \in \mathbb{C} - \{0\}$, $k \ge 2$, of functions of bounded boundary rotation of complex order. Later Noor et al. [63] generalized this concept and introduced the class $V_k(\alpha, b)$ and $R_k(\alpha, b)$ and studied mapping properties of these classes under certain integral operator. In [15], Janowski introduced the class $P[A, B]$. For A and B, $-1 \le B < A \le 1$, a function p analytic in E with $p(0) = 1$ belongs to the class $P[A, B]$ if $p(z) \prec \frac{1+Az}{1+Bz}$, where $\prec$ denotes the usual meaning of subordination. The class $P_k[A, B]$ is defined similarly to the class P_k given by (14) by taking p_1, p_2 from the class $P[A, B]$. Analogous to the class $P[A, B]$, Noor [31] defined the class $V_k[A, B]$ and $R_k[A, B]$ as follows.

DEFINITION 2.4. A function f analytic in E and given by (1) is said to be in the class $V_k[A, B]$, $-1 \le B < A \le 1$, $k \ge 2$ if and only if $\frac{(zf'(z))'}{f'(z)} \in P_k[A, B]$.

DEFINITION 2.5. A function f analytic in E and of the form (1) is said to be in the class $R_k[A, B]$, $-1 \le B < A \le 1$, $k \ge 2$, if and only if $\frac{zf'(z)}{f(z)} \in P_k[A, B]$.

For $k = 2$, we have classes $S^*[A, B]$ and $C[A, B]$ defined in [15]. Clearly $S^*[A, B] \subset S^*(\frac{1-A}{1-B}) \subset S^*[1, -1] = S^*$ and $C[A, B] \subset C(\frac{1-A}{1-B}) \subset C[1, -1] = C$.

Noor [34] obtained the sharp results of radius of convexity and starlikeness for the classes $V_k[A, B]$ and $R_k[A, B]$ respectively. In the same paper, Noor extended the results of [31]. Here we only give the radius of convexity for the class $V_k[A, B]$ and observe that by taking $A = 1 - 2\alpha, B = -1$, we obtain the sharp results given by (20).

THEOREM 2.3. Let $f \in V_k[A, B]$. Then f maps $|z| < 1$, onto a convex domain, where

$$r_1 = \frac{2}{\frac{k}{2}(A-B) + \sqrt{\frac{k^2}{4}(B-A)^2 + 4AB}} \tag{22}$$

This result is sharp.

The behaviour of these classes under certain linear operators are investigated in [36, 59, 57]. The linear combination of functions belonging to the class $V_k[A, B]$ and $R_k[A, B]$ are studied in [73].

In 1994, Noor [39], generalizes this concept to define the class $V_k^b[A, B]$ of functions of bounded boundary rotation of complex order related with class $P_k[A, B]$. In the same article, Noor also generalizes the result of Brannan [3] as follows:

THEOREM 2.4. For $b \neq 0$ real, $f \in V_k^b[A, B]$ if and only if there exist two functions $s_1, s_2 \in S^*[A, B]$ such that

$$f'(z) = \frac{\left(\frac{s_1(z)}{z}\right)^{b\left(\frac{k+2}{4}\right)}}{\left(\frac{s_2(z)}{z}\right)^{b\left(\frac{k-2}{4}\right)}}. \tag{23}$$

For $b = 1$, this result coincides with the expression (9).

3 Generalizations of Close-to-Convexity

In 1983, Noor [28] introduced the class T_k of analytic functions in which the concept of close-to-convexity is generalized. A necessary condition for a function $f \in T_k$, a radius of convexity problem and coefficient result are also solved in this article. Some more problems related to radius of convexity problems for the classes T_k and V_k are discussed in [71]. The class T_k was first considered by Noor in [26] in 1980 but was thoroughly explored later in [28].

DEFINITION 3.1. Let f be analytic in E. Then $f \in T_k$, $k \geq 2$, if there exists a function $g \in V_k$ such that

$$\mathrm{Re}\frac{f'(z)}{g'(z)} > 0. \tag{24}$$

$T_2 \equiv K$ is the class of close-to-convex functions introduced by Kaplan [16]. Here we only give the rate of growth which was proved by Noor in [26].

THEOREM 3.1. Let $f \in T_k$. Then for $n \geq 1$,

$$||a_n| - |a_{n+1}|| \leq c(k) n^{\frac{k}{2}-1},\ k \geq 2, \tag{25}$$

where $c(k)$ is a constant and depends only on k.

In [26], the class K_{kk} is defined which is the generalization of functions of bounded boundary rotation.

DEFINITION 3.2. Let f be analytic in E and of the form (1). Then $f \in K_{kk}$ if there exists a function $g \in V_k$ such that for $z = re^{i\theta}$ in E

$$\int_0^{2\pi} \left|\mathrm{Re}\frac{f'(z)}{g'(z)}\right| d\theta \leq k\pi,\ k \geq 2. \tag{26}$$

Clearly $K_{2k} \equiv T_k$ and $K_{22} \equiv K$.

The geometrical interpretation, coefficient results and Hankel determinant problem are solved in [26]. The concept of order for the classes T_k and K_{kk} was first given by Noor [37] and [38] respectively. Rate of growth, Hankel determinant, arc-length problem are also studied for these classes. The rate of growth problem for the class K_{kk} is given as follows:

THEOREM 3.2. Let $f \in K_{kk}$ and of the form (1) Then

$$||a_n| - |a_{n+1}|| \leq A(k) n^{\frac{k}{2}-1},\ k \geq 2, \tag{27}$$

where $A(k)$ is a constant depending upon k only. The function f_0 given by

$$f_0(z) = \frac{k}{2(k+2)} \left\{ \left(\frac{1+z}{1-z} \right)^{\frac{k}{2}+1} - 1 \right\} \tag{28}$$

shows that the exponent $\left(\frac{k}{2} - 1\right)$ is best possible.

For $k = 2$, we have a result of Theorem 3.1. The class $T_k[A, B]$ was first introduced by Noor [36], we define it as follows.

DEFINITION 3.3. Let $f \in A$ be analytic in E and be given by (1). Then f is said to belong to the class $T_k[A, B]$, $-1 \leq B < A < 1$, $k \geq 2$, if and only if there exists a function $g \in V_k[A, B]$ such that

$$\frac{f'(z)}{g'(z)} \in P[A, B]. \tag{29}$$

THEOREM 3.3 ([36]). Let $f \in T_k[A, B]$ with respect to $h \in V_k[A, B]$. Let $g \in R_k[A, B]$ and for $\alpha + \beta = 1$, α, $\beta \geq 0$, let F be defined as

$$F(z) = \int_0^z f'^{\alpha} \left(\frac{g(t)}{t} \right)^{\beta} dt, \tag{30}$$

where F is close-to-convex with respect to H defined by

$$H(z) = \int_0^z (h'(t))^{\alpha} \left(\frac{g(t)}{t} \right)^{\beta} dt, \quad \text{for all } |z| < r_1, \tag{31}$$

where r_1 is given by

$$r_1 = \frac{4}{[k(A-B) + \sqrt{k^2(B-A)^2 + 16AB}]}. \tag{32}$$

Recently this result is generalized by Noor [57] for the class $T_k[A, B, \rho]$. The classes related to T_k are studied with reference to certain integral operators in [83].

The generalized concept of strongly-close-to-convexity is given by Noor [44, 55] and Noor et al. [80].

DEFINITION 3.4. Let f be analytic in E and given by (1). Then $f \in K(k,\alpha)$ if and only if for $k \geq 2, \alpha \geq 0$, there exists a function $g \in V_k$ such that

$$\left|\arg \frac{f'(z)}{g'(z)}\right| \leq \alpha \frac{\pi}{2}. \tag{33}$$

For $k = 2$, $K(2,\alpha)$ is the class of strongly close-to-convex functions of order α.

THEOREM 3.4. Let f be analytic in E and of the form of (1) with $k+\alpha > 4, \alpha < 2$. Then for $n \geq 1$,

$$||a_{n+1}| - |a_n|| \leq c(k,\alpha) n^{\frac{k}{2}+\alpha-2}, \tag{34}$$

where $c(k,\alpha)$ is a constant.

Noor further generalized the concept of strongly close-to-convex and defined the class $\widetilde{T}_k(\rho,\alpha)$ as follows.

DEFINITION 3.5. Let f be analytic in E and of the form (1). Then $f \in \widetilde{T}_k(\rho,\alpha)$ if and only if for $k \geq 2$, $\alpha \geq 0$, there exists a function $g \in V_k(\rho)$ such that

$$\left|\arg \frac{f'(z)}{g'(z)}\right| \leq \alpha \frac{\pi}{2}. \tag{35}$$

For $k = 2, \rho = 0$, $\widetilde{T_2}(0,\alpha)$ is a class of strongly close-to-convex functions of order α in the sense of Pommerenke [91]. For $\rho = 0$, the class $\widetilde{T_k}(0,\alpha)$ is defined by Noor [44].

A necessary condition, distortion results, a radius problem, coefficient results and Hankel determinant problem for this class are studied in [55]. Here we only give the rate of growth of coefficients for the class $\widetilde{T_k}(\rho,\alpha)$.

THEOREM 3.5. Let $f \in \widetilde{T}_k(\rho,\alpha)$ as given in (1) with $k > \left(\frac{2-\alpha}{1-\rho} + 2\right), \alpha < 2$. Then for $n \geq 1$,

$$||a_{n+1}| - |a_n|t| \leq c(k,\rho,\alpha) n^{\frac{(k-2)(1-\rho)}{2}+\alpha-1}, \tag{36}$$

where $c(k,\rho,\alpha)$ is a constant.

SPECIAL CASES: (i) For $\rho = 0$, we have the result of Theorem 3.3. (ii) For $\alpha = 1$ and $\rho = 0$, the above result reduces to Theorem 3.1.

4 Some More Classes Related with V_k, R_k and T_k

In this section, we are going to highlight different analytic classes which generalize the classes of functions of bounded boundary rotation, bounded radius rotation and the class T_k.

4.1 The Class of Bazilevic Functions

In 1955, Bazilevic [2] introduced the class of Bazilevic functions. Later, Thomas [96] defined the class of Bazilevic function of type β. Noor [27] obtained the coefficient result for the Bazilevic functions of type β. Noor and Al-Bani [64] generalized the idea of Thomas [96] to introduce the class $B_k(\beta)$ in [27]. Arc-length, Hankel determinant, coefficient problems and some other results are also solved for this generalized class in [64].

definition 4.1.1. Let f be analytic in E and be given by (1). Then f belongs to the class $B_k(\beta)$, $\beta > 0$, if there exists a function $g \in R_k, k \geq 2$, such that

$$\operatorname{Re}\left(\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right) > 0,\ z \in E. \tag{37}$$

For $\beta = 1, \beta_k(1) \equiv T_k$ and $B_2(1) \equiv K$, the class of close-to-convex functions.

THEOREM 4.1.1 ([64]). Let $f \in B_k(\beta)$, $0 \leq \beta < 1, k \geq 2$ and be given by (1). Then for $n \geq 2$, $k > \frac{5}{\beta} - 2$,

$$||a_{n+1}| - |a_n|| \leq O(1)M^{1-\beta}(1 - \frac{1}{n})n^{\beta(\frac{k}{2}+1)-2}. \tag{38}$$

where $M(r) = \max\limits_{|z|=r} |f(z)|$ and $O(1)$ depends only on k and β.

COROLLARY 4.1.1. If $\beta = 1, f \in T_k$ and we obtain a known [26] result for $k > 3$,

$$||a_{n+1}| - |a_n|| \leq O(1)n^{\frac{k}{2}-1}. \tag{39}$$

Noor [48] generalized the class of Bazilevic function and defined it as follows:

DEFINITION 4.1.2 ([48]). Let $f \in A$. Then $f \in B_k(\lambda, \alpha, \rho)$ if and only if

$$\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha-1}\right] \in P_k(\rho),\ z \in E, \tag{40}$$

where $\alpha > 0$, $\lambda > 0$, $k \geq 2$ and $0 \leq \rho < 1$.

For $k = 2$ and with different choices of λ, α and ρ we have different analytic classes studied in [5, 6, 10, 85]. In particular, $B_2(1, \alpha, \rho)$ is the class of Bazilevic functions.

THEOREM 4.1.2. Let λ, $\alpha > 0$, $0 \leq \rho < 1$ and let $f \in B_k(\lambda, \alpha, \rho)$. Then $\left(\frac{f(z)}{z}\right)^{\alpha} \in P_k(\rho_1)$, where ρ_1 is given by

$$\rho_1 = \rho + (1-\rho)(2\gamma - 1)$$

and

$$\gamma = \int_0^1 (1 + t^{\frac{\lambda}{\alpha}})^{-1} dt.$$

Noor et al. [61, 76, 75] generalizes this result under certain linear operators.

4.2 Analytic Classes of Functions of Bounded Mocanu Variation

In 1969, Mocanu [22] introduced the class M_α of bounded Mocanu variation also known as α-convex or α-starlike functions which generalizes the classes of starlike and convex univalent functions. Using the same criteria, Noor [81] defined the class Q_α which generalize the class of close-to-convex and quasi-convex univalent functions. For detailed study of quasi-convex univalent functions see [84, 29]. Noor [33] introduced the class $Q_{mk}[\alpha; A, B]$ which generalizes both M_α and Q_α. For further generalizations see [41, 66, 51, 69].

DEFINITION 4.2.1. Let f be analytic in E and given by (1). Then $f \in Q_{mk}[\alpha; A, B], -1 \leq B < A \leq 1$, if there exists a $g \in V_m[A, B]$ such that for $\alpha > 0$ and $0 < r < 1$,

$$\left[(1-\alpha)\frac{f'(z)}{g'(z)} + \alpha\frac{(zf'(z))'}{g'(z)}\right] \in P_k[A, B]. \tag{41}$$

For $m = 2$, we have $Q_{mk} = Q_k[\alpha, A, B]$. The following theorem shows the integral preserving property for the class $Q_k[\alpha, A, B]$ and also generalizes the results in [30] and [32].

THEOREM 4.2.1. Let $f \in Q_k[\alpha; A, B]$ and for $0 < \beta \leq 1$, let F be defined by

$$F(z) = \frac{1}{\beta} z^{1-\frac{1}{\beta}} \int_0^z t^{\frac{1}{\beta}-2} f(t)dt. \tag{42}$$

Then $F \in Q_k[\alpha; A, B]$.

Using the m-fold symmetric concept Noor [58] defined the class $Q_k(\alpha, \beta, m, \gamma)$. We define it as follows:

DEFINITION 4.2.2. Let f be analytic in E with $\frac{f(z)f'(z)}{z} \neq 0$ given by

$$f(z) = z + \sum_{n=m+1}^{\infty} a_n z^n,$$

and let

$$J_k(\alpha, \beta, m, f(z)) = \left[(1-\alpha)\frac{zf'(z)}{f(z)} + \frac{\alpha}{1-\beta}\left\{(1-\beta) + \frac{zf''(z)}{f'(z)}\right\}\right] \tag{43}$$

for real α, $\beta \in [\frac{-1}{2}, 1)$. Then $f \in Q_k(\alpha, \beta, m, \gamma)$ if and only if $J_k(\alpha, \beta, m, f(z)) \in P_k(\gamma), 0 \leq \gamma < 1$, $k \geq 2$.

This class generalizes several known classes of analytic functions. We list some of these as follows:

(i) For $k = 2, \beta = 0, \alpha$ real, $\gamma = 0$, we obtain the class $M_m(\alpha)$ of m-fold symmetric alpha-starlike functions discussed in some detail in [8].

(ii) $Q_k(0, \beta, m, \gamma) \equiv R_k(\gamma) \subset R_k$.

(iii) $Q_k(1, 0, 1, \gamma) \equiv V_k(\gamma) \subset V_k$.

(iv) The class $Q_k(\alpha, \beta, 1, 0) \equiv B_k(\alpha, \beta)$ was introduced and studied in [74].

(v) $Q_2(1, 0, m, 0) \equiv C(m), Q_2(0, \beta, m, 0) \equiv S^*(m)$, where $C(m)$ and $S^*(m)$ are respectively the class of m-symmetric convex and m-symmetric starlike functions in E.

The following theorem shows that the class $Q_k(\alpha, \beta, m, \gamma)$ is the class of univalent function by restricting the parameter k.

THEOREM 4.2.2. Let for $\alpha > 0$, $f \in Q_k(\alpha, \beta, m, \gamma)$. Then f is univalent in E for

$$k \leq \frac{2[(m + 2\beta) + (1 - \beta)(1 - \gamma)]}{(1 - \beta)(1 - \gamma)}.$$

SPECIAL CASE: For $m = 1, \gamma = 0$, this result has been proved in [74].

4.3 Analytic Classes of Functions of Bounded Radius Rotation with Respect to Symmetrical Points

In 1959, Sakaguchi [93] defined the class of starlike functions with respect to symmetrical points. The class of starlike univalent functions with respect to symmetrical points includes the class of convex and odd starlike functions with respect to origin. By using the concept of symmetrical points Noor et al. [77] introduced the generalized class of bounded radius rotation of order α with respect to symmetrical points which is defined as follows:

DEFINITION 4.4.1. Let $f \in A$ and defined by (1). Then f is said to be of bounded radius rotation of order $\alpha, 0 \leq \alpha < 1$, with respect to symmetrical points, if and only if

$$\frac{2zf'(z)}{f(z) - f(-z)} \in P_k(\alpha) \text{ for } z \in E. \tag{44}$$

The class of such functions is denoted by $R_k^s(\alpha)$. Clearly, for $\alpha = 0$, the class $R_k^s(\alpha) \equiv R_k^s$, see [42], the class of functions of bounded radius rotation with respect to symmetrical points, basic properties such as coefficient results, and arc-length and radius problems for class $R_k^s(\alpha)$. The following result gives necessary conditions for a function f to belong to $R_k^s(\alpha)$.

THEOREM 4.4.1. Let $f \in R_k^s(\alpha), 0 \leq \alpha < 1$. Then with $z = re^{i\theta}$ and $\theta_1 < \theta_2, 0 \leq \beta < 1$

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta > -(1 - \alpha)(k - 1)\pi. \tag{45}$$

As a special case for $\alpha = 0$, we have a result proved in [42] for the class R_k^s. Apart from this, certain analytic classes and their properties such as inclusion results, integral preserving property and radii problems have been investigated by Noor, see details [40, 53, 54, 67]. Noor's recent work elaborates and generalizes these concepts more clearly. For more generalizations, we refer to [60, 78, 82, 65].

FUTURE WORK: The study encompasses classes of functions of bounded boundary rotation and bounded radius rotation which have brought tremendous progress in

Geometric Function Theory. In order to explore the geometric properties of analytic classes, the major tools such as convolution and subordination have been extensively used in this area but there are still many problems which are unsolved. The convolution preserving property for $f \in C$ and $g \in S^*, k > 2$, the sufficient condition for Theorem 4.4.1, are open problems and new motivation for researchers in this field.

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