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LEGENDRE CURVES ON THREE-DIMENSIONAL QUASI-SASAKIAN MANIFOLDS WITH SEMI SYMMETRIC METRIC CONNECTION

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ABSTRACT. The object of the present paper is to study Legendre curves on three-dimensional quasi-Sasakian manifolds with semi-symmetric metric connection.

1. INTRODUCTION

In the study of contact manifolds, Legendre curve play an important role e.g., a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves. Legendre curves on contact manifolds have been studied by C. Baikoussis and D. E. Blair [1]. Legendre curves with Pseudo-Hermitian connection have been studied by J. T. Cho [5]. The first author of the paper has studied Legendre curves in the papers [11], [12]. Again Legendre curves on three-dimensional quasi-Sasakian manifold has been studied in the paper [2]. In this paper we are interested to study Legendre curves on three-dimensional quasi-Sasakian manifolds with respect to semi-symmetric metric connection. The notion of quasi-Sasakian manifolds was given by D. E. Blair in the paper [4]. Again Z. Olszak [10] studied quasi-Sasakian manifolds. Semi-symmetric metric connection was studied by K. Yano [14]. Semi-symmetric connection was introduced by Schouten [7]. Later Hayden [8] has introduced the idea of metric connection with torsion in a Riemannian manifold. A. Sharfuddin and S. I. Hussain [13] introduced the study of almost contact manifolds with semi-symmetric metric connection. The present paper is organized as follows:

After the introduction we give some preliminaries in Section 2. Section 3 is devoted to study biharmonic Legendre curves on three-dimensional quasi-Sasakian manifolds with respect to semi symmetric metric connection. In

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Section 4, we study locally ϕ-symmetric Legendre curves on three-dimensional quasi-Sasakian manifolds with respect to semi-symmetric metric connection.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) i.e., ϕ is a 1-1 tensor field, ξ is a unit vector field, η is a 1-form and g is a Riemannian metric such that [3]

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta(\phi) = 0 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ g(X, \xi) = \eta(X) \tag{2.2}$$

for all $X, Y \in \chi(M)$.

An almost contact metric manifold of dimension three is quasi-Sasakian if and only if

$$\nabla_X \xi = -\beta\phi X, \tag{2.3}$$

for $X \in \chi(M)$ and a function β defined on the manifold [10].

As a consequence of (2.3), we have [9]

$$(\nabla_X \phi)Y = \beta(g(X, Y)\xi - \eta(Y)X),\ X, Y \in \chi(M) \tag{2.4}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y) = -\beta g(\phi X, Y) \tag{2.5}$$

$$(\nabla_X \eta)\xi = -\beta\eta(\phi X) = 0 \tag{2.6}$$

The curvature tensor of a three-dimensional quasi-Sasakian manifold is given by [6]

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)[(\frac{r}{2} - \beta^2)X + (3\beta^2 - \frac{r}{2})\eta(X)\xi + \eta(X)(\phi\,\mathrm{grad}\,\beta) \\ &- d\beta(\phi X)\xi] - g(X,Z)[(\frac{r}{2} - \beta^2)Y + (3\beta^2 - \frac{r}{2})\eta(Y)\xi \\ &+ \eta(Y)(\phi\,\mathrm{grad}\,\beta) - d\beta(\phi Y)\xi] + [(\frac{r}{2} - \beta^2)g(Y,Z) \\ &+ (3\beta^2 - \frac{r}{2})\eta(Y)\eta(Z) - \eta(Y)d\beta(\phi Z) - \eta(Z)d\beta(\phi Y)]X \\ &- [(\frac{r}{2} - \beta^2)g(X,Z) + (3\beta^2 - \frac{r}{2})\eta(X)\eta(Z) - \eta(X)d\beta(\phi Z) \\ &- \eta(Z)d\beta(\phi X)]Y - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{2.7}$$

A curve γ on a manifold M is called Legendre curve if it satisfies [1]

$$\eta(\dot{\gamma}) = 0 \tag{2.8}$$

The semi symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ on an almost contact metric manifold are related by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi \tag{2.9}$$

for all vector fields X, Y on M.

The torsion tensor of a semi symmetric metric connection on an almost contact metric manifold is given by

$$\tilde{T}(X,Y) = \eta(Y)X - \eta(X)Y \tag{2.10}$$

A curve γ on M is called Frenet curve with respect to semi-symmetric metric connections if it satisfies

$$\tilde{\nabla}_T T = \tilde{k}N \tag{2.11}$$

$$\tilde{\nabla}_T N = -\tilde{k}T + \tilde{\tau}B \tag{2.12}$$

$$\tilde{\nabla}_T B = -\tilde{\tau}N \tag{2.13}$$

where $\tilde{k}, \tilde{\tau}$ are the curvature and torsion of the curve with respect to semi symmetric metric connection, $\{T, N, B\}$ is an orthonormal frame with $\dot{\gamma} = T$.

3. Biharmonic Legendre curves with respect to semi symmetric metric connection

Definition 3.1. A Legendre curve on three-dimensional quasi-Sasakian manifold will be called biharmonic with respect to semi-symmetric metric connection if it satisfies [5]

$$\tilde{\nabla}^3_T T + \tilde{\nabla}_T \tilde{\tau}(\tilde{\nabla}_T T, T)T + \tilde{R}(\tilde{\nabla}_T T, T)T = 0 \tag{3.1}$$

where $\tilde{\tau}$ is torsion of semi symmetric connection and T is tangent vector field of the curve.

Let us consider a Legendre curve γ and T be the tangent. We take $T,\ \phi T,\ \xi$ as the orthonormal right handed system where $\phi T = -N,\ \phi N = T$. For semi-symmetric metric connection, we have $\tilde{\nabla}_T \tilde{\tau}(\tilde{\nabla}_T T, T)T = 0$.

Hence (3.1) reduces to

$$\tilde{\nabla}^3_T T + \tilde{k}\tilde{R}(N,T)T = 0. \tag{3.2}$$

Let $\tilde{R}$ and R be the curvature tensor of a three-dimensional quasi-Sasakian manifold with respect to semi-symmetric metric connection and Levi-Civita connection respectively. Then the relation between $\tilde{R}$ and R is given by [14]

$$\begin{aligned} \tilde{R}(X,Y)Z = R(X,Y)Z &- L(Y,Z)X + L(X,Z)Y \\ &+ 2g(\nabla_Y X, Z)\xi - 2g(\nabla_X Y, Z)\xi + \eta(Z)([X,Y]) \\ &+ \eta(X)g(Y,Z)\xi + \eta(Y)g(X,Z)\xi, \end{aligned} \tag{3.3}$$

where

$$L(Y,Z) = (\nabla_Y \eta)Z - \eta(Y)\eta(Z) + g(Y,Z) \tag{3.4}$$

Now using (2.5) in (3.4) we get

$$L(Y,Z) = -\beta g(\phi Y, Z) - \eta(Y)\eta(Z) + g(Y,Z) \tag{3.5}$$

Using (3.5) in (3.3) we get

(3.6)

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + \beta(g(\phi Y,Z)X - g(\phi X,Z)Y) \\ &+ \eta(Z)(\eta(Y)X - \eta(X)Y) - (g(Y,Z)X - g(X,Z)Y) \\ &+ 2(g(\nabla_Y X,Z) - g(\nabla_X Y,Z))\xi + (\eta(X)g(Y,Z)\xi + \eta(Y)g(X,Z)\xi) \\ &+ \eta(Z)([X,Y])\end{aligned}$$

Since we have considered Frenet Frame as $T, \phi T, \xi$ where $\phi T = -N$, so for a Legendre curve we get $\eta(T) = 0$, $\eta(N) = 0$. Using this fact and putting $X = N$, $Y = T$, $Z = T$ in (3.6) we get

$$\tilde{R}(N,T)T = R(N,T)T - N + \beta T + 2[g(\nabla_T N,T) - g(\nabla_N T,T)]\xi \tag{3.7}$$

Now putting $X = N$, $Y = T$, $Z = T$ in (2.7) we get

$$R(N,T)T = \frac{r}{2}N - 2\beta^2 N - d\beta(\phi N)\xi \tag{3.8}$$

From (3.7) and (3.8) after some simplification and setting $\xi = B$ we get

$$\tilde{R}(N,T)T = \frac{r}{2}N - 2\beta^2 N - d\beta(\phi N)B - N + \beta T - 2\tilde{k}B \tag{3.9}$$

Again by Serret-Frenet formula we get,

$$\tilde{\nabla}^3{}_T T = -3\tilde{k}\tilde{k}'T + (\tilde{k}'' - \tilde{k}^3 - \tilde{k}\tilde{\tau}^2)N + (2\tilde{\tau}\tilde{k}' + \tilde{k}\tilde{\tau}')B \tag{3.10}$$

From (3.9) and (3.10) we get,

$$\begin{aligned}\tilde{\nabla}^3{}_T T + \tilde{k}\tilde{R}(N,T)T &= (-3\tilde{k}\tilde{k}' - \tilde{k}\beta)T + (\tilde{k}'' - \tilde{k}^3 - \tilde{k}\tilde{\tau}^2 + \tilde{k}\frac{r}{2} - 2\tilde{k}\beta^2 + \tilde{k})N \\ &+ (2\tilde{\tau}\tilde{k}' + \tilde{k}\tilde{\tau}' - \tilde{k}d\beta(\phi N) + 2\tilde{k}^2)B.\end{aligned}$$

If the Legendre curve is biharmonic, then we have $\tilde{\nabla}^3{}_T T + \tilde{k}\tilde{R}(N,T)T = 0$. So we have

$$-3\tilde{k}\tilde{k}' - \tilde{k}\beta = 0 \tag{3.11}$$

$$\tilde{k}'' - \tilde{k}^3 - \tilde{k}\tilde{\tau}^2 + \tilde{k}\frac{r}{2} - 2\tilde{k}\beta^2 + \tilde{k} = 0 \tag{3.12}$$

$$2\tilde{\tau}\tilde{k}' + \tilde{k}\tilde{\tau}' - \tilde{k}d\beta(\phi N) + 2\tilde{k}^2 = 0. \tag{3.13}$$

In view of (3.11), we obtain the following theorem:

Theorem 3.1. *The curvature of a non-geodesic biharmonic Legendre curve on a three-dimensional quasi-Sasakian manifold with respect to semi-symmetric connection is given by* $\tilde{k} = -\frac{1}{3}\int \beta ds$, *where* s *is the arc length parameter.*

4. Locally ϕ-symmetric Legendre curves

Definition 4.1. With respect to semi-symmetric metric connection a Legendre curve on a three-dimensional quasi-Sasakian manifold is called locally ϕ-symmetric if it satisfies [11]

$$\phi^2(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T,T)T=0 \tag{4.1}$$

Now putting $X=\tilde{\nabla}_T T$, $Y=Z=T$ in (3.6) and (2.7) and then using Serret-Frenet formula, after some calculations we get

$$\tilde{R}(\tilde{\nabla}_T T,T)T=\beta\tilde{k}T+(\frac{r}{2}\tilde{k}-2\beta\tilde{k}-\tilde{k})N-(\tilde{k}d\beta(\phi N)+2\tilde{k}^2)B. \tag{4.2}$$

Again putting $X=B$, $Y=Z=T$ in (3.6) and (2.7) and then using $\phi T=-N$ we get

$$\tilde{R}(B,T)T=\beta^2B+\phi\operatorname{grad}\beta+d\beta(N)T. \tag{4.3}$$

By definition of covariant differentiation of $\tilde{R}$ and using Serret-Frenet formula, we get

$$\begin{aligned}(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T,T)T=&\tilde{\nabla}_T\tilde{R}(\tilde{\nabla}_T T,T)T\\&-\tilde{k}\tilde{\tau}\tilde{R}(B,T)T-\tilde{k}\tilde{R}(N,T)T-\tilde{k}^2\tilde{R}(N,T)N\end{aligned} \tag{4.4}$$

Again putting $X=N$, $Y=T$, $Z=N$ in (3.6) and (2.7) and setting $\xi=B$ we get

$$\tilde{R}(N,T)N=2\beta^2T-\frac{r}{2}T+d\beta(N)B-\beta N+T-2g(\nabla_N T,N)B \tag{4.5}$$

Now using (4.2) and Serret-Frenet formula we get

$$\begin{aligned}\tilde{\nabla}_T\tilde{R}(\tilde{\nabla}_T T,T)T=&[(\beta\tilde{k})'-\frac{r}{2}\tilde{k}^2+2\beta\tilde{k}^2+\tilde{k}^2]T\\&+[\beta\tilde{k}^2+(\frac{r}{2}\tilde{k})'-2(\beta\tilde{k})'-\tilde{k}'+\tilde{k}d\beta(\phi N)\tilde{\tau}\\&+2\tilde{k}^2\tilde{\tau}]N+[\frac{r}{2}\tilde{k}\tilde{\tau}-2\beta\tilde{k}\tilde{\tau}-\tilde{k}\tilde{\tau}\\&-\tilde{k}\tilde{\nabla}_T(d\beta(\phi N))\tilde{k}'d\beta(\phi N)-4\tilde{k}\tilde{k}']B\end{aligned} \tag{4.6}$$

Now from (3.9), (4.3), (4.4), (4.5), (4.6) we get

$$\begin{aligned}(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T,T)T=&[(\beta\tilde{k})'+2\beta\tilde{k}^2-\tilde{k}\tilde{\tau}d\beta(N)-\tilde{k}\beta-2\beta^2\tilde{k}]T\\&+[\beta\tilde{k}^2+(\frac{r}{2}\tilde{k})'-2(\beta\tilde{k})'-\tilde{k}'+\tilde{k}d\beta(\phi N)\tilde{\tau}\\&+2\tilde{k}^2\tilde{\tau}-\tilde{k}\frac{r}{2}+2\tilde{k}\beta^2+\tilde{k}^2\beta]N\\&+[\frac{r}{2}\tilde{k}\tilde{\tau}-2\beta\tilde{k}\tilde{\tau}-\tilde{k}\tilde{\tau}-\tilde{k}\tilde{\nabla}_T(d\beta(\phi N))\\&-\tilde{k}'d\beta(\phi N)-4\tilde{k}\tilde{k}'-\tilde{k}\tilde{\tau}\beta^2+\tilde{k}d\beta(\phi N)+2\tilde{k}^2\\&-\tilde{k}^2d\beta(N)+2\tilde{k}^2g(\nabla_N T,N)]B-\tilde{k}\tilde{\tau}\phi\operatorname{grad}\beta\end{aligned} \tag{4.7}$$

Applying ϕ^2 on both sides, we get,

$$
\begin{aligned}
\phi^2(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T)T = &-[(\beta\tilde{k})' + 2\beta\tilde{k}^2 - \tilde{k}\tilde{\tau}d\beta(N) - \tilde{k}\beta - 2\beta^2\tilde{k}]T \\
&- [\beta\tilde{k}^2 + (\frac{r}{2}\tilde{k})' - 2(\beta\tilde{k})' - \tilde{k}' + \tilde{k}d\beta(\phi N)\tilde{\tau} \\
&+ 2\tilde{k}^2\tilde{\tau} - \tilde{k}\frac{r}{2} + 2\tilde{k}\beta^2 + \tilde{k}^2\beta]N \\
&- \tilde{k}\tilde{\tau}\phi^3 \operatorname{grad}\beta
\end{aligned} \tag{4.8}
$$

If the curves are locally ϕ-symmetric, then $\tilde{k}\tilde{\tau}\phi^3 \operatorname{grad}\beta = 0$.

Let $\tilde{k} \neq 0$ and β is not constant. Then $\tilde{\tau} = 0$. So, the torsion with respect to semi-symmetric connection of a locally ϕ-symmetric Legendre curve on a three-dimensional quasi-Sasakian manifold is zero.

Theorem 4.1. *A non-geodesic locally ϕ-symmetric Legendre curve with respect to semi-symmetric metric connection on a three-dimensional quasi-Sasakian manifold with non-constant structure function is a plane curve.*

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