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ON THE CONVERGENCE OF CESÀRO MEANS OF NEGATIVE ORDER OF WALSH-FOURIER SERIES

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ABSTRACT. In this paper we investigate the convergence of Cesàro means of negative order of Walsh-Fourier series of functions of generalized bounded oscillation.

Let $r_0(x)$ be a function defined on $R := (-\infty, \infty)$ by

$$r_0(x) = \begin{cases} 1, \text{if } x \in \left[0, \frac{1}{2}\right) \\ -1, \text{if } x \in \left[\frac{1}{2,} 1\right) \end{cases}, \qquad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \qquad n \geq 1 \text{ and } x \in [0,1).$$

Let $w_0, w_1, \ldots$ represent the Walsh functions, i.e., $w_0(x) = 1$ and if $k = 2^{n_1} + ... + 2^{n_s}$ is a positive integer with $n_1 > n_2 > \cdots > n_s$ then $w_k(x) = r_{n_1}(x) \times \cdots \times r_{n_s}(x)$.

The idea of using products of Rademacher's functions to define the Walsh system originated from Paley [16].

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n, \text{ if } x \in \left[0, \frac{1}{2^n}\right) \\ 0, \text{ if } x \in \left[\frac{1}{2^n}, 1\right). \end{cases}$$

Suppose that f is a Lebesgue integrable function on $[0,1]$ and 1-periodic. Then its Walsh-Fourier series is defined by

$$\sum_{k=0}^{\infty} \widehat{f}(k) w_k(x),$$

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where

$$\widehat{f}(k)=\int_0^1 f(t)\,w_k(t)\,dt$$

is called the k-th Walsh-Fourier coefficient of the function f. Denote by $S_n(f,x)$ the n-th partial sum of the Walsh-Fourier series of the function f, namely

$$S_n(f,x)=\sum_{k=0}^{n-1}\widehat{f}(k)w_k(x).$$

The Cesàro (C,α)-means of the Walsh-Fourier series are defined as

$$\sigma_n^\alpha(f,x)=\frac{1}{A_n^\alpha}\sum_{k=0}^{n}A_{n-k}^\alpha\widehat{f}(k)\,w_k(x),$$

where

$$A_0^\alpha=1,$$
$$A_n^\alpha=\frac{(\alpha+1)\cdots(\alpha+n)}{n!},\qquad \alpha\neq-1,-2,\ldots.$$

Let $C([0,1])$ denote the space of continuous functions f with period 1. If $f\in C([0,1])$, then the function

$$w(\delta,f)=\sup\{|f(x')-f(x'')|:|x'-x''|\leq\delta,\ x',x''\in[0,1]\}$$

is called the modulus of continuity of the function f. The modulus of continuity of an arbitrary function $f\in C([0,1])$ has the following properties:

1) $\omega(0)=0$,
2) $\omega(\delta)$ is nondecreasing,
3) $\omega(\delta)$ is continuous on $[0,1]$,
4) $\omega(\delta_1+\delta_2)\leq\omega(\delta_1)+\omega(\delta_2)$ for $0\leq\delta_1\leq\delta_2\leq\delta_1+\delta_2\leq1$.

An arbitrary function $\omega(\delta)$ which is defined on $[0,1]$ and has properties 1) – 4) is called a modulus of continuity. If the modulus of continuity $\omega(\delta)$ is given, then H^ω denotes the class of functions $f\in C([0,1])$ for which

$$\omega(\delta,f)=O(\omega(\delta))\quad\text{as}\quad\delta\to0.$$

$C_w([0,1])$ is the collection of functions $f\colon[0,1)\to R$ that are uniformly continuous from the dyadic topology of $[0,1)$ to the usual topology of R, or for short: uniformly W-continuous.

Let f be defined on $[0,1)$. We shall represent the dyadic modulus of continuity by

$$\dot{\omega}(\delta,f)=\sup_{0\leq h\leq\delta}\sup_x|f(x\oplus h)-f(x)|,$$

where $\oplus$ denotes dyadic addition (see [12] or [18]).

The problems of summability of Cesàro means of the Walsh-Fourier series were studied in [4], [7], [10], [9], [8], [16], [18], [17].

Tevzadze [19] has studied the uniform convergence of Cesàro means of negative order of the Walsh-Fourier series. In particular, in terms of modulus of continuity and variation of function $f \in C_w([0,1])$ he has proved the criterion for the uniform summability by the Cesàro method of negative order of Fourier series with respect to the Walsh system.

In [9] Goginava investigated the problem of estimating the deviation of $f \in L_p$ from its Cesàro means of negative order of Walsh-Fourier series in the L_p-metric, $p \in [1,\infty)$. Analogous results for Walsh-Kaczmarz system were proved by Nagy [15] and Gát, Nagy [6].

In his monograph [23, part 1, chapter 4] Zhizhiashvili investigated the behaviour of Cesàro means of negative order of trigonometric Fourier series in detail.

The notion of a function-bounded variation was introduced by Jordan [13]. Generalizing this notion Wiener [21] considered the class of function V_p. Young [22] introduced the notion of the function of bounded Φ-variation. Waterman [20] studied the class of function of bounded Λ-variation, and Chanturia [3] defined the notion of the modulus of variation of a function. In 1990, Kita and Yoneda [14] introduced the notion of the generalized Wiener's class $BV(p(n)\uparrow p)$. Generalizing the class $BV(p(n)\uparrow p)$, Akhobadze [1, 2] considered the classes of function $BV(p(n)\uparrow p,\varphi)$ and $B\Lambda(p(n)\uparrow p,\varphi)$.

Definition 1. [11] Let $1 \leq p(n) \uparrow p$ as $n \to \infty$ where $1 \leq p \leq \infty$. We say that a function belongs to the $BO(p(n)\uparrow p)$ class if

$$O(f;p(n)\uparrow p) := \sup_n \left\{ \sum_{l=0}^{2^n-1} \sup_{t,u\in[l2^{-n}.(l+1)2^{-n})} |f(t)-f(u)|^{p(n)} \right\}^{\frac{1}{p(n)}} < \infty.$$

When $p(n) = p$ for all n, $BO(p(n)\uparrow p)$ coincides with the class of p-bounded fluctuation BF_p [18].

Estimates of the Fourier coefficients of functions of bounded fluctuation with respect to the Vilenkin system were studied by Gát and Toledo [5].

In [11] Goginava proved that the following statements are true.

Theorem 1. *Let f be a function in the class $BO(p(n)\uparrow\infty)$ and*

$$\dot{\omega}\left(\frac{1}{2^n},f\right) = o\left(\frac{1}{p(n+1)\log_2 p(n+1)}\right) \text{ as } n\to\infty.$$

Then the Walsh-Fourier series of the function f converges uniformly in $[0,1]$.

Theorem 2. *Let $p(2n) \leq cp(n)$, $n \in P$ and $p(n)\log_2 p(n) = o(n)$ as $n\to\infty$. If ω satisfies the condition*

$$\lim_{n\to\infty}\sup \omega\left(\frac{1}{n}\right) p([\log_2 n]) \log_2 p([\log_2 n]) = c_0 > 0,$$

then there exists a function in the class $H^\omega \cap BO(p(n)\uparrow\infty)$ for which the Walsh-Fourier series diverges at some point.

The theorem of Tevzadze [19] implies that if $p < \frac{1}{\alpha}$ and $f \in BF_p \cap C_\omega$, then the Cesàro mean $\sigma_n^{-\alpha}(f)$ of Walsh-Fourier series uniformly converges to the function f. On the other hand, for $p=\frac{1}{\alpha}$ there exists a continuous function f for which $\sigma_n^{-\alpha}(f,0)$ diverges. On the basis of the above facts the following problems arise naturally:

Let $f \in BO\left(p(n)\uparrow \frac{1}{\alpha}\right)$, $0<\alpha<1$. Under what condition on the sequence $\{p(n): n\geq 1\}$ the uniform convergence of Cesàro $(C,-\alpha)$ means of Walsh-Fourier series of the function f holds?

The following theorem is true.

Theorem 3. *Let $f \in C_w([0,1]) \cap BO\left(p(n)\uparrow \frac{1}{\alpha}\right)$, $0<\alpha<1$, $2^k \leq n \leq 2^{k+1}$. Then*

$$\left\|\sigma_n^{-\alpha}(f)-f\right\|_c \leq c(\alpha)\left\{\sum_{r=0}^{k} 2^{r-k}\dot{\omega}\left(\frac{1}{2^r},f\right)_c + \frac{\left(\dot{\omega}\left(\frac{1}{2^k},f\right)\right)^{1-\alpha p(k)}}{1-\alpha p(k)}\right\}.$$

Corollary 1. *Let $f \in C_w([0,1]) \cap BO\left(p(n)\uparrow \frac{1}{\alpha}\right)$, $0<\alpha<1$ and*

$$\frac{\left(\dot{\omega}\left(\frac{1}{2^k},f\right)\right)^{1-\alpha p(k)}}{1-\alpha p(k)} \to 0 \quad as \quad k\to\infty.$$

Then

$$\left\|\sigma_n^{-\alpha}(f)-f\right\|_c \to 0.$$

In order to prove Theorem 3 we need the following lemmas proved by Goginava in [9, 8].

Lemma 1 (Goginava [9])**.** *Let $f\in C_w([0,1])$. Then for every $\alpha\in(0,1)$ the following estimation holds*

$$\frac{1}{A_n^{-\alpha}}\left\|\int_0^1 \sum_{\nu=0}^{2^{k-1}-1} A_{n-\nu}^{-\alpha} w_\nu(u)\left[f(\cdot\oplus u)-f(\cdot)\right]du\right\|_c \leq c(p,\alpha)\sum_{r=0}^{k-1} 2^{r-k}\dot{\omega}\left(1/2^r,f\right)_p,$$

where $2^k\leq n<2^{k+1}$.

Lemma 2 (Goginava [8])**.** *Let $f\in C_w([0,1])$ and $2^k\leq n<2^{k+1}$. Then for every $\alpha\in(0,1)$ the following estimations hold*

$$\frac{1}{A_n^{-\alpha}}\left|\int_0^1 \sum_{\nu=2^{k-1}}^{2^k-1} A_{n-\nu}^{-\alpha} w_\nu(u)\left[f(\cdot\oplus u)-f(\cdot)\right]du\right|$$
$$\leq c(\alpha)\left(\sum_{j=1}^{2^{k-1}-1}\frac{1}{j^{1-\alpha}}\left|f\left(x\oplus\frac{2j}{2^k}\right)-f\left(x\oplus\frac{2j+1}{2^k}\right)\right|\right),$$

$$\frac{1}{A_n^{-\alpha}}\left|\int_0^1 \sum_{\nu=2^k}^{n} A_{n-\nu}^{-\alpha} w_\nu(u)\left[f(\cdot\oplus u)-f(\cdot)\right]du\right|$$
$$\leq c(\alpha)\left(\sum_{j=1}^{2^k}\frac{1}{j^{1-\alpha}}\left|f\left(x\oplus\frac{2j}{2^{k+1}}\right)-f\left(x\oplus\frac{2j+1}{2^{k+1}}\right)\right|\right).$$

Proof of Theorem 3. We can write

$$\begin{aligned}\sigma_n^{-\alpha}(f,x)-f(x)&=\frac{1}{A_n^{-\alpha}}\int_0^1\sum_{\nu=0}^{n}A_{n-\nu}^{-\alpha}w_\nu(x)\left[f(x\oplus u)-f(x)\right]du\\ &=\frac{1}{A_n^{-\alpha}}\int_0^1\sum_{\nu=0}^{2^{k-1}-1}A_{n-\nu}^{-\alpha}w_\nu(x)\left[f(x\oplus u)-f(x)\right]du\\ &\quad+\frac{1}{A_n^{-\alpha}}\int_0^1\sum_{\nu=2^{k-1}}^{2^k-1}A_{n-\nu}^{-\alpha}w_\nu(x)\left[f(x\oplus u)-f(x)\right]du\\ &\quad+\frac{1}{A_n^{-\alpha}}\int_0^1\sum_{\nu=2^k}^{n}A_{n-\nu}^{-\alpha}w_\nu(x)\left[f(x\oplus u)-f(x)\right]du\\ &=I+II+III.\end{aligned}\tag{1}$$

From Lemmas 1 and 2 we have

$$\|I\|_c\le c(\alpha)\sum_{\nu=0}^{k-1}2^{r-k}\omega\left(\frac{1}{2^r},f\right)_c,\tag{2}$$

$$|II|\le c(\alpha)\left(\sum_{j=1}^{2^{k-1}-1}\frac{1}{j^{1-\alpha}}\left|f\left(x\oplus\frac{2j}{2^k}\right)-f\left(x\oplus\frac{2j+1}{2^k}\right)\right|\right)$$

and

$$|III|\le c(\alpha)\left(\sum_{j=1}^{2^k-1}\frac{1}{j^{1-\alpha}}\left|f\left(x\oplus\frac{2j}{2^{k+1}}\right)-f\left(x\oplus\frac{2j+1}{2^{k+1}}\right)\right|\right).$$

Using Abel's transformation, we get

$$\begin{aligned}|III|\le c(\alpha)&\left(\sum_{j=1}^{2^k-2}\left(\frac{1}{j^{1-\alpha}}-\frac{1}{(j+1)^{1-\alpha}}\right)\right.\\ &\times\sum_{l=1}^{j}\left|f\left(x\oplus\frac{2l}{2^{k+1}}\right)-f\left(x\oplus\frac{2l+1}{2^{k+1}}\right)\right|\\ &\left.+\frac{1}{(2^k-1)^{1-\alpha}}\sum_{j=1}^{2^k-1}\left|f\left(x\oplus\frac{2j}{2^{k+1}}\right)-f\left(x\oplus\frac{2j+1}{2^{k+1}}\right)\right|\right)\\ =&III_1+III_2.\end{aligned}\tag{3}$$

Let $\varepsilon_k := \alpha p_k < 1$, $s_k := \frac{p(k)}{\varepsilon_k}$, $\frac{1}{s_k} + \frac{1}{t_k} = 1$. Then using Hölder's inequality for III_2 we can write

$$
\begin{aligned}
III_2 &= \frac{1}{(2^k-1)^{1-\alpha}} \sum_{j=1}^{2^k-1} \left| f\left(x \oplus \frac{2j}{2^{k+1}}\right) - f\left(x \oplus \frac{2j+1}{2^{k+1}}\right) \right|^{\varepsilon_k} \\
&\qquad \times \left| f\left(x \oplus \frac{2j}{2^{k+1}}\right) - f\left(x \oplus \frac{2j+1}{2^{k+1}}\right) \right|^{1-\varepsilon_k} \\
&\leq \frac{c(\alpha)}{2^{k(1-\alpha)}} \left(\sum_{j=1}^{2^k-1} \left| f\left(x \oplus \frac{2j}{2^{k+1}}\right) - f\left(x \oplus \frac{2j+1}{2^{k+1}}\right) \right|^{p(k)} \right)^{\frac{\varepsilon_k}{p(k)}} \\
&\quad \times \left(\sum_{j=1}^{2^k-1} \left| f\left(x \oplus \frac{2j}{2^{k+1}}\right) - f\left(x \oplus \frac{2j+1}{2^{k+1}}\right) \right|^{(1-\varepsilon_k)t_k} \right)^{\frac{1}{t_k}} \\
&\leq \frac{c(\alpha)}{2^{k(1-\alpha)}} \left(BO\left(f, p(k) \uparrow \frac{1}{\alpha}\right)\right)^{\varepsilon_k} \left(\dot{\omega}\left(f, \frac{1}{2^k}\right)\right)^{1-\varepsilon_k} 2^{\frac{k}{t_k}} \\
&\leq c(\alpha) \left(BO\left(f, p(k) \uparrow \frac{1}{\alpha}\right)\right)^{\varepsilon_k} \left(\dot{\omega}\left(f, \frac{1}{2^k}\right)\right)^{1-\varepsilon_k} 2^{k\left(\alpha - \frac{1}{s_k}\right)} \\
&= c(\alpha) \left(BO\left(f, p(k) \uparrow \frac{1}{\alpha}\right)\right)^{\varepsilon_k} \left(\dot{\omega}\left(f, \frac{1}{2^k}\right)\right)^{1-\varepsilon_k} 2^{k\left(\alpha - \frac{\varepsilon_k}{p(k)}\right)} \\
&= c(\alpha) \left(BO\left(f, p(k) \uparrow \frac{1}{\alpha}\right)\right)^{\varepsilon_k} \left(\dot{\omega}\left(f, \frac{1}{2^k}\right)\right)^{1-\alpha p(k)} \longrightarrow 0
\end{aligned}
$$

as $k \to \infty$.

Fix $m_0(k)$ and define it later

$$
\begin{aligned}
III_1 &\leq c(\alpha) \sum_{j=1}^{m_0(k)} \frac{1}{j^{2-\alpha}} \sum_{l=1}^{j} \left| f\left(x \oplus \frac{2l}{2^{k+1}}\right) - f\left(x \oplus \frac{2l+1}{2^{k+1}}\right) \right| \\
&\quad + \sum_{j=m_0(k)+1}^{2^k-1} \frac{1}{j^{2-\alpha}} \sum_{l=1}^{j} \left| f\left(x \oplus \frac{2l}{2^{k+1}}\right) - f\left(x \oplus \frac{2l+1}{2^{k+1}}\right) \right| \\
&\leq c(\alpha) \left\{ \sum_{j=1}^{m_0(k)} \frac{1}{j^{2-\alpha}} j \dot{\omega}\left(\frac{1}{2^k}, f\right) \right. \\
&\quad \left. + \sum_{j=m_0(k)+1}^{2^k-1} \frac{1}{j^{1+1/p(k)-\alpha}} \left(\sum_{l=1}^{j} \left| f\left(x \oplus \frac{2l}{2^{k+1}}\right) - f\left(x \oplus \frac{2l+1}{2^{k+1}}\right) \right|^{p(k)} \right)^{\frac{1}{p(k)}} \right\} \\
&\leq c(\alpha) \left\{ (m_0(k))^{\alpha} \dot{\omega}\left(\frac{1}{2^k}, f\right) + \frac{m_0(k)^{\alpha - \frac{1}{p(k)}}}{\frac{1}{p(k)} - \alpha} BO\left(f, p(k) \uparrow \frac{1}{\alpha}\right) \right\}.
\end{aligned}
$$

Set

$$
m_0(k) = \left(\frac{1}{\dot{\omega}\left(\frac{1}{2^k}, f\right)} \right)^{p(k)}.
$$

Then we have

(4)
$$III_1 \leq c(\alpha)\left\{\dot{\omega}\left(\frac{1}{2^k},f\right)^{1-\alpha p(k)}+\frac{\dot{\omega}\left(\frac{1}{2^k},f\right)^{1-\alpha p(k)}}{\frac{1}{p(k)}-\alpha}\right\} \leq c(\alpha)\frac{\dot{\omega}\left(\frac{1}{2^k},f\right)^{1-\alpha p(k)}}{1-\alpha p(k)}.$$

Combining (3) – (4) we have

$$|III| \leq c(\alpha)\frac{\dot{\omega}\left(\frac{1}{2^k},f\right)^{1-\alpha p(k)}}{1-\alpha p(k)}. \tag{5}$$

Analogously we can prove that

$$|II| \leq c(\alpha)\frac{\dot{\omega}\left(\frac{1}{2^k},f\right)^{1-\alpha p(k)}}{1-\alpha p(k)}. \tag{6}$$

Combining (1), (2), (5) and (6) we complete the proof of Theorem 3. □

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