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COMPLEX LAGRANGE SPACES WITH (γ, β)-METRIC

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Abstract. The purpose of the present paper is to introduce the notion of (γ, β)-metric in a complex Lagrange space, where γ is a cubic-root metric and β is a differential $(1, 0)$-form. Several geometric objects of such spaces such as fundamental metric tensor, its inverse, Euler-Lagrange equations, complex semispray coefficients, complex nonlinear connection and Chern–Lagrange connection are discussed.

1. Introduction

R. Miron [4] and B. Nicolaescu [8, 9] studied Lagrange spaces with (α, β)-metric. T. N. Pandey and V. K. Chaubey [10] introduced the concept of (γ, β)-metric in a Lagrange space, where γ is a cubic-root metric and β is a 1-form defined by $\gamma = \sqrt[3]{a_{ijk}(x)y^i y^j y^k}$ and $\beta = b_i(x)y^i$, respectively. In 2013, S. K. Shukla and P. N. Pandey [11] further extended the theory of Lagrange spaces with (γ, β)-metric. N. Aldea and G. Munteanu [1] introduced and worked on complex Finsler spaces with (α, β)-metric. The authors [3] of the present paper further studied complex Randers spaces. G. Munteanu [5] initiated the study of a complex Lagrange space in 1998. Later on, in 2002, various analysis of complex Lagrange space was done by G. Munteanu [6].

In the present paper, the notion of (γ, β)-metric in a complex Lagrange space, where γ is a cubic-root metric and β is a differential $(1, 0)$-form, is introduced. We determine the fundamental metric tensor, its inverse, Euler-Lagrange equations, complex semispray coefficients, complex nonlinear connection and Chern–Lagrange connections for a complex Lagrange space with (γ, β)-metric.

2. Preliminaries

Let M be a complex manifold of dimension n. Let $(z^k)_{k=\overline{1,n}}$ be local coordinates in a chart (U, z^k) and $T'M$ be its holomorphic tangent bundle. $T'M$ has

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a natural structure of complex manifold such that (z^k,η^k) are local coordinates in a chart on U belonging to $T'M$. A complex Lagrangian L [7] on $T'M$ is a smooth real valued function $L\colon T'M\to\mathbb{R}$ such that

$$g_{i\bar{j}}=\dot{\partial}_i\dot{\partial}_{\bar{j}}L,\qquad \dot{\partial}_i\equiv\frac{\partial}{\partial\eta^i},\qquad \dot{\partial}_{\bar{j}}\equiv\frac{\partial}{\partial\eta^{\bar{j}}}\tag{1}$$

is a non-degenerated metric ($\det g_{i\bar{j}}\neq 0$) and determines a Hermitian metric structure. A complex Lagrange space is a pair $L^n=(M,L(z,\eta))$. The existence of a complex Lagrange function L involves the study of the variational problem on curves. Let $c:[0,1]\to M$ be a holomorphic curve and $L(z,\eta)$ be the complex Lagrangian on $T'M$. The Euler-Lagrange equations for a geodesic are given by

$$E_i(L)\equiv\frac{\partial L}{\partial z^i}-\frac{d}{dt}\left(\frac{\partial L}{\partial\eta^i}\right)=0,\qquad \eta^i=\frac{d}{dt}z^i.\tag{2}$$

The coefficients of the complex semispray S of a complex Lagrange space $L^n=(M,L(z,\eta))$ are

$$G^k(z,\eta)=\frac{1}{2}g^{\bar{i}k}(\partial_j\dot{\partial}_{\bar{i}}L)\eta^j.\tag{3}$$

The coefficients of the complex nonlinear connection (c.n.c.) (Cartan Connection) [7] of a complex Lagrange space $L^n=(M,L(z,\eta))$ are

$$\overset{c}{N^i_j}=\dot{\partial}_jG^i.\tag{4}$$

Also, the Chern–Lagrange connection $\overset{CL}{N^k_j}$ [7] is defined as

$$\overset{CL}{N^k_j}=g^{\bar{i}k}(\partial_j\dot{\partial}_{\bar{i}}L).\tag{5}$$

These two connections are related by

$$\overset{c}{N^k_j}=\frac{1}{2}\dot{\partial}_j\overset{CL}{N^k_0}.$$

In this paper, we study a complex Lagrange space whose Lagrangian L is a function of $\gamma(z,\eta)$ and $|\beta(z,\eta)|$, i.e.,

$$L(z,\eta)=\gamma(z,\eta)+|\beta|(z,\eta),\tag{6}$$

where

$$\gamma=\sqrt[3]{a_{i\bar{j}\bar{k}}\eta^i\bar{\eta}^j\bar{\eta}^k}\tag{7}$$

and

$$|\beta(z,\eta)|=\sqrt{\beta(z,\eta)\overline{\beta(z,\eta)}}\text{ with }\beta(z,\eta)=b_i(z)\eta^i.\tag{8}$$

The space $L^n=(M,L(z,\eta))$ is called the complex Lagrange space with $(\gamma,|\beta|)$-metric.

3. Fundamental metric tensor

Differentiating (7) partially with respect to η^l and $\overline{\eta}^m$ and using the symmetry of $a_{i\bar{j}\,\bar{k}}$ in its indices, we get

$$\dot{\partial}_l\gamma = \frac{a_l}{3\gamma^2}, \qquad \dot{\partial}_{\overline{m}}\gamma = \frac{2a_{\overline{m}}}{3\gamma^2}, \tag{9}$$

where $a_l = a_{l\bar{j}\,\bar{k}}\,\overline{\eta}^j\,\overline{\eta}^k$ and $a_{\overline{m}} = a_{i\bar{j}\,\overline{m}}\eta^i\overline{\eta}^j$. Again, differentiating the first equation of (3.1) partially with respect to $\overline{\eta}^p$, we obtain

$$\partial_l\dot{\partial}_{\overline{p}}\gamma = \frac{2a_{l\overline{p}}}{3\gamma^2} - \frac{4a_l a_{\overline{p}}}{9\gamma^5}, \tag{10}$$

where $a_{l\overline{p}} = a_{l\overline{p}\,\bar{k}}\,\overline{\eta}^k$. Differentiation of (8) with respect to η^l and $\overline{\eta}^m$ gives

$$\dot{\partial}_l|\beta| = \frac{\overline{\beta}b_l}{2|\beta|}, \qquad \dot{\partial}_{\overline{m}}|\beta| = \frac{\beta b_{\overline{m}}}{2|\beta|}. \tag{11}$$

Further differentiating the first equation of (3.3) partially with respect to $\overline{\eta}^p$, we have

$$\partial_l\dot{\partial}_{\overline{p}}|\beta| = \frac{b_l b_{\overline{p}}}{4|\beta|}. \tag{12}$$

This leads to

Proposition 1. *In a complex Lagrange space with (γ, β)-metric,* (9), (10), (11), *and* (12) *hold.*

The moments of Lagrangian $L(z, \eta)$ are defined as

$$p_i = \frac{1}{2}\dot{\partial}_i L. \tag{13}$$

Since the Lagrangian $L(z, \eta)$ is a function of γ and $|\beta|$, (13) implies

$$p_i = \frac{1}{2}(L_\gamma\dot{\partial}_i\gamma + L_{|\beta|}\dot{\partial}_i|\beta|), \tag{14}$$

where $L_\gamma = \partial_\gamma L$, $L_|\beta| = \partial_{|\beta|}L$, $\partial_\gamma \equiv \partial/\partial\gamma$, and $\partial_{|\beta|} = \partial/\partial|\beta|$.

Using the first equation of (3.1) and (3.3) in (14), we have

$$p_i = \left(\frac{1}{6}\gamma^{-2}L_\gamma a_i + \frac{1}{4}\overline{\beta}|\beta|^{-1}L_{|\beta|}b_i\right).$$

Thus, we have

Theorem 1. *In a complex Lagrange space L^n with (γ, β)-metric, the moments of Lagrangian $L(z, \eta)$ are given by*

$$p_i = \rho a_i + \rho_1 b_i,$$

where

$$\rho = \frac{1}{6}\gamma^{-2}L_\gamma \tag{15}$$

and

$$\rho_1 = \frac{1}{4}\overline{\beta}|\beta|^{-1}L_{|\beta|}. \tag{16}$$

The scalars ρ and ρ_1 appearing in Theorem 1 are called the *principal invariants* of the space L^n. Differentiating (15) and (16) partially with respect to η^j and $\overline{\eta}^l$, we respectively have

$$\begin{aligned}
\partial_j\rho &= \frac{1}{18}\gamma^{-4}\left(L_{\gamma\,\gamma} - 2\gamma^{-1}L_\gamma\right)\, a_j + \frac{1}{12}\overline{\beta}|\beta|^{-1}\gamma^{-2}L_{\gamma\,|\beta|}b_j,\\
\dot{\partial}_{\bar{j}}\rho &= \frac{1}{9}\gamma^{-4}\left(L_{\gamma\,\gamma} - 2\gamma^{-1}L_\gamma\right)\, a_{\bar{j}} + \frac{1}{12}\beta|\beta|^{-1}\gamma^{-2}L_{\gamma\,|\beta|}b_{\bar{j}},\\
\partial_j\rho_1 &= \frac{1}{12}\overline{\beta}|\beta|^{-1}\gamma^{-2}L_{\gamma\,|\beta|}a_j + \frac{1}{8}\overline{\beta}\beta^{-1}\left(L_{|\beta|||\beta|} + |\beta|^{-1}L_{|\beta|}\right)\, b_j,\\
\dot{\partial}_{\bar{j}}\rho_1 &= \frac{1}{6}\overline{\beta}|\beta|^{-1}\gamma^{-2}L_{\gamma\,|\beta|}a_{\bar{j}} + \frac{1}{8}|\beta|^{-1}\left(L_{|\beta|||\beta|} + |\beta|^{-1}L_{|\beta|}\right)\, b_{\bar{j}},
\end{aligned}$$

where

$$L_{\gamma\,\gamma} = \frac{\partial^2 L}{\partial\gamma^2},\qquad L_{\gamma\,|\beta|} = \frac{\partial^2 L}{\partial\gamma\partial|\beta|} = \frac{\partial^2 L}{\partial|\beta|\partial\gamma} = L_{|\beta|\,\gamma},\qquad L_{|\beta|\,|\beta|} = \frac{\partial^2 L}{\partial|\beta|^2}.$$

Thus, we have

Theorem 2. *The derivatives of the principal invariants of a complex Lagrange space L^n with (γ, β)-metric are given by*

$$\partial_j\rho = \frac{1}{2}\rho_{-2}a_j + \overline{\beta}\beta^{-1}\rho_{-1}b_j,\qquad \dot{\partial}_{\bar{j}}\rho = \rho_{-2}a_{\bar{j}} + \rho_{-1}b_{\bar{j}}, \tag{17}$$

and

$$\partial_j\rho_1 = \overline{\beta}\beta^{-1}\left(\rho_{-1}a_j + \rho_0 b_j\right),\qquad \dot{\partial}_{\bar{j}}\rho_1 = 2\overline{\beta}\beta^{-1}\rho_{-1}a_{\bar{j}} + \rho_0 b_{\bar{j}} \tag{18}$$

with

$$\begin{aligned}
\rho_{-2} &= \frac{1}{9}\gamma^{-4}\left(L_{\gamma\,\gamma} - 2\gamma^{-1}L_\gamma\right),\\
\rho_{-1} &= \frac{1}{12}\beta|\beta|^{-1}\gamma^{-2}L_{\gamma\,|\beta|},\\
\rho_0 &= \frac{1}{8}\left(L_{|\beta|||\beta|} + |\beta|^{-1}L_{|\beta|}\right).
\end{aligned} \tag{19}$$

The energy of the complex Lagrangian $L(z, \eta)$ is defined as

$$E_L = \eta^i\dot{\partial}_i L - L. \tag{20}$$

Using (6) in (20), we have

$$E_L = \eta^i(L_\gamma\,\dot{\partial}_i\gamma + L_{|\beta|}\,\dot{\partial}_i|\beta|) - L. \tag{21}$$

Since γ and $|\beta|$ are positively homogeneous of degree one in η^i, in view of Euler's theorem on homogeneous functions, we conclude

$$\eta^i\dot{\partial}_i\gamma = \frac{\gamma}{3},\qquad \eta^i\dot{\partial}_i|\beta| = \frac{|\beta|}{2}. \tag{22}$$

Using (22) in (21), we get

$$E_L = \frac{\gamma}{3} L_\gamma + \frac{|\beta|}{2} L_{|}\beta| - L. \tag{23}$$

This leads to

Theorem 3. *The energy of the Lagrangian $L(z, \eta)$ in a complex Lagrange space with (γ, β)-metric is given by* (23).

Next, we calculate the fundamental metric tensor $g_{i\bar{j}}(z, \eta)$ of a complex Lagrange space with (γ, β)-metric. In view of (6), (1) implies

$$\begin{aligned} g_{i\bar{j}} = \frac{1}{2} \Big[& \left(L_{\gamma\gamma} \dot{\partial}_i \gamma + L_{\gamma|\beta|} \dot{\partial}_i |\beta| \right) \dot{\partial}_{\bar{j}} \gamma + L_\gamma \dot{\partial}_i \dot{\partial}_{\bar{j}} \gamma \\ & + \left(L_{|\beta|\gamma} \dot{\partial}_i \gamma + L_{|\beta||\beta|} \dot{\partial}_i |\beta| \right) \dot{\partial}_{\bar{j}} |\beta| + L_{|\beta|} \dot{\partial}_i \dot{\partial}_{\bar{j}} |\beta| \Big]. \end{aligned} \tag{24}$$

Now using (15) and (19) in (24), we have

$$g_{i\bar{j}}(z, \eta) = 4\rho a_{i\bar{j}} + 2\rho_{-2} a_i a_{\bar{j}} + 2\rho_{-1} \beta^{-1} (2\overline{\beta} a_{\bar{j}} b_i + \beta a_i b_{\bar{j}}) + 2\rho_0 b_i b_{\bar{j}}. \tag{25}$$

A simple calculation shows that

$$(2\overline{\beta} a_{\bar{j}} b_i + \beta a_i b_{\bar{j}}) = \frac{3\gamma^2 |\beta|}{2L} \eta_i \overline{\eta}_j - \frac{3\gamma^2 |\beta|}{2} b_i b_{\bar{j}} - \frac{4|\beta|}{3\gamma^2} a_i a_{\bar{j}}, \tag{26}$$

where $\eta^i = \dot{\partial}_i L$ and $\overline{\eta}^j = \dot{\partial}_{\bar{j}} L$. In view of (26), (25) reduces to

$$g_{i\bar{j}}(z, \eta) = 4\rho a_{i\bar{j}} + q_{-2} a_i a_{\bar{j}} + q_{-1} \eta_i \eta_{\bar{j}} + q_0 b_i b_{\bar{j}} \tag{27}$$

with

$$q_{-2} = 2 \left(\rho_{-2} - \frac{4|\beta| \rho_{-1}}{3\beta\gamma^2} \right), \quad q_{-1} = \frac{3\gamma^2 |\beta|}{\beta L} \rho_{-1}, \quad q_0 = 2\rho_0 - \frac{3\gamma^2 |\beta|}{\beta} \rho_{-1}.$$

Further (27) can be written as

$$g_{i\bar{j}}(z, \eta) = 4\rho a_{i\bar{j}} + c_i c_{\bar{j}}, \tag{28}$$

where

$$c_i = r_{-1} a_i + r_0 b_i$$

such that

$$r_0 r_{-1} = q_{-1}, \qquad (r_{-1})^2 = q_{-2}, \qquad r_0^2 = q_0.$$

Thus, we have

Theorem 4. *The expression for the fundamental metric tensor $g_{i\bar{j}}$ of a complex Lagrange space with (γ, β)-metric is given by* (28).

Using a proposition given by D. Bao, S. S. Chern, and Z. Shen [2], the inverse $g^{\bar{j}i}$ of the fundamental tensor $g_{i\bar{j}}$ is given by

$$g^{\bar{j}i} = \frac{1}{4\rho} \left(a^{\bar{j}i} - \frac{1}{4\rho + c^2} c^i c^{\bar{j}} \right), \tag{29}$$

where

$$c^i = a^{\bar{j}i} c_{\bar{j}}, \qquad c^2 = a^{\bar{j}i}\, c_i c_{\bar{j}}.$$

This leads to

Theorem 5. *The inverse $g^{\bar{j}i}$ of the fundamental tensor $g_{i\bar{j}}$ of a complex Lagrange space with (γ,β)-metric is given by* (29).

4. Euler-Lagrange equations

In view of (6), (2) reduces to

$$E_i(L) \equiv L_\gamma E_i(\gamma) + L_{|\beta|}E_i(|\beta|) - \left(L_{\gamma\,\gamma}\frac{d\gamma}{dt} + L_{\gamma\,|\beta|}\frac{d|\beta|}{dt}\right)\frac{\partial\gamma}{\partial\eta^i} - \left(L_{|\beta|\,\gamma}\frac{d\gamma}{dt} + L_{|\beta|\,|\beta|}\frac{d|\beta|}{dt}\right)\frac{\partial|\beta|}{\partial\eta^i} = 0. \tag{30}$$

Also

$$E_i(\gamma^3) = 3\gamma^2 E_i(\gamma) - 3\,\frac{\partial\gamma}{\partial\eta^i}\frac{d\gamma^2}{dt} \tag{31}$$

and

$$E_i(|\beta|^2) = 2|\beta|E_i(|\beta|) - 2\,\frac{\partial|\beta|}{\partial\eta^i}\frac{d|\beta|}{dt}. \tag{32}$$

Substituting values of $E_i(\gamma)$ and $E_i(|\beta|)$ from (31) and (32) in (30), we obtain

$$E_i(L) \equiv 2\rho E_i(\gamma^3) + \frac{2}{\overline{\beta}}\rho_1 E_i(|\beta|^2) + 6\rho\frac{\partial\gamma}{\partial\eta^i}\frac{d\gamma^2}{dt} + \frac{4}{\overline{\beta}}\rho_1\frac{\partial|\beta|}{\partial\eta^i}\frac{d|\beta|}{dt} - \frac{\partial\gamma}{\partial\eta^i}\left(L_{\gamma\,\gamma}\frac{d\gamma}{dt} + L_{\gamma\,|\beta|}\frac{d|\beta|}{dt}\right) - \frac{\partial|\beta|}{\partial\eta^i}\left(L_{|\beta|\,\gamma}\frac{d\gamma}{dt} + L_{|\beta|\,|\beta|}\frac{d|\beta|}{dt}\right). \tag{33}$$

This leads to

Theorem 6. *The Euler-Lagrange equations of a complex Lagrange space with (γ,β)-metric are given by* (33).

For the natural parametrization of the curve $c : t \in [0,1] \mapsto z^i(t) \in M$ with respect to the cubic-root metric γ, $\gamma\left(z, \frac{dz}{dt}\right) = 1$.

Thus, we have

Theorem 7. *In the natural parametrization, the Euler-Lagrange equations of a complex Lagrange space with (γ,β)-metric are*

$$E_i(L) \equiv 2\rho E_i(\gamma^3) + \frac{2}{\overline{\beta}}\rho_1 E_i(|\beta|^2) + \frac{4}{\overline{\beta}}\rho_1\frac{\partial|\beta|}{\partial\eta^i}\frac{d|\beta|}{dt} - L_{\gamma\,|\beta|}\frac{\partial\gamma}{\partial\eta^i}\frac{d|\beta|}{dt} - L_{|\beta|\,|\beta|}\frac{\partial|\beta|}{\partial\eta^i}\frac{d|\beta|}{dt} = 0.$$

If $|\beta|$ is constant along the integral curve of the Euler-Lagrange equations with natural parametrization, then the Euler-Lagrange equations of the complex Lagrange space with (γ, β)-metric are given by

$$E_i(L) \equiv 2\rho E_i(\gamma^3) + \frac{2}{\overline{\beta}}\rho_1 E_i(|\beta|^2) = 0. \tag{34}$$

This leads to the following theorem.

Theorem 8. *If $|\beta|$ is constant along the integral curve of the Euler-Lagrange equations with natural parametrization, then the Euler-Lagrange equations of the complex Lagrange space with (γ, β)-metric are given by* (34).

5. Complex canonical semispray

The coefficients of the complex canonical semispray of a complex Lagrange space with (γ, β)-metric is given by (3) together with (6).

Differentiating (7) and (8) partially with respect to z^h, we have

$$\partial_h \gamma = A_h \gamma^{-2}, \qquad \partial_h |\beta| = \frac{\overline{\beta}}{2|\beta|} B_h + \frac{\beta}{2|\beta|} C_h, \tag{35}$$

where

$$A_h = \frac{1}{3}(\partial_h a_{i\overline{j}\overline{k}})\eta^i \overline{\eta}^j \overline{\eta}^k, \qquad B_h = (\partial_h b_i)\eta^i, \qquad C_h = (\partial_h b_{\overline{j}})\overline{\eta}^j.$$

Substituting (35), (15) and (16) in $\partial_k L = L_\gamma \partial_k \gamma + L_{|\beta|}\partial_k |\beta|$, we get

$$\partial_k L = 6\rho A_k + 2\rho_1 \left(B_k + \frac{\beta}{\overline{\beta}} C_k \right). \tag{36}$$

Differentiating (36) partially with respect to $\overline{\eta}^h$, we have

$$\begin{aligned} \dot{\partial}_{\overline{h}} \partial_k L = & \left(6\rho_{-2} A_k + \frac{4\overline{\beta}}{\beta} \rho_{-1} B_k + 4\rho_{-1} C_k \right) a_{\overline{h}} \\ & + \left(6\rho_{-1} A_k + 2\rho_0 B_k + 2\rho_0 \frac{\beta}{\overline{\beta}} C_k - 2\rho_1 \frac{\beta}{\overline{\beta}^2} C_k \right) b_{\overline{h}} \\ & + \left(6\rho A_{k\overline{h}} + 2\rho_1 B_{k\overline{h}} + 2\rho_1 \frac{\beta}{\overline{\beta}} C_{k\overline{h}} \right), \end{aligned} \tag{37}$$

where

$$A_{k\overline{h}} = \dot{\partial}_{\overline{h}} A_k, \qquad B_{k\overline{h}} = \dot{\partial}_{\overline{h}} B_k, \qquad C_{k\overline{h}} = \dot{\partial}_{\overline{h}} C_k. \tag{38}$$

Contracting (37) with η^k, we obtain

$$(\dot{\partial}_{\overline{h}}\partial_k L)\eta^k = \left(6\rho_{-2}A_0 + \frac{4\overline{\beta}}{\beta}\rho_{-1}B_0 + 4\rho_{-1}C_0\right)a_{\overline{h}} + \left(6\rho_{-1}A_0 + 2\rho_0 B_0 + 2\rho_0\frac{\beta}{\overline{\beta}}C_0 - 2\rho_1\frac{\beta}{\overline{\beta}^2}C_0\right)b_{\overline{h}} + \left(6\rho A_{0\overline{h}} + 2\rho_1 B_{0\overline{h}} + 2\rho_1\frac{\beta}{\overline{\beta}}C_{0\overline{h}}\right), \tag{39}$$

where

$$\begin{aligned} A_0 &= A_k(z,\eta)\eta^k, & B_0 &= B_k(z,\eta)\eta^k, \\ C_0 &= C_k(z,\eta)\eta^k, & A_{0\overline{h}} &= A_{k\overline{h}}(z,\eta)\eta^k, \\ B_{0\overline{h}} &= B_{k\overline{h}}(z,\eta)\eta^k, & C_{0\overline{h}} &= C_{k\overline{h}}(z,\eta)\eta^k. \end{aligned} \tag{40}$$

Putting (39) in (3), we have

$$G^i = g^{\overline{h}i}\Big[\left(3\rho_{-2}A_0 + \frac{2\overline{\beta}}{\beta}\rho_{-1}B_0 + 2\rho_{-1}C_0\right)a_{\overline{h}} + \left(3\rho_{-1}A_0 + \rho_0 B_0 + \rho_0\frac{\beta}{\overline{\beta}}C_0 - \rho_1\frac{\beta}{\overline{\beta}^2}C_0\right)b_{\overline{h}} + \left(3\rho A_{0\overline{h}} + \rho_1 B_{0\overline{h}} + \rho_1\frac{\beta}{\overline{\beta}}C_{0\overline{h}}\right)\Big]. \tag{41}$$

This leads to the next theorem.

Theorem 9. *The coefficients of the complex canonical semispray of a complex Lagrange space with (γ,β)-metric are given by* (41).

6. Canonical complex nonlinear connection and Chern–Lagrange connection

In this section, we find out the coefficients of the complex nonlinear connection $\overset{c}{N}{}^k_j$ and Chern–Lagrange connection $\overset{CL}{N}{}^k_j$ of a complex Lagrange space with $(\gamma,|\beta|)$-metric. Partial differentiation of $g^{\overline{h}i}g_{\overline{h}j} = \delta^i_j$ with respect to η^j yields

$$\dot{\partial}_j g^{\overline{h}i} = -g^{\overline{h}r}C^i_{rj}. \tag{42}$$

Partial differentiation of the quantities appearing in (19) and (40) with respect to η^j, we get

$$
(43)\qquad
\begin{aligned}
\dot\partial_j \rho_{-2} &= \mu_{-3} a_j + \mu_{-2} b_j, & \dot\partial_j \rho_{-1} &= \frac{1}{2}\beta\overline{\beta}^{-1}\mu_{-2} a_j + \mu_{-1} b_j,\\
\dot\partial_j \rho_0 &= \mu_{-1} a_j + \mu_0 b_j, & \dot\partial_j A_0 &= A_j + A_{0j},\\
\dot\partial_j C_0 &= C_j, & \dot\partial_j A_{0\overline{h}} &= 2A_{0\overline{h}j} + A_{j\overline{h}},\\
\dot\partial_j C_{0\overline{h}} &= C_{j\overline{h}}, & \dot\partial_j a_{\overline{h}} &= 2\, a_{j\overline{h}},\\
\dot\partial_j B_0 &= \mathfrak{S}_{(kj)}\{\partial_k b_j\}\eta^k, & \dot\partial_j B_{0\overline{h}} &= B_{j\overline{h}},
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_{-3} &= \frac{1}{27}\gamma^{-8}\left(\gamma^2 L_{\gamma\gamma\gamma} - 6\gamma L_{\gamma\gamma} + 10 L_\gamma\right),\\
\mu_{-2} &= \frac{1}{18}\gamma^{-4}\overline{\beta}|\beta|^{-1}\left(L_{\gamma\gamma|\beta|} - 2\gamma^{-1}L_{\gamma|\beta|}\right),\\
\mu_{-1} &= \frac{1}{24}\gamma^{-2}|\beta|^{-1}\left(|\beta| L_{\gamma|\beta||\beta|} + L_{\gamma|\beta|}\right),\\
\mu_0 &= \frac{1}{16}\overline{\beta}|\beta|^{-3}\left(|\beta|^2 L_{|\beta||\beta||\beta|} + |\beta| L_{|\beta||\beta|} - L_{|\beta|}\right),\\
A_{0\overline{h}j} &= A_{r\overline{h}j}\eta^r,\ A_{r\overline{h}j} = \partial_r a_{\overline{h}j}
\end{aligned}
$$

and $\mathfrak{S}_{(kj)}$ denotes the interchange of the indices k and j, and addition. Now, applying (41) in (4), we get

$$
(44)\qquad
\begin{aligned}
\overset{c}{N}{}^i_j = \frac{1}{2}\left(\dot\partial_j g^{\overline{h}i}\right)\Bigg[&\left(3\rho_{-2}A_0 + \frac{2\overline{\beta}}{\beta}\rho_{-1}B_0 + 2\rho_{-1}C_0\right)a_{\overline{h}}\\
&+\left(3\rho_{-1}A_0 + \rho_0 B_0 + \rho_0\frac{\beta}{\overline{\beta}}C_0 - \rho_1\frac{\beta}{\overline{\beta}^2}C_0\right)b_{\overline{h}}\\
&+\left(3\rho A_{0\overline{h}} + \rho_1 B_{0\overline{h}} + \rho_1\frac{\beta}{\overline{\beta}}C_{0\overline{h}}\right)\Bigg]\\
&+ g^{\overline{h}i}\dot\partial_j\Bigg[\left(3\rho_{-2}A_0 + \frac{2\overline{\beta}}{\beta}\rho_{-1}B_0 + 2\rho_{-1}C_0\right)a_{\overline{h}}\\
&+\left(3\rho_{-1}A_0 + \rho_0 B_0 + \rho_0\frac{\beta}{\overline{\beta}}C_0 - \rho_1\frac{\beta}{\overline{\beta}^2}C_0\right)b_{\overline{h}}\\
&+\left(3\rho A_{0\overline{h}} + \rho_1 B_{0\overline{h}} + \rho_1\frac{\beta}{\overline{\beta}}C_{0\overline{h}}\right)\Bigg].
\end{aligned}
$$

Using (17), (18), (38), (40), (42), and (43) in (44) and simplifying, we have

(45)
$$
\begin{aligned}
\overset{c}{N}{}^i_j = &-C^i_{rj}G^r + g^{\overline{h}i}\Bigg[\rho_{-2}\left(3(A_{0j}+A_j)a_{\overline{h}} + 6A_0a_{j\overline{h}} + \frac{3}{2}A_{0\overline{h}}a_j\right)\\
&+\rho_{-1}\Big\{\left(3A_{0j}+3A_j - a_j\overline{\beta}^{\,-1}C_0\right)b_{\overline{h}} + 4(\overline{\beta}\beta^{-1}B_0 + C_0)a_{j\overline{h}}\\
&+(3\overline{\beta}\beta^{-1}A_{0\overline{h}} - 2\overline{\beta}\beta^{-2}B_0a_{\overline{h}})b_j + 2(\overline{\beta}\beta^{-1}\mathfrak{S}_{(kj)}\,(\partial_k b_j)\,\eta^k + C_j)a_{\overline{h}}\\
&+\overline{\beta}\beta^{-1}(B_{0\overline{h}} + \overline{\beta}^{\,-1}\beta C_{0\overline{h}})a_j\Big\} + \rho_0\Big\{\left(\mathfrak{S}_{(kj)}\,(\partial_k b_j)\,\eta^k + \beta\overline{\beta}^{-1}C_j\right)b_{\overline{h}}\\
&+\overline{\beta}\beta^{-1}(B_{0\overline{h}} + \overline{\beta}^{\,-1}\beta C_{0\overline{h}})b_j\Big\} + \rho_1\Big\{\overline{\beta}^{\,-1}C_{0\overline{h}}b_j - \beta^{-2}(b_jC_0 - \beta C_j)b_{\overline{h}}\\
&+\overline{\beta}^{\,-1}\beta C_{j\overline{h}} + B_{j\overline{h}}\Big\} + 3\rho(2A_{0\overline{h}j} + A_{j\overline{h}}) + 3\mu_{-3}a_ja_{\overline{h}}A_0\\
&+\mu_{-2}\Big\{(B_0 + \overline{\beta}^{\,-1}\beta C_0)a_ja_{\overline{h}} + 3A_0b_ja_{\overline{h}}\Big\} + \mu_{-1}\Big\{(B_0+\overline{\beta}^{\,-1}\beta C_0)a_jb_{\overline{h}}\\
&+2(\overline{\beta}\beta^{-1}B_0 + C_0)b_ja_{\overline{h}} + 3A_0b_jb_{\overline{h}}\Big\} + \mu_0(B_0 + \overline{\beta}^{\,-1}\beta C_0)b_jb_{\overline{h}}\Bigg].
\end{aligned}
$$

Also, using (37) in (5), we obtain

$$
\begin{aligned}
\overset{CL}{N}{}^k_j = 2g^{\bar{i}k}\Bigg[&\left(3\rho_{-2}A_j + \frac{2\overline{\beta}}{\beta}\rho_{-1}B_j + 2\rho_{-1}C_0\right)a_{\bar{i}}\\
&+\left(3\rho_{-1}A_j + \rho_0B_j + \rho_0\frac{\beta}{\overline{\beta}}C_j - \rho_1\frac{\beta}{\overline{\beta}^2}C_j\right)b_{\bar{i}}\\
&+\left(3\rho A_{j\bar{i}} + \rho_1B_{j\bar{i}} + \rho_1\frac{\beta}{\overline{\beta}}C_{j\bar{i}}\right)\Bigg].
\end{aligned}
\tag{46}
$$

Thus, we can state the following theorem.

Theorem 10. *The coefficients of the complex nonlinear connection and Chern–Lagrange connection of a complex Lagrange space with (γ,β)-metric are given by* (45) *and* (46)*, respectively.*

7. Conclusions

In the present paper, we have developed the theory of complex Lagrange spaces with (γ,β)-metric. It plays a significant role in the expansion of the earlier works of G. Munteanu [5]-[6]. For some geometric objects, expressions are obtained which are further useful in the development of the current space. The results obtained are useful in the study of connections, holomorphic curvature, complex nonlinear connections and torsions in such spaces. In Section

5 and Section 6, expressions for complex canonical semispray, complex nonlinear connections and Chern–Lagrange connections are obtained, which will be applicable in geodesic correspondence between two complex Lagrange spaces with different (γ, β)-metrics on the same underlying complex manifold.

References

[1] N. Aldea, G. Munteanu, *On complex Finsler spaces with Randers metric*, J. Korean Math. Soc., 46(5) (2009), 949-966.

[2] D. Bao, S. S. Chern, Z. Shen, *An introduction to Riemann-Finsler geometry*, Graduate Texts in Mathematics, 200, Springer-Verlag, New York, 2000.

[3] S. Kumari, P. N. Pandey, *On a complex Randers space*, Nat. Acad. Sci. Lett., 42 (2019), 123-130.

[4] R. Miron, *Finsler-Lagrange spaces with (α, β)-metrics and Ingarden spaces*, Rep. Math. Phys., 58(3) (2006), 417-431.

[5] G. Munteanu, *Complex Lagrange spaces*, Balkan J. Geom. Appl., 3(1) (1998), 61-71.

[6] G. Munteanu, *Lagrange geometry via complex Lagrange geometry*, Novi Sad J. Math., 32(2) (2002), 141-154.

[7] G. Munteanu, *Complex spaces in Finsler, Lagrange and Hamilton geometries*, Fundamental Theories of Physics, 141, Kluwer Academic Publisher, Dordrecht, 2004.

[8] B. Nicolaescu, *The variational problem in Lagrange spaces endowed with (α, β)-metrices*, Proc. of the 3rd International Colloq. in Engg. and Numerical Physics, Bucharest, Romania, (2005), 202-207.

[9] B. Nicolaescu, *Nonlinear connection in Lagrange spaces with (α, β)-metrics*, Diff. Geom. Dyn. Sys., 8 (2006), 196-199.

[10] T. N. Pandey, V. K. Chaubey, *The variational problem in Lagrange spaces endowed with (γ, β)-metrics*, Int. J. Pure Appl. Math., 71(4) (2011), 633-638.

[11] S. K. Shukla, P. N. Pandey, *Lagrange spaces with (γ, β)-metric*, Geometry, 1(7) (2013), pp. 7.

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