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VARIABLE LEBESGUE SPACES AND CONTINUOUS WAVELET TRANSFORMS

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Dedicated to Professor Ferenc Schipp on the occasion of his 75th birthday, to Professor William Wade on the occasion of his 70th birthday and to Professor Péter Simon on the occasion of his 65th birthday.

Abstract. In this paper we summarize the previous results in the topic of variable Lebesgue space. We present the basic properties of the variable Lebesgue spaces and investigate norm and almost everywhere convergence of the inverse continuous wavelet transform in the variable Lebesgue space.

1. Introduction

The topic of variable Lebesgue spaces is a new chapter of mathematics and is studied intensively nowadays. In this paper we summarize the main results. For an exponent function $p\colon \mathbb{R}^d \to [1,\infty)$, we consider the variable $L_{p(\cdot)}(\mathbb{R}^d)$ spaces. The variable Lebesgue spaces have a lot of common properties with the classical Lebesgue spaces. For example, the variable $L_{p(\cdot)}(\mathbb{R}^d)$ spaces are equipped with the norm $\|\cdot\|_{p(\cdot)}$ are Banach spaces. The analogue of Hölder's inequality remains true. Similarly to the classical Lebesgue spaces, if the measure of $\mathbb{X} \subset \mathbb{R}^d$ is finite, then the spaces $L_{q(\cdot)}(\mathbb{X}) \subset L_{p(\cdot)}(\mathbb{X})$ if and only if the exponent function $p(\cdot) \leq q(\cdot)$ almost everywhere. Or, if the exponent function $q(\cdot)$ is the conjugate function of $p(\cdot)$ and $\sup\{p(x) : x \in \mathbb{R}^d\} < \infty$, then the dual space $L^*_{p(\cdot)}(\mathbb{R}^d)$ and the space $L_{q(\cdot)}(\mathbb{R}^d)$ are equivalent. Though, there are some properties of the classical Lebesgue spaces, which do not remain true in the variable Lebesgue spaces. For instance, the translation invariance, Young's inequality, or the boundedness of the strong maximal operator (see Kováčik and Rákosník [11], Cruz-Uribe and Fiorenza [4], Diening, Hästö and

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Růžička [5], Cruz-Uribe, Firorenza and Neugebauer [3], Cruz-Uribe, Fiorenza, Martell and Pérez [2]).

The so-called θ-summation method is studied intensively in the literature (see e.g. Butzer and Nessel [1], Trigub and Belinsky [15], Gát [8], Goginava [9], Simon [13] and Weisz [16, 17]). For an integrable function θ on $\mathbb{R}^d$, the θ-summation is defined by

$$\sigma_T^\theta f(x) = \int_{\mathbb{R}^d} f(x-t) T^d \theta(Tt)\, dt.$$

By a suitable chosen of θ, we get back the well-known summability methods, like Fejér, Riesz, Weierstrass, Abel, etc. Feichtinger and Weisz [6, 7, 19] have proved that the θ-means $\sigma_T^\theta f$ converge to f almost everywhere and in norm as $T \to \infty$, whenever f is in the $L_p(\mathbb{R}^d)$ space or in a Wiener amalgam space. The points of the set of the almost everywhere convergence are characterized as the Lebesgue points. In [14] we proved that in case of $T \to 0$, the θ-means converge to 0 pointwise if $f \in L_p$ $(1 \le p \le \infty)$ and in norm, whenever $f \in L_p$ $(1 < p < \infty)$ and θ has a radial majorant.

Some similar results are known in the variable Lebesgue spaces (see e.g. Cruz-Uribe and Fiorenza [4], Szarvas and Weisz [14]). Under some conditions on the exponent function $p(\cdot)$ and θ, the θ-means of f converge to f almost everywhere and in norm for all $f \in L_{p(\cdot)}$ as $T \to \infty$ and converge to 0 pointwise and in norm as $T \to 0$.

Recently Li and Sun [12] have proved that under some conditions the inverse continuous wavelet transform converges to Cf at every Lebesgue points of f for any $f \in L_p(\mathbb{R}^d)$ $(1 \le p < \infty)$, where C is a constant depending on g and γ. If $1 < p < \infty$, then the convergence holds in the $L_p(\mathbb{R}^d)$-norm for all $f \in L_p(\mathbb{R}^d)$. Under some other conditions Weisz [18] has proved similar results. These results were generalized for variable Lebesgue spaces in [14].

In this paper we will present the basic properties of the variable Lebesgue spaces and investigate the norm and almost everywhere convergence of the continuous wavelet transform in variable Lebesgue spaces. The solution was that we traced back the problem to the summability of Fourier transforms.

2. Variable Lebesgue spaces

First of all, we will investigate the classical $L_p(\mathbb{R}^d)$ spaces. Let us fix $d \ge 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \ne \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \cdots \times \mathbb{Y}$ taken with itself d-times. The space $L_p(\mathbb{R}^d)$ equipped with the norm

$$\|f\|_p := \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{1/p} \quad (1 \le p \le \infty),$$

is the classical Lebesgue space. We use the notation $|I|$ for the Lebesgue measure of the set I. The set of *locally integrable functions* is denoted by $L_1^{loc}(\mathbb{R}^d)$.

A function $p(\cdot)$ belongs to $\mathcal{P}(\mathbb{R}^d)$ if $p\colon \mathbb{R}^d \to [1,\infty]$ and $p(\cdot)$ is measurable. Then we say that $p(\cdot)$ is an exponent function. Let

$$p_- := \inf\{p(x) : x \in \mathbb{R}^d\} \quad \text{and} \quad p_+ := \sup\{p(x) : x \in \mathbb{R}^d\}.$$

Set

$$\Omega_\infty^{p(\cdot)} := \Omega_\infty := \{x \in \mathbb{R}^d : p(x) = \infty\}.$$

The modular generated by $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ is defined by

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^d \setminus \Omega_\infty} |f(x)|^{p(x)}\, dx + \|f\|_{L_\infty(\Omega_\infty)},$$

where f is a measurable function. A measurable function f belongs to the $L_{p(\cdot)}(\mathbb{R}^d)$ space if there exists $\lambda > 0$ such that $\varrho_{p(\cdot)}(f/\lambda) < \infty$. We can see that the modular $\varrho_{p(\cdot)}$ is not a norm. Define the $L_{p(\cdot)}(\mathbb{R}^d)$-norm by

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

Then $\|\cdot\|_{p(\cdot)}$ is a norm and the space $(L_{p(\cdot)}(\mathbb{R}^d), \|\cdot\|_{p(\cdot)})$ is a normed space. In case $p(\cdot) = p$ is a constant, then we get back the usual $L_p(\mathbb{R}^d)$ spaces.

The variable Lebesgue spaces are special cases of much more general function spaces, the so-called Musielak-Orlicz spaces. Let $\Phi\colon \mathbb{R}^d \times [0,\infty) \to [0,\infty]$ be given such that for all $x \in \mathbb{R}^d$, $\Phi(x,\cdot)$ is non-decreasing, continuous and convex on the set where it is finite. Assume that $\Phi(x,0) = 0$, $\Phi(x,t) > 0$ $(t > 0)$, $\lim_{t\to\infty} \Phi(x,t) = \infty$ and for all $t \ge 0$, $\Phi(\cdot,t)$ is a measurable function. Let

$$\|f\|_{L_{\Phi(\cdot,\cdot)}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^d} \Phi\left(x, \frac{|f(x)|}{\lambda}\right) dx \le 1\right\},$$

which is the so-called Luxemburg-norm. Then a measurable function f belongs to the Musielak-Orlicz space if $\|f\|_{L_{\Phi(\cdot,\cdot)}} < \infty$. If we take $\Phi(x,t) := t^p$ $(1 \le p < \infty)$, $\Phi(x,t) := t^p w(x)$ $(1 \le p < \infty)$, $\Phi(x,t) := t^{p(x)}$, $\Phi(x,t) := t^{p(x)} w(x)$ or $\Phi(x,t) = \Phi(t)$, then we get the classical Lebesgue spaces, the weighted Lebesgue spaces, the variable Lebesgue spaces, the weighted variable Lebesgue spaces or the Orlicz spaces back. Here the function w is a so-called weight function, i.e. w is positive and locally integrable.

3. Basic properties of the variable Lebesgue spaces

In this chapter we review some basic properties of the variable Lebesgue spaces (the proofs and other interesting facts about variable Lebesgue spaces can be found in [4] or in [5]). Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $f, f_n \in L_{p(\cdot)}(\mathbb{R}^d)$ $(n \in \mathbb{N})$. We say that (f_n) converges to f *in modular* if for some $\lambda > 0$, $\lim_{n\to\infty} \varrho(\lambda(f - f_n)) = 0$.

We say that (f_n) converges to f *in the $L_{p(\cdot)}(\mathbb{R}^d)$-norm*, if $\lim_{n\to\infty} \|f - f_n\|_{p(\cdot)} = 0$. The norm convergence is stronger than the modular convergence (see [4, p.44]).

Theorem 1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $f, f_n \in L_{p(\cdot)}(\mathbb{R}^d)$ $(n \in \mathbb{N})$. If (f_n) converges to f in the $L_{p(\cdot)}(\mathbb{R}^d)$-norm, then (f_n) converges to f in modular.*

Although, if we take a condition on the exponent function, we get (see [4, p.44])

Theorem 2. *In case of $p_+ < \infty$, the sequence (f_n) $(n \in \mathbb{N})$ converges to f in modular if and only if (f_n) converges to f in the $L_{p(\cdot)}(\mathbb{R}^d)$-norm.*

One of the most important properties of the variable Lebesgue spaces is that they are *Banach spaces* (see [4, p.55]).

Theorem 3. *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, then the space $(L_{p(\cdot)}(\mathbb{R}^d), \|\cdot\|_{p(\cdot)})$ is a Banach space.*

The dense subsets of the classical Lebesgue spaces are very important. For the variable Lebesgue spaces we have ([4, p.56])

Theorem 4. *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $p_+ < \infty$, then the set of bounded functions with compact support is dense in $L_{p(\cdot)}(\mathbb{R}^d)$.*

The well-known *Hölder's inequality* remains true in the variable Lebesgue spaces too (see e.g. [4, p.27]). Namely, if $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ exponent functions are such that $1/p(x) + 1/q(x) = 1$ for all $x \in \mathbb{R}^d$, then we say that $q(\cdot)$ is the *conjugate function of* $p(\cdot)$.

Theorem 5 (Hölder's inequality)**.** *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $1/p(x) + 1/q(x) = 1$. Then for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$ and $g \in L_{q(\cdot)}(\mathbb{R}^d)$, $fg \in L_1(\mathbb{R}^d)$ and*

$$\int_{\mathbb{R}^d} |f(x)g(x)|\, dx \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}.$$

Let $\Omega \subset \mathbb{R}^d$, $|\Omega| < \infty$. For the classical spaces the $L_p(\Omega)$ spaces decrease as p increase. In the variable Lebesgue spaces we have an analogous theorem ([4, p.41]).

Theorem 6. *Suppose that $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$ and $|\Omega \setminus \Omega_\infty^{p(\cdot)}| < \infty$. Then $L_{q(\cdot)}(\Omega) \subset L_{p(\cdot)}(\Omega)$ if and only if $p(\cdot) \leq q(\cdot)$ almost everywhere.*

Another embedding theorem is (see [5, p.83])

Theorem 7. *If $p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $p(\cdot) \leq q(\cdot) \leq r(\cdot)$ almost everywhere, then*

$$L_{p(\cdot)}(\mathbb{R}^d) \cap L_{r(\cdot)}(\mathbb{R}^d) \hookrightarrow L_{q(\cdot)}(\mathbb{R}^d) \hookrightarrow L_{p(\cdot)}(\mathbb{R}^d) + L_{r(\cdot)}(\mathbb{R}^d).$$

Moreover, if $g \in L_{q(\cdot)}(\mathbb{R}^d)$, then

$$\|g\|_{q(\cdot)} \leq 2 \max\{\|g\|_{p(\cdot)}, \|g\|_{r(\cdot)}\}.$$

and

$$\inf_{\substack{g=g_1+g_2,\, g_1 \in L_{p(\cdot)}(\mathbb{R}^d), \\ g_2 \in L_{r(\cdot)}(\mathbb{R}^d)}} \{\|g_1\|_{p(\cdot)} + \|g_2\|_{r(\cdot)}\} \leq 2\|g\|_{q(\cdot)}.$$

The *duality* results are similar to the case of the classical Lebesgue spaces. It is known that the dual space of $L_{p(\cdot)}(\mathbb{R}^d)$ is $L^*_{p(\cdot)}(\mathbb{R}^d)$ equipped with the norm $\|\Phi\| := \sup_{\|f\|_{p(\cdot)}\le 1} |\Phi(f)|$, where $L^*_{p(\cdot)}(\mathbb{R}^d)$ contains all bounded linear functionals $\Phi\colon L_{p(\cdot)}(\mathbb{R}^d) \to \mathbb{R}$.

Theorem 8. *Let g be a measurable function and define the linear functional $\Phi_g\colon L_{p(\cdot)}(\mathbb{R}^d) \to \mathbb{R}$ by*

$$\Phi_g(f) := \int_{\mathbb{R}^d} f(x)g(x)\,dx.$$

If $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ is the conjugate function of $p(\cdot)$, then $\Phi_g \in L^*_{p(\cdot)}(\mathbb{R}^d)$ if and only if $g \in L_{q(\cdot)}(\mathbb{R}^d)$.

Moreover, if p_+ is finite, then essentially there are no other bounded linear functionals in the dual space just Φ_g (see e.g. [4, p.63]).

Theorem 9. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $q(\cdot)$ is the conjugate function of $p(\cdot)$. Then the followings are equivalent:*

(1) $p_+ < \infty$,
(2) *For any $\Phi \in L^*_{p(\cdot)}(\mathbb{R}^d)$ there exists a unique $g \in L_{q(\cdot)}(\mathbb{R}^d)$ such that $\Phi = \Phi_g$.*

We get the next results as immediate consequences:

Corollary 1. *In case of $p_+ < \infty$, the space $L^*_{p(\cdot)}(\mathbb{R}^d)$ and the space $L_{q(\cdot)}(\mathbb{R}^d)$ are equivalent.*

Corollary 2. *Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$. Then $L_{p(\cdot)}(\mathbb{R}^d)$ is reflexive if and only if $1 < p_- \le p_+ < \infty$.*

We have seen that there are some important properties of the classical Lebesgue spaces which remain true for the variable Lebesgue spaces. However, there are some important statements that are not true for the variable Lebesgue spaces. For example, the classical Lebesgue spaces are translation invariant spaces, i.e. for all $f \in L_p(\mathbb{R}^d)$ $(1 \le p \le \infty)$ and for all $h \in \mathbb{R}^d$, $T_h f \in L_p(\mathbb{R}^d)$ and $\|f\|_p = \|T_h f\|_p$, where $T_h f(x) = f(x-h)$ $(x \in \mathbb{R}^d)$. This property is used to prove norm-convergence (for example the norm-convergence of θ-summation). But in the variable Lebesgue spaces, this never holds (see [4, p.203]).

Theorem 10. *Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $h \in \mathbb{R}^d$. Then the translation operator T_h is bounded on $L_{p(\cdot)}(\mathbb{R}^d)$ if and only if $p(\cdot)$ is constant. Moreover, if $p(\cdot)$ is non-constant, then for all $h \in \mathbb{R}^d$ there exists $f \in L_{p(\cdot)}(\mathbb{R}^d)$ such that $T_h \notin L_{p(\cdot)}(\mathbb{R}^d)$.*

A Banach space B is called *homogeneous Banach space*, if

(1) for all $f \in B$ and $x \in \mathbb{R}^d$, $T_x f \in B$ and $\|T_x f\|_B = \|f\|_B$,
(2) the function $x \mapsto T_x f$ from $\mathbb{R}^d$ to B is continuous for all $f \in B$,

(3) the functions in B are *uniformly locally integrable*, i.e. for every compact set $K \subset \mathbb{R}^d$ there exists a constant C_K such that

$$\int_K |f| \, d\lambda \leq C_K \|f\|_B \quad (f \in B).$$

Corollary 3. *In case $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ is not a constant, then $L_{p(\cdot)}(\mathbb{R}^d)$ is not a homogeneous Banach space.*

The convolution of the measurable functions f and g is defined by

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x-t)g(t) \, dt = \int_{\mathbb{R}^d} f(t)g(x-t) \, dt.$$

The so-called *Young's inequality* is a well-known result in the classical Lebesgue spaces:

$$\|f * g\|_r \leq \|f\|_p \|g\|_q$$

for all $f \in L_p(\mathbb{R}^d)$ and $g \in L_q(\mathbb{R}^d)$, where $1 \leq p, q, r \leq \infty$, and $1/r + 1 = 1/p + 1/q$. In case of $q = 1$ we obtain the inequality $\|f * g\|_p \leq \|f\|_p \|g\|_1$. Unfortunately, since the spaces $L_{p(\cdot)}(\mathbb{R}^d)$ are not translation invariant, unless $p(\cdot)$ is a constant, we get that the Young's inequality does not hold in the variable Lebesgue spaces ([4, 204]):

Theorem 11. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $1 < p_- \leq p_+ < \infty$. Then*

$$\|f * g\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)} \|g\|_1$$

for some $C > 0$ and for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$, $g \in L_1(\mathbb{R}^d)$ if and only if $p(\cdot)$ is a constant.

Another surprising result for the classical Hardy-Littlewood maximal operator is Theorem 15.

The interpolation is another central topic of the classical Lebesgue spaces. The *Riesz-Thorin's theorem* remains true for the variable Lebesgue spaces ([4, p.128]).

Theorem 12. *Let $p_i(\cdot), q_i(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ $(i = 1, 2)$. If T is a sublinear operator such that for all $f \in L_{p_i(\cdot)}(\mathbb{R}^d)$*

$$\|Tf\|_{q_i(\cdot)} \leq M_i \|f\|_{p_i(\cdot)} \quad (i = 1, 2,),$$

then for each $0 < \theta < 1$ and for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$,

$$\|Tf\|_{q(\cdot)} \leq C M_1^{\theta} M_2^{1-\theta} \|f\|_{p(\cdot)},$$

where $p(\cdot), q(\cdot)$ are defined by

$$\frac{1}{p(x)} = \frac{\theta}{p_1(x)} + \frac{1-\theta}{p_2(x)}, \quad \frac{1}{q(x)} = \frac{\theta}{q_1(x)} + \frac{1-\theta}{q_2(x)} \quad (x \in \mathbb{R}^d).$$

In the classical Lebesgue spaces, the so-called *Marcinkiewicz-theorem* says that

Theorem 13 (Marcinkiewicz). *Let* $1 \leq p_1 \leq p_2 < \infty$ *If* T *is a sublinear operator for which*

$$\sup_{t>0} \left\| t\chi_{\{x\in\mathbb{R}^d : Tf(x)>t\}} \right\|_{p_i} \leq C\|f\|_{p_i}$$

for all $f \in L_{p_i}(\mathbb{R}^d)$ $(i = 1, 2)$, *then for any* $p_1 < q < p_2$

$$\|Tf\|_q \leq C\|f\|_q$$

for all $f \in L_q(\mathbb{R}^d)$.

In the topic of variable Lebesgue spaces it is an open question that if the sublinear operator T satisfies

$$\sup_{t>0} \left\| t\chi_{\{x\in\mathbb{R}^d : Tf(x)>t\}} \right\|_{p_i(\cdot)} \leq C \|f\|_{p_i(\cdot)} \quad (i = 1, 2)$$

for all $f \in L_{p_i(\cdot)}(\mathbb{R}^d)$, then the inequality

$$\|Tf\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)}$$

does hold for any $f \in L_{p(\cdot)}(\mathbb{R}^d)$, where $1/p(x) = \theta/p_1(x) + (1-\theta)/p_2(x)$ $(0 < \theta < 1,\ x \in \mathbb{R}^d)$?

4. Maximal operators

Maximal operators are playing central role in Fourier analysis and in approximation theory. The classical Hardy-Littlewood maximal operator is defined by

$$Mf(x) := \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f| \, d\lambda \quad (x \in \mathbb{R}^d),$$

where $f \in L_1^{loc}(\mathbb{R}^d)$ and the supremum is taken over all cubes $Q \subset \mathbb{R}^d$ with sides parallel to the axes. The following result is known for the classical Lebesgue spaces:

$$\|Mf\|_p \leq C \|f\|_p$$

for all $f \in L_p(\mathbb{R}^d)$ $(1 < p \leq \infty)$ and

$$\sup_{t>0} t\lambda \left(x \in \mathbb{R}^d : Mf(x) > t \right) \leq C \|f\|_1$$

for all $f \in L_1(\mathbb{R}^d)$. To generalize this result for variable Lebesgue spaces we will introduce the *log-Hölder continuous functions.* We say that $r(\cdot)$ is locally log-Hölder continuous if there exists a constant C_0 such that for all $x, y \in \mathbb{R}^d$, $\|x - y\|_2 < 1/2$,

$$|r(x) - r(y)| \leq \frac{C_0}{-\log(\|x - y\|_2)}.$$

We denote this set by $LH_0(\mathbb{R}^d)$.

We say that $r(\cdot)$ is log-Hölder continuous at infinity if there exist constants C_∞ and r_∞ such that for all $x \in \mathbb{R}^d$

$$|r(x) - r_\infty| \leq \frac{C_\infty}{\log(e + \|x\|_2)}.$$

We write briefly $r(\cdot) \in LH_\infty(\mathbb{R}^d)$. Let

$$LH(\mathbb{R}^d) := LH_0(\mathbb{R}^d) \cap LH_\infty(\mathbb{R}^d).$$

For example, if $r(\cdot)$ is in $LH_0(\mathbb{R}^d)$, then $r(\cdot)$ is uniformly continuous and bounded on every bounded set $E \subset \mathbb{R}^d$. Moreover, if $r(\cdot)$ is in $LH_\infty(\mathbb{R}^d)$, then $r(\cdot)$ is bounded (the proofs can be found in [4, p.15]). With the help of this property, we get sufficient condition for the boundedness of the Hardy-Littlewood maximal operator (see [4, p.89]).

Theorem 14. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $1/p(\cdot) \in LH(\mathbb{R}^d)$.*

(1) *Then for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$*

$$\sup_{t>0} \left\| t\chi_{\{x\in\mathbb{R}^d : Mf(x)>t\}} \right\|_{p(\cdot)} \leq C \left\| f \right\|_{p(\cdot)}.$$

(2) *If in addition $p_- > 1$, then for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$*

$$\left\| Mf \right\|_{p(\cdot)} \leq C \left\| f \right\|_{p(\cdot)}.$$

However, the integral inequality never holds in the variable Lebesgue spaces (see [4, p.108]).

Theorem 15. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $p_+ < \infty$. Then the inequalities*

$$\int_{\{x\in\mathbb{R}^d : Mf(x)>t\}} t^{p(x)}\, dx \leq C \int_{\mathbb{R}^d} |f(x)|^{p(x)}\, dx,$$

$$\int_{\mathbb{R}^d} Mf(x)^{p(x)}\, dx \leq C \int_{\mathbb{R}^d} |f(x)|^{p(x)}\, dx$$

are true if and only if $p(\cdot)$ is a constant almost everywhere.

It is known that in case of $p = 1$, the Hardy-Littlewood maximal operator is not bounded on $L_1(\mathbb{R}^d)$. Analogously, if $p_- = 1$, then the maximal operator M is not bounded on the space $L_{p(\cdot)}(\mathbb{R}^d)$ (see [4, p.90]).

We can define other maximal operators too. For example, the strong maximal operator is given by

$$M_s f(x) := \sup_{x\in I} \frac{1}{|I|} \int_I |f|\, d\lambda \quad (x \in \mathbb{R}^d),$$

where $f \in L_1^{loc}(\mathbb{R}^d)$ and the supremum is taken over all rectangles $I \subset \mathbb{R}^d$ with sides parallel to the axes. It is known that

$$\left\| M_s f \right\|_p \leq C \left\| f \right\|_p$$

for all $f \in L_p(\mathbb{R}^d)$ $(1 < p \leq \infty)$ and

$$\sup_{t>0} t\lambda\left(x \in \mathbb{R}^d : M_s f(x) > t, |x| \leq C_0\right) \leq C + C \int_{\mathbb{R}^d} |f|(\log^+ |f|)^{d-1}\, d\lambda$$

for all $f \in L_1(L\log L)^{d-1}(\mathbb{R}^d)$. In contrast to the previous theorem, the strong maximal operator is never bounded on the variable $L_{p(\cdot)}(\mathbb{R}^d)$ spaces unless $p(\cdot)$ is a constant (see Kokilashvili and Meskhi [10]):

Theorem 16. *Let $1 < p_- \leq p_+ < \infty$. Then the strong maximal operator M_s is bounded on $L_{p(\cdot)}(\mathbb{R}^d)$ if and only if $p(\cdot)$ is constant.*

5. θ-SUMMATION

In this chapter we will introduce a general method for approximation, the so-called θ-summation. Let $\theta \in L_1(\mathbb{R}^d)$ be a radial function. The *θ-means* of $f \in L_p(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$ are defined by

$$\sigma_T^\theta f(x) := (f * \theta_T)(x) = \int_{\mathbb{R}^d} f(x-t)\theta_T(t)\, dt,$$

where

$$\theta_T(x) := T^d \theta(Tx) \quad (x \in \mathbb{R}^d, T > 0).$$

A point $x \in \mathbb{R}^d$ is called a Lebesgue point of $f \in L_1^{loc}(\mathbb{R}^d)$ if

$$\lim_{h\to 0} \frac{1}{|B(0,h)|} \int_{B(0,h)} |f(x+u) - f(x)|\, du = 0,$$

where

$$B(a,\delta) := \{x \in \mathbb{R}^d : \|x-a\|_2 < \delta\}.$$

It is known that if $f \in L_p(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$ or $f \in L_{p(\cdot)}(\mathbb{R}^d)$ $(p(\cdot) \in \mathcal{P}(\mathbb{R}^d))$, then almost every $x \in \mathbb{R}^d$ is a Lebesgue point of f (see e.g. Feichtinger and Weisz [7] and Szarvas and Weisz [14]).

We say that η is a radial majorant of f, if η is radial, non-increasing as a function on $(0,\infty)$, non-negative, bounded, $|f| \leq \eta$ and $\eta \in L_1(\mathbb{R}^d)$. If in addition, $\eta(\cdot)\ln(|\cdot|+2) \in L_1(\mathbb{R}^d)$, then we say that η is a radial log-majorant of f. The next theorem can be found in Feichtinger and Weisz [6] and Szarvas and Weisz [14].

Theorem 17. *Suppose that $\theta \in L_1(\mathbb{R}^d)$.*

(1) *Then for all $f \in L_p(\mathbb{R}^d)$ $(1 \leq p < \infty)$*

$$\lim_{T\to\infty} \sigma_T^\theta f = \int_{\mathbb{R}^d} \theta(x)\, dx \cdot f \qquad \text{in the } L_p(\mathbb{R}^d)\text{-norm.}$$

If θ has a radial majorant, then the convergence holds for all Lebesgue points.

(2) *If θ has a radial majorant, then for all $f \in L_p(\mathbb{R}^d)$ $(1 < p < \infty)$*

$$\lim_{T\to 0} \sigma_T^\theta f = 0 \qquad \text{in the } L_p(\mathbb{R}^d)\text{-norm.}$$

This convergence holds for all $x \in \mathbb{R}^d$ pointwise and for all $1 \leq p \leq \infty$.

We have generalized this result for variable Lebesgue spaces in [14]:

Theorem 18. *Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and θ has a radial majorant.*

(1) *Then for all Lebesgue points of* $f \in L_{p(\cdot)}(\mathbb{R}^d)$,

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = \int_{\mathbb{R}^d} \theta(y)\, dy \cdot f(x).$$

(2) *If in addition* $p_+ < \infty$, *then*

$$\lim_{T\to 0} \sigma_T^\theta f(x) = 0$$

for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$ *and for all* $x \in \mathbb{R}^d$.

To prove norm-convergence, we have to take other conditions. The first statement of the next theorem can be found in Cruz-Uribe and Fiorenza [4, p.199]. We have proved the second one in [14].

Theorem 19. *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $p_+ < \infty$, $1/p(x) + 1/q(x) = 1$. *Suppose that* θ *has a radial majorant and the maximal operator is bounded on* $L_{q(\cdot)}(\mathbb{R}^d)$.

(1) *Then for all* $f \in L_{p(\cdot)}(\mathbb{R}^d)$

$$\lim_{T\to\infty} \sigma_T^\theta f = \int_{\mathbb{R}^d} \theta(x)\, dx \cdot f \qquad \text{in the } L_{p(\cdot)}(\mathbb{R}^d)\text{-norm.}$$

(2) *If in addition* $p_- > 1$, *then*

$$\lim_{T\to 0} \sigma_T^\theta f = 0 \qquad \text{in the } L_{p(\cdot)}(\mathbb{R}^d)\text{-norm}$$

for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$.

Note that if $1/p(\cdot) \in LH(\mathbb{R}^d)$ and $p_+ < \infty$, then $1/q(\cdot) \in LH(\mathbb{R}^d)$ and $q_- > 1$, hence the maximal operator is bounded on $L_{q(\cdot)}(\mathbb{R}^d)$. Therefore if $1/p(\cdot) \in LH(\mathbb{R}^d)$, $p_+ < \infty$ and θ has a radial majorant, then the statements of the previous theorem remain true.

6. Continuous wavelet transform

In this chapter we will present an application of the previous results. Namely, we will rewrite the continuous wavelet transform as a θ-summation and we will get convergence theorems for the continuous wavelet transform. The *continuous wavelet transform* of f with respect to a wavelet g is defined by

$$W_g f(x,s) := |s|^{-d/2} \int_{\mathbb{R}^d} f(t)\overline{g(s^{-1}(t-x))}\, dt = \langle f, T_x D_s g\rangle,$$

$(x \in \mathbb{R}^d, s \in \mathbb{R}, s \neq 0)$, when the integral does exist. Here T_x the translation operator and D_s is the dilatation operator, i.e.,

$$T_x f(t) = f(t-x) \quad \text{and} \quad D_s f(t) = |s|^{-d/2} f\left(\frac{t}{s}\right) \quad (x,t \in \mathbb{R}^d,\ 0 \neq s \in \mathbb{R}).$$

We suppose that $g, \gamma \in L_2(\mathbb{R}^d)$ and

$$\int_0^\infty |\widehat{g}(s\omega)||\widehat{\gamma}(s\omega)|\, \frac{ds}{s} < \infty$$

for almost every $\omega \in \mathbb{R}^d$ with $\|\omega\|_2 = 1$. If

$$C_{g,\gamma} := \int_0^\infty \overline{\widehat{g}}(s\omega)\widehat{\gamma}(s\omega) \frac{ds}{s}$$

is independent of ω, then the inversion formula holds for all $f \in L_2(\mathbb{R}^d)$:

$$\int_0^\infty \int_{\mathbb{R}^d} W_g f(x,s) T_x D_s \gamma \frac{dxds}{s^{d+1}} = C_{g,\gamma} \cdot f,$$

where the equality is understood in a vector-valued weak sense. Consider the operators

$$\rho_S f := \int_S^\infty \int_{\mathbb{R}^d} W_g f(x,s) T_x D_s \gamma \frac{dxds}{s^{d+1}}$$

and

$$\rho_{S,T} f := \int_S^T \int_{\mathbb{R}^d} W_g f(x,s) T_x D_s \gamma \frac{dxds}{s^{d+1}},$$

where $0 < S < T < \infty$. Let

$$C'_{g,\gamma} := -\int_{\mathbb{R}^d} (g^* * \gamma)(x) \ln(|x|)\, dx,$$

where $g^*(x) = \overline{g(-x)}$ is the involution operator. In case of g and γ both have radial log-majorants, then $C'_{g,\gamma}$ is finite (see Li and Sun [12]).

If g and γ are radial, $\int_{\mathbb{R}^d} (g^* * \gamma)(x)\, dx = 0$, and both have a radial log-majorant, then for any $f \in L_p(\mathbb{R}^d)$ $(1 \le p < \infty)$

$$\lim_{S \to 0_+, T \to \infty} \rho_{S,T} f(x) = \lim_{S \to 0_+} \rho_S f(x) = C'_{g,\gamma} f(x)$$

at every Lebesgue point of f (see Li and Sun [12]). Moreover, if $1 < p < \infty$, then the convergence holds in the $L_p(\mathbb{R}^d)$-norm for all $f \in L_p(\mathbb{R}^d)$. In case of $p = 1$, then

$$\lim_{S \to 0_+} \rho_S f = C'_{g,\gamma} f \quad \text{in the } L_1(\mathbb{R}^d)\text{-norm}$$

for all $f \in L_1(\mathbb{R}^d)$. Under some similar conditions Weisz [18] proved similar results. We investigated the same questions on variable Lebesgue spaces and we proved similar theorems (see [14]). Under some conditions $C_{g,\gamma} = C'_{g,\gamma}$ (see Li and Sun [12]). Our main result in [14] is, that under some conditions we trace back $\varrho_S f$ to a θ-summation:

Theorem 20. *Suppose that g, γ have radial log-majorants and*

$$\int_{\mathbb{R}^d} (g^* * \gamma)(x)\, dx = 0.$$

If $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $p_+ < \infty$, then for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$

$$\varrho_S f = \sigma^\theta_{1/S} f \quad (S > 0),$$

where θ is a suitable function, which has radial majorant.

Using this result we proved the convergence of $\varrho_S f$ and $\varrho_{S,T} f$ at Lebesgue points, almost everywhere and in the $L_{p(\cdot)}(\mathbb{R}^d)$-norm.

Theorem 21. *Suppose that g, γ have radial log-majorants and*

$$\int_{\mathbb{R}^d} (g^* * \gamma)(x)\, dx = 0.$$

If $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $p_+ < \infty$, then for all Lebesgue points of $f \in L_{p(\cdot)}(\mathbb{R}^d)$,

(1)
$$\lim_{S \to 0_+} \varrho_S f(x) = C'_{g,\gamma} \cdot f(x).$$

(2)
$$\lim_{S \to 0_+, T \to \infty} \varrho_{S,T} f(x) = C'_{g,\gamma} \cdot f(x).$$

Since almost every $x \in \mathbb{R}^d$ is a Lebesgue point of $f \in L_{p(\cdot)}(\mathbb{R}^d)$ ($p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$), we get the almost everywhere convergence of $\varrho_S f$ and $\varrho_{S,T} f$ as a corollary.

Theorem 22. *Suppose that g, γ have radial log-majorants and*

$$\int_{\mathbb{R}^d} (g^* * \gamma)(x)\, dx = 0.$$

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $p_+ < \infty$, $1/p(x) + 1/q(x) = 1$. If the maximal operator is bounded on $L_{q(\cdot)}(\mathbb{R}^d)$, then for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$,

(1)
$$\lim_{S \to 0_+} \varrho_S f = C'_{g,\gamma} \cdot f \quad \text{in the } L_{p(\cdot)}(\mathbb{R}^d)\text{-norm.}$$

(2) *If in addition $p_- > 1$, then for all $f \in L_{p\cdot}(\mathbb{R}^d)$*
$$\lim_{S \to 0_+, T \to \infty} \varrho_{S,T} f = C'_{g,\gamma} \cdot f \quad \text{in the } L_{p(\cdot)}(\mathbb{R}^d)\text{-norm.}$$

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