Applied Mathematics E-Notes, 12(2012), 14-22 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Dichotomy Of Poincare Maps And Boundedness Of Some Cauchy Sequences*

Akbar Zada[†], Sadia Arshad[‡], Gul Rahmat[§], Rohul Amin[¶]

Received 14 March 2011

Abstract

Let $\{U(p,q)\}_{p\geq q\geq 0}$ be the N-periodic discrete evolution family of $m\times m$ matrices having complex scalers as entries generated by $L(\mathbb{C}^m)$-valued, N-periodic sequence of $m\times m$ matrices (A_n) where $N\geq 2$ is a natural number. We proved that the Poincare map $U(N,0)$ is dichotomic if and only if the matrix $V_\mu = \sum_{\nu=1}^{N} U(N,\nu)e^{i\mu\nu}$ is invertible and there exists a projection P which commutes with the map $U(N,0)$ and the matrix V_μ, such that for each $\mu\in\mathbb{R}$ and each vector $b\in\mathbb{C}^m$ the solutions of the discrete Cauchy sequences $x_{n+1} = A_n x_n + e^{i\mu n}Pb,\ x_0 = 0$ and $y_{n+1} = A_n^{-1}y_n + e^{i\mu n}(I-P)b,\ y_0 = 0$ are bounded.

1 Introduction

It is well-known, see [2], that a matrix A is dichotomic, i.e. its spectrum does not intersect the unit circle if and only if there exists a projector, i.e. an $m\times m$ matrix P satisfying $P^2 = P$, which commutes with A and has the property that for each real number μ and each vector $b\in\mathbb{C}^m$, the following two discrete Cauchy problems

$$\begin{cases} x_{n+1} = Ax_n + e^{i\mu n}Pb, & n\in\mathbb{Z}_+ \\ x_0 = 0 \end{cases} \tag{1}$$

and

$$\begin{cases} y_{n+1} = A^{-1}y_n + e^{i\mu n}(I-P)b, & n\in\mathbb{Z}_+ \\ y_0 = 0 \end{cases} \tag{2}$$

have bounded solutions. In particular, the spectrum of A belongs to the interior of the unit circle if and only if for each real number μ and each m-vector b, the solution of the Cauchy problem (1) is bounded. Continuous version of the above result is given in [4].

On the other hand, in [3], it is shown that an N-periodic evolution family $\mathcal{U} = \{U(p,q)\}_{p\geq q\geq 0}$ of bounded linear operators acting on a complex space X, is uniformly

*Mathematics Subject Classifications: 35B35

[†]Department of Mathematics, Abdul Wali Khan University, Mardan, Pakistan
[‡]Abdus Salam School of Mathematical Sciences (ASSMS), GCU, Lahore, Pakistan
[§]Abdus Salam School of Mathematical Sciences (ASSMS), GCU, Lahore, Pakistan
[¶]Department of Mathematics, University of Peshawar, Peshawar, Pakistan

exponentially stable, i.e. the spectral radius of the Poincare map $U(N,0)$ is less than one, if and only if for each real number μ and each N-periodic sequence (z_n) decaying to $n=0$, we have

$$\sup_{n\geq 1}\left\|\sum_{k=1}^{n} e^{i\mu k}U(n,k)z_{k-1}\right\| = M(\mu,b) < \infty.$$

Recently in [1], it is proved that the spectral radius of the matrix $U(N,0)$ is less than one, if for each real μ and each m-vector b, the operator $V_\mu := \sum_{\nu=1}^{N} e^{i\mu\nu}U(N,\nu)$ is invertible and

$$\sup_{n\geq 1}\left\|\sum_{j=1}^{kN} e^{i\mu(j-1)}U(kN,j)b\right\| < \infty.$$

This note is a continuation of the latter quoted paper. In fact, we prove that the matrix $U(N,0)$ is dichotomic if and only if for each real μ and each m-vector b, the operator $V_\mu := \sum_{\nu=1}^{N} e^{i\mu\nu}U(N,\nu)$ is invertible and solutions of the two discrete Cauchy sequences like $(A, Pb, x_0, 0)$ are bounded.

2 Preliminary Results

Consider the following Cauchy Problem

$$\begin{cases} z_{n+1} = Az_n, & z_n \in \mathbb{C}^m, \quad n \in \mathbb{Z}_+ \\ z_n(0) = z_0. \end{cases} \tag{3}$$

where A is an $m\times m$ matrix. It is easy to check that the solution of (3) is $A^n z_0$.

Consider the following lemma which is used in Theorem 1.

LEMMA 1. Let $N \geq 1$ be a natural number. If q_n is a polynomial of degree n and $\Delta^N q_n = 0$ for all $n = 0,1,2\ldots$ where $\Delta z_n = z_{n+1} - z_n$ then q is a $\mathbb{C}^m$-valued polynomial of degree less than or equal to $N-1$.

For proof see [2].

Let p_A be the characteristic polynomial associated with the matrix A and let $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, $k \leq m$ be its spectrum. There exist integer numbers $m_1, m_2, \ldots, m_k \geq 1$ such that

$$p_A(\lambda) = (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\ldots(\lambda-\lambda_k)^{m_k}, \quad m_1+m_2+\cdots+m_k = m.$$

Then in [2] we have the following theorem.

THEOREM 1. For each $z \in \mathbb{C}^m$ there exists $w_j \in W_j := \ker(A-\lambda_j I)^{m_j}$, $(j \in \{1,2,\ldots,k\})$ such that

$$A^n z = A^n w_1 + A^n w_2 + \cdots + A^n w_k.$$

Moreover, if $w_j(n) := A^n w_j$ then $w_j(n) \in W_j$ for all $n \in \mathbb{Z}_+$ and there exist a $\mathbb{C}^m$-valued polynomials $q_j(n)$ with $\deg(q_j) \leq m_j - 1$ such that

$$w_j(n) = \lambda_j^n q_j(n), \quad n \in \mathbb{Z}_+,\ j \in \{1,2,\ldots,k\}.$$

FROOF. Indeed from the Cayley-Hamilton theorem and using the well known fact that

$$\ker[pq(A)] = \ker[p(A)] \oplus \ker[q(A)]$$

whenever the complex valued polynomials p and q are relatively prime, it follows that

$$\mathbb{C}^m = W_1 \oplus W_2 \oplus \cdots \oplus W_k. \tag{4}$$

Let $z \in \mathbb{C}^m$. For each $j \in \{1, 2, \ldots, k\}$ there exists a unique $w_j \in W_j$ such that

$$z = w_1 + w_2 + \cdots + w_k$$

and then

$$A^n z = A^n w_1 + A^n w_2 + \cdots + A^n w_k, \quad n \in \mathbb{Z}_+.$$

Let $q_j(n) = \lambda_j^{-n} w_j(n)$. Successively one has

$$\begin{aligned}
\Delta q_j(n) &= \Delta(\lambda_j^{-n} w_j(n)) \\
&= \Delta(\lambda_j^{-n} A^n w_j) \\
&= \lambda_j^{-(n+1)} A^{n+1} w_j - \lambda_j^{-n} A^n w_j \\
&= \lambda_j^{-(n+1)} (A - \lambda_j I) A^n w_j.
\end{aligned}$$

Again taking Δ,

$$\begin{aligned}
\Delta^2 q_j(n) &= \Delta[\Delta q_j(n)] \\
&= \Delta[\lambda_j^{-(n+1)} (A - \lambda_j I) A^n w_j] \\
&= \lambda_j^{-(n+2)} (A - \lambda_j I) A^{(n+1)} w_j - \lambda_j^{-(n+1)} (A - \lambda_j I) A^n w_j \\
&= \lambda_j^{-(n+2)} (A - \lambda_j I)^2 A^n w_j.
\end{aligned}$$

Continuing up to m_j we get $\Delta^{m_j} q_j(n) = \lambda_j^{-(n+m_j)} (A - \lambda_j I)^{m_j} A^n w_j$. But $w_j(n)$ belongs to W_j for each $n \in \mathbb{Z}_+$. Thus $\Delta^{m_j} q_j(n) = 0$. Using Lemma 1, we can say that the degree of polynomial $q_j(n)$ is less than or equal to $m_j - 1$.

3 Dichotomy and Boundedness

A family $\mathcal{U} = \{U(p,q) : (p,q) \in \mathbb{Z}_+ \times \mathbb{Z}_+\}$ of an $m \times m$ complex valued matrices is called discrete periodic evolution family if it satisfies the following properties.

1. $U(p,q)U(q,r) = U(p,r)$ for all $p \geq q \geq r \geq 0$;
2. $U(p,p) = I$ for all $p \geq 0$ and
3. there exists a fixed $N \geq 2$ such that $U(p+N, q+N) = U(p,q)$ for all $p, q \in \mathbb{Z}_+, \ p \geq q$.

Let us consider the following discrete Cauchy problem:

$$\begin{cases} z_{n+1} = A_n z_n + e^{i\mu n} b, \quad n \in \mathbb{Z}_+ \\ \quad z_0 = 0, \end{cases}$$

where the sequence (A_n) is N-periodic, i.e. $A_{n+N} = A_n$ for all $n \in \mathbb{Z}_+$ and a fixed $N \geq 2$. Let

$$U(n,j) = \begin{cases} A_{n-1}A_{n-2}\cdots A_j & \text{if } j \leq n-1, \\ I & \text{if } j = n, \end{cases}$$

then, the family $\{U(n,j)\}_{n\geq j\geq 0}$ is a discrete N-periodic evolution family and the solution (z_n) of the Cauchy problem $(A_n, \mu, b)_0$ is given by:

$$z_n = \sum_{j=1}^{n} U(n,j) e^{i\mu(j-1)} b$$

Let us denote by $C_1 = \{z \in \mathbb{C} : |z| = 1\}$, $C_1^+ = \{z \in \mathbb{C} : |z| > 1\}$ and $C_1^- = \{z \in \mathbb{C} : |z| < 1\}$. Clearly $\mathbb{C} = C_1 \cup C_1^+ \cup C_1^-$. Then with the help of above partition of $\mathbb{C}$ for matrix A we give the following definition:

DEFINITION 1. The matrix A is called:

(i) *stable* if $\sigma(A)$ is the subset of C_1^- or, equivalently, if there exist two positive constants N and ν such that $\|A^n\| \leq Ne^{-\nu n}$ for all $n = 0, 1, 2 \ldots,$

(ii) *expansive* if $\sigma(A)$ is the subset of C_1^+ and

(iii) *dichotomic* if $\sigma(A)$ have empty intersection with set C_1.

It is clear that any expansive matrix A whose spectrum consists of $\lambda_1, \lambda_2, \ldots, \lambda_k$ is an invertible one and its inverse is stable, because

$$\sigma(A^{-1}) = \{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_k}\} \subset C_1^-.$$

Let $L := U(N, 0)$, $V_\mu = \sum\limits_{\nu=1}^{N} U(N,\nu)e^{i\mu\nu}$ and $A_iA_j = A_jA_i$ for any $i, j \in \{1, 2, \ldots, n\}$. We recall that a linear map P acting on $\mathbb{C}^m$ is called projection if $P^2 = P$.

THEOREM 2. Let $N \geq 2$ be a fixed integer number. The matrix L is dichotomic if and only if the matrix V_μ is invertible and there exists a projection P having the property $PL = LP$ and $PV_\mu = V_\mu P$ such that for each $\mu \in \mathbb{R}$ and each vector $b \in \mathbb{C}^m$ the solutions of the following discrete Cauchy problems

$$\begin{cases} x_{n+1} = A_n x_n + e^{i\mu n} Pb, \quad n \in \mathbb{Z}_+ \\ \quad x_0 = 0 \end{cases} \tag{5}$$

and

$$\begin{cases} y_{n+1} = A_n^{-1} y_n + e^{i\mu n}(I-P)b, \quad n \in \mathbb{Z}_+ \\ \quad y_0 = 0\,. \end{cases} \tag{6}$$

are bounded.

PROOF. *Necessity:* Working under the assumption that L is a dichotomic matrix we may suppose that there exists $\eta \in \{1, 2, \ldots, \xi\}$ such that

$$|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_\eta| < 1 < |\lambda_{\eta+1}| \leq \cdots \leq |\lambda_\xi|.$$

Having in mind the decomposition of $\mathbb{C}^m$ given by (4) let us consider

$$X_1 = W_1 \oplus W_2 \oplus \cdots \oplus W_\eta, \quad X_2 = W_{\eta+1} \oplus W_{\eta+2} \oplus \cdots \oplus W_\xi.$$

Then $\mathbb{C}^m = X_1 \oplus X_2$. Define $P : \mathbb{C}^m \to \mathbb{C}^m$ by $Px = x_1$, where $x = x_1 + x_2$, $x_1 \in X_1$ and $x_2 \in X_2$. It is clear that P is a projection. Moreover for all $x \in \mathbb{C}^m$ and all $n \in \mathbb{Z}_+$, this yields

$$PL^k x = P(L^k(x_1 + x_2)) = P(L^k(x_1) + L^k(x_2)) = L^k(x_1) = L^k Px,$$

where the fact that X_1 is an L^k- invariant subspace, was used. Then $PL^k = L^k P$. Similarly by using the fact that X_1 and X_2 are V_μ invariant subspaces we can prove that $PV_\mu = V_\mu P$. We know that the solution of the Cauchy problem (5) is:

$$x_n = \sum_{j=1}^{n} U(n, j) e^{i\mu(j-1)} Pb.$$

Put $n = Nk + r$, where $r = 0, 1, 2, \ldots, N - 1$. Then

$$x_{Nk+r} = \sum_{j=1}^{Nk+r} U(Nk + r, j) e^{i\mu(j-1)} Pb.$$

Let

$$\mathcal{A}_\nu = \{\nu, \nu + N, \ldots, \nu + (k - 1)N\}, \quad \text{where} \;\; \nu \in \{1, 2, \ldots, N\}$$

and

$$\mathcal{R} = \{kN + 1, kN + 2, \ldots, kN + r\}.$$

Then

$$\mathcal{R} \cup (\cup_{\nu=1}^{N} A_\nu) \;= \{1, 2, \ldots, n\}.$$

Thus

$$\begin{aligned} x_{Nk+r} &= e^{-i\mu} \sum_{\nu=1}^{N} \sum_{j \in \mathcal{A}_\nu} U(Nk + r, j) e^{i\mu j} Pb + e^{-i\mu} \sum_{j \in \mathcal{R}} U(Nk + r, j) e^{i\mu j} Pb \\ &= e^{-i\mu} \sum_{\nu=1}^{N} \sum_{s=0}^{k-1} U(Nk + r, \nu + sN) e^{i\mu(\nu+sN)} Pb + \\ & \quad e^{-i\mu} \sum_{\rho=1}^{r} U(Nk + r, Nk + \rho) e^{i\mu(kN+\rho)} Pb \end{aligned}$$

$$
\begin{aligned}
&= e^{-i\mu}\sum_{\nu=1}^{N}\sum_{s=0}^{k-1}U(r,0)U(N,0)^{(k-s-1)}U(N,\nu)e^{i\mu(\nu+sN)}Pb+ \\
&\quad e^{-i\mu}\sum_{\rho=1}^{r}U(r,\rho)e^{i\mu(kN+\rho)}Pb.
\end{aligned}
$$

Let $z_\mu = e^{i\mu N}$, also we know that $L = U(N,0)$, thus

$$
\begin{aligned}
x_{Nk+r} &= e^{-i\mu}U(r,0)\sum_{s=0}^{k-1}L^{(k-s-1)}z_\mu^s\sum_{\nu=1}^{N}U(N,\nu)e^{i\mu\nu}Pb+ \\
&\quad e^{-i\mu}z_\mu^k\sum_{\rho=1}^{r}U(r,\rho)e^{i\mu\rho}Pb \\
&= e^{-i\mu}U(r,0)\big(L^{k-1}z_\mu^0+L^{k-2}z_\mu^1+\cdots+L^0z_\mu^{k-1}\big)\sum_{\nu=1}^{N}U(N,\nu)e^{i\mu\nu}Pb \\
&\quad +e^{-i\mu}z_\mu^k\sum_{\rho=1}^{r}U(r,\rho)e^{i\mu\rho}Pb.
\end{aligned}
$$

We know that $\sum_{\nu=1}^{N}U(N,\nu)e^{i\mu\nu} = V_\mu$ thus

$$
\begin{aligned}
x_{Nk+r} &= e^{-i\mu}U(r,0)\big(L^{k-1}z_\mu^0+L^{k-2}z_\mu^1+\cdots+L^0z_\mu^{k-1}\big)V_\mu Pb+ \\
&\quad e^{-i\mu}z_\mu^k\sum_{\rho=1}^{r}U(r,\rho)e^{i\mu\rho}Pb.
\end{aligned}
$$

By our assumption we know that L is dichotomic and $|z_\mu| = 1$ thus z_μ is contained in the resolvent set of L therefore the matrix $(z_\mu I - L)$ is an invertible matrix. Thus

$$
\begin{aligned}
x_{Nk+r} &= e^{-i\mu}U(r,0)(z_\mu I-L)^{-1}(z_\mu^k I-L^k)V_\mu Pb+e^{-i\mu}z_\mu^k\sum_{\rho=1}^{r}U(r,\rho)e^{i\mu\rho}Pb \\
&= e^{-i\mu}U(r,0)(z_\mu I-L)^{-1}(z_\mu^k I-L^k)PV_\mu b+e^{-i\mu}z_\mu^k\sum_{\rho=1}^{r}U(r,\rho)e^{i\mu\rho}Pb.
\end{aligned}
$$

We know that V_μ is a surjective map, so there exists b' such that $V_\mu b = b'$ then

$$
x_{Nk+r} = e^{-i\mu}U(r,0)(z_\mu I-L)^{-1}(z_\mu^k I-L^k)Pb'+e^{-i\mu}z_\mu^k\sum_{\rho=1}^{r}U(r,\rho)e^{i\mu\rho}Pb.
$$

Taking norm of both sides

$$
\|x_{Nk+r}\| = \|e^{-i\mu}U(r,0)(z_\mu I-L)^{-1}(z_\mu^k I-L^k)Pb'+e^{-i\mu}z_\mu^k\sum_{\rho=1}^{r}U(r,\rho)e^{i\mu\rho}Pb\|
$$

$$\begin{aligned}
\|x_{Nk+r}\| &\leq \|U(r,0)(z_\mu I - L)^{-1} z_\mu^k Pb'\| + \|U(r,0)(z_\mu I - L)^{-1} PL^k b'\| + \\
&\quad \sum_{\rho=1}^{r} \|U(r,\rho)Pb\| \\
&= \|U(r,0)\|\|(z_\mu I - L)^{-1}\|\|Pb'\| + \|U(r,0)\|\|(z_\mu I - L)^{-1}\|\|PL^k b'\| \\
&\quad + \sum_{\rho=1}^{r} \|U(r,\rho)Pb\|.
\end{aligned}$$

Using THEOREM 1, We have

$$L^k b' = \lambda_1^k p_1(k) + \lambda_2^k p_2(k) + \cdots + \lambda_\xi^k p_\xi(k),$$

Thus

$$PL^k b' = \lambda_1^k p_1(k) + \lambda_2^k p_2(k) + \cdots + \lambda_\eta^k p_\eta(k),$$

where each $p_i(k)$ are $\mathbb{C}^m$-valued polynomials with degree at most $(m_i - 1)$ for any $i \in \{1, 2, \ldots, \xi\}$. From hypothesis we know that $|\lambda_i| < 1$ for each $i \in \{1, 2, \ldots, \eta\}$. Thus $\|PL^k b'\| \to 0$ when $k \to \infty$ and so x_{Nk+r} is bounded for any $r = 0, 1, 2, \ldots, N-1$. Thus x_n is bounded. For the second Cauchy problem: We have

$$y_n = \sum_{j=1}^{n} U^{-1}(n,j) e^{i\mu(j-1)} (I-P)b.$$

where

$$U^{-1}(n,j) = \begin{cases} A_{n-1}^{-1} A_{n-2}^{-1} \cdots A_j^{-1} & \text{if } j \leq n-1, \\ I & \text{if } j = n. \end{cases}$$

It is easy to check that $U^{-1}(n,j)$ is also a discrete evaluation family. By putting $n = Nk + r$, where $r = 0, 1, 2, \ldots, N-1$. Then

$$y_{Nk+r} = \sum_{j=1}^{Nk+r} U^{-1}(Nk+r, j) e^{i\mu(j-1)} (I-P)b.$$

As $A_i A_j = A_j A_i$ for all $i, j \in \{1, 2, \ldots, n\}$ thus $L^{-1} = U^{-1}(N, 0)$. By similar procedure as above we obtained that

$$\begin{aligned}
\|y_{Nk+r}\| &= \|U^{-1}(r,0)\|\|(z_\mu I - L^{-1})^{-1}\|\|(I-P)V_\mu(b)\| + \\
&\quad \|U^{-1}(r,0)\|\|(z_\mu I - L^{-1})^{-1}\|\|L^{-k}(I-P)V_\mu(b)\| + \\
&\quad \sum_{\rho=1}^{r} \|U^{-1}(r,\rho)(I-P)b\|.
\end{aligned}$$

??$(I-P)V_\mu b \in X_2$ the assertion would follow.?? But

$$X_2 = W_{\eta+1} \oplus W_{\eta+2} \oplus \cdots \oplus W_\xi.$$

Each vector from X_2 can be represented as a sum of $\xi - \eta$ vectors $w_{\eta+1}, w_{\eta+2}, \ldots, w_\xi$. It would be sufficient to prove that $L^{-k} w_j \to 0$, for any $j \in \{\eta+1, \ldots, \xi\}$. Let $W \in$

$\{W_{\eta+1}, W_{\eta+2}, \dots, W_\xi\}$, say $W = \ker(L-\lambda I)^\gamma$, where $\gamma \geq 1$ is an integer number and $|\lambda| > 1$. Consider $r_1 \in W \setminus \{0\}$ such that $(L-\lambda I)r_1 = 0$ and let $r_2, r_3, \dots, r_\gamma$ given by $(L-\lambda I)r_j = r_{j-1}, \;\; j = 2, 3, \dots, \gamma$. Then $B := \{r_1, r_2, \dots, r_\gamma\}$ is a basis in Y. It is then sufficient to prove that $L^{-k}r_j \to 0, \;\;$ for any $j = 1, 2, \dots, \gamma$. For $j = 1$ we have that $L^{-k}r_1 = \frac{1}{\lambda^k}r_1 \to 0$. For $j = 2, 3, \dots, \gamma$, denote $X_k = L^{-k}r_j$. Then $(L-\lambda I)^\gamma X_k = 0$ i.e.

$$X_k - C_\gamma^1 X_{k-1}\alpha + C_\gamma^2 X_{k-2}\alpha^2 + \cdots + C_\gamma^\gamma X_{k-\gamma}\alpha^\gamma = 0, \;\; \text{for all} \;\; k \geq \gamma \tag{7}$$

where $\alpha = \frac{1}{\lambda}$. Passing for instance at the components, it follows that there exists a $\mathbb{C}^m$-valued polynomial P_γ having degree at most $\gamma - 1$ and verifying (7) such that $X_k = \alpha^k P_\gamma(k)$. Thus $X_k \to 0$ as $k \to \infty$, i.e. $L^{-k}r_j \to 0$ for any $j \in \{1, 2, \dots, \gamma\}$. Thus (y_n) is bounded.

Sufficiency: Suppose to the contrary that the matrix L is not dichotomic. Then $\sigma(L) \cap \Gamma_1 \neq \phi$. Let $\omega \in \sigma(L) \cap \Gamma_1$. Then there exists a nonzero $y \in \mathbb{C}^m$ such that $Ly = \omega y$. It is easy to see that $L^k y = w^k y$. Choose $\mu_0 \in \mathbb{R}$ such that $e^{i\mu_0 N} = \omega$. We know that

$$\begin{aligned} x_{Nk+r}(\mu_0, b) \;\; &= \;\; e^{-i\mu_0}U(r,0)\big(L^{k-1}z^0_{\mu_0} + L^{k-2}z^1_{\mu_0} + \cdots + L^0 z^{k-1}_{\mu_0}\big)PV_{\mu_0}b + \\ & \;\;\;\;\;\; e^{-i\mu_0}z^k\sum_{\rho=1}^{r} U(r,\rho)e^{i\mu_0\rho}\;Pb\;. \end{aligned}$$

But V_{μ_0} is surjective, thus there exists $b_0 \in \mathbb{C}^m$ such that $V_{\mu_0}b_0 = y$, so

$$\begin{aligned} x_{Nk+r}(\mu_0, b_0) \;\; &= \;\; e^{-i\mu_0}U(r,0)\big(L^{k-1}z^0_{\mu_0} + L^{k-2}z^1_{\mu_0} + \cdots + L^0 z^{k-1}_{\mu_0}\big)Py + \\ & \;\;\;\;\;\; e^{-i\mu_0}z^k\sum_{\rho=1}^{r} U(r,\rho)e^{i\mu_0\rho}Pb_0 \\ &= \;\; e^{-i\mu_0}U(r,0)\big(PL^{k-1}yz^0_{\mu_0} + PL^{k-2}yz^1_{\mu_0} + \cdots + PL^0 yz^{k-1}_{\mu_0}\big) + \\ & \;\;\;\;\;\; e^{-i\mu_0}z^k\sum_{\rho=1}^{r} U(r,\rho)e^{i\mu_0\rho}Pb \\ &= \;\; e^{-i\mu_0}U(r,0)P\big(L^{k-1}yz^0_{\mu_0} + L^{k-2}yz^1_{\mu_0} + \cdots + L^0 yz^{k-1}_{\mu_0}\big) + \\ & \;\;\;\;\;\; e^{-i\mu_0}z^k\sum_{\rho=1}^{r} U(r,\rho)e^{i\mu_0\rho}Pb \\ &= \;\; e^{-i\mu_0}U(r,0)P[ke^{-i\mu_0}z^{k-1}{}_{\mu_0}] + e^{-i\mu_0}z^k\sum_{\rho=1}^{r} U(r,\rho)e^{i\mu_0\rho}Pb \end{aligned}$$

Clearly

$$x_{kN}(\mu_0, b_0) \to \infty \text{ when } \;\; k \to \infty.$$

Thus a contradiction arises. In [1] an example, in terms of stability is given which shows that the assumption on invertibility of V_μ, for each real number μ, cannot be removed.

References

[1] S. Arshad, C. Buse, A. Nosheen and A. Zada, Connections between the stability of a Poincare map and boundedness of certain associate sequences, Electronic Journal of Qualitative Theory of Differential Equations, 16(2011), 1–12.

[2] C. Buse and A. Zada, Dichotomy and bounded-ness of solutions for some discrete Cauchy problems, Proceedings of IWOTA–2008, Operator Theory, Advances and Applications, (OT) Series Birkhäuser Verlag, Eds: J. A. Ball, V. Bolotnikov, W. Helton, L. Rodman and T. Spitkovsky, 203(2010), 165–174.

[3] C. Buse, P. Cerone, S. S. Dragomir and A. Sofo, Uniform stability of periodic discrete system in Banach spaces, J. Difference Equ. Appl., 12(11)(2005), 1081–1088.

[4] A. Zada, A characterization of dichotomy in terms of boundedness of solutions for some Cauchy problems, Electronic Journal of Differential Equations, 94(2008), 1–5.