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POINTWISE CONVERGENCE OF LOGARITHMIC MEANS OF FOURIER SERIES

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Dedicated to Professor Ferenc Schipp on the occasion of his 75th birthday, to Professor William Wade on the occasion of his 70th birthday and to Professor Péter Simon on the occasion of his 65th birthday.

Abstract. In this paper we characterize the set of convergence of the general logarithmic means of trigonometric Fourier series.

Let $\mathbb{T} := [-\pi, \pi]$ denotes an interval in the 1-dimensional Euclidean space $\mathbb{R}$. The notation $a \lesssim b$ in the paper stands for $a \leq cb$, where c is an absolute constant.

We denote by $L_p(\mathbb{T})$ the class of all measurable functions f that are 2π-periodic with respect to all variables and satisfy

$$\|f\|_p := \left(\int_{\mathbb{T}} |f|^p \right)^{\frac{1}{p}} < \infty. \quad 1 \leq p \leq \infty.$$

In case $p = \infty$, by $L_p(\mathbb{T})$ we mean $C(\mathbb{T})$, endowed with the supremum norm.

Let $f \in L_1(\mathbb{T})$. The Fourier series of f with respect to the trigonometric system is the series

$$S[f] := \sum_{n=-\infty}^{+\infty} \hat{f}(n) e^{inx},$$

where

$$\hat{f}(n) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-inx} dx$$

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are the Fourier coefficients of the function f. The partial sums are defined as follows:

$$S_N(f,x) := \sum_{n=-N}^{N} \hat{f}(n) e^{inx}.$$

In the literature, there is known the notion of the Riesz's logarithmic means of a Fourier series. The nth Riesz logarithmic mean of the Fourier series of an integrable function f is defined by

$$\frac{1}{l_n}\sum_{k=0}^{n}\frac{S_k(f)}{k+1}, \quad l_n := \sum_{k=0}^{n}\frac{1}{k+1},$$

where $S_k(f)$ is the partial sum of its Fourier series. This Riesz's logarithmic means with respect to the trigonometric system has been studied by a lot of authors. We mention for instance the papers of Szász and Yabuta [6],[8]. This mean with respect to the Walsh, Vilenkin system is discussed by Simon, and Gát [5],[1].

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of f are defined by

$$\frac{1}{\sum_{k=0}^{n} q_k}\sum_{k=0}^{n} q_k S_{n-k}(f)$$

If $q_k = \frac{1}{k+1}$, then we get the Nörlund logarithmic means:

$$L_n(f;x) := \frac{1}{l_n}\sum_{k=0}^{n}\frac{S_{n-k}(f)}{k+1}.$$

Although, it is a kind of "reverse" Riesz's logarithmic means. In [2, 3] it is proved some convergence and divergence properties of the logarithmic means of Walsh-Fourier series of functions in the class of continuous functions, and in the Lebesgue space L.

In one of his last papers [7] Tkebuchava constructed a set of logarithmic summation methods which contains both of the above mentioned logarithmic summation methods as limit cases. Namely, for any integers n, n_0 such that $0 \leq n_0 \leq n$ let Tkebuchava's means T_{n,n_0} be defined by

$$t_{n,n_0}(f;x) := \frac{1}{l(n,n_0)}\left(\sum_{k=0}^{n_0-1}\frac{S_k(f,x)}{n_0-k+1} + S_{n_0}(f,x) + \sum_{k=n_0+1}^{n}\frac{S_k(f,x)}{k-n_0+1}\right),$$

where

$$l(n,n_0) := \sum_{k=0}^{n_0-1}\frac{1}{n_0-k+1} + 1 + \sum_{k=n_0+1}^{n}\frac{1}{k-n_0+1}.$$

It is clear, that $l(n,n_0) \asymp \log n$. This summation method includes the Riesz (for $n_0 = 0$) and Nörlund (for $n_0 = n$) logarithmic methods, too.

Define the kernels F_{n,n_0} of Tkebuchava's means by

$$F_{n,n_0} := \frac{1}{l(n,n_0)}\left(\sum_{k=0}^{n_0-1}\frac{D_k}{n_0-k+1}+D_{n_0}+\sum_{k=n_0+1}^{n}\frac{D_k}{k-n_0+1}\right).$$

Tkebuchava [7] gave estimates of kernels. Namely, the following theorem holds.

Theorem T. *Let* $0 \le n_0 \le n$. *Then*

$$1+\frac{\log^2(n_0+2)}{\log(n+2)} \lesssim \|F_{n,n_0}\|_{L_1(\mathbb{T})} \lesssim 1+\frac{\log^2(n_0+2)}{\log(n+2)}.$$

Analogical result for Walsh-Fourier series is proved by Gát and Nagy [4]. Moreover, the general logarithmic means of quadratical partial sums also treated for double Walsh-Fourier series and trigonometric Fourier series. The last trigonometric Fourier case was partially solved, only.

Set $t_{n,n_0}f := F_{n,n_0} * f$ the logarithmic means of Fourier series of $f \in L_1(\mathbb{T})$. The following theorem is proved by Tkebuchava [7]

Theorem T2. *The following conditions are equivalent :*

a) $\log n_0(n) = O\left(\sqrt{\log n}\right)$ *;*

b) $\|t_{n,n_0}f - f\|_{C(\mathbb{T})} \to 0$ *as* $n \to \infty$ $\forall f \in C(\mathbb{T})$*;*

c) $\|t_{n,n_0}f - f\|_{L_1(\mathbb{T})} \to 0$ *as* $n \to \infty$ $\forall f \in L_1(\mathbb{T})$.

Definition 1. Let $f \in L_1(\mathbb{T})$, then x is a Lebesgue point of f if

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{\varepsilon}|f(x+t)-f(x)|\,dt = 0.$$

Theorem L (see [9])**.** *Let* $f \in L_1(\mathbb{T})$*, then*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{\varepsilon}|f(x+t)-f(x)|\,dt = 0 \quad \text{a.e. } x \quad \text{in } \mathbb{T}.$$

In this paper we prove that the following is true.

Theorem 1. *Let* $f \in L_1(\mathbb{T})$ *and* $\log n_0(n) = O\left(\sqrt{\log n}\right)$*, then*

$$\lim_{n\to\infty} t_{n,n_0}(f,x) = f(x) \tag{1}$$

at each Lebesgue point x of f. Thus (1) holds a.e. in $\mathbb{T}$.

In order to prove Theorem 1 we need the following lemma.

Lemma 1. *The following estimation holds*

$$|F_{n,n_0}(x)| \lesssim n\mathbb{I}_{[0,\frac{1}{n})}(x)+\frac{1}{x\log n}\mathbb{I}_{[\frac{1}{n},\frac{1}{n_0})}(x) \tag{2}$$
$$+\frac{n_0}{\log n}\mathbb{I}_{[\frac{1}{n},\frac{1}{n_0})}(x)\log\frac{1}{x}+\frac{1}{x\log n}\mathbb{I}_{[\frac{1}{n_0},\pi]}(x)\log\frac{1}{x}.$$

Proof. First, we note that

$$F_{n,n_0}(x)=\frac{1}{l(n,n_0)}\left(\sum_{j=2}^{n_0+1}\frac{D_{n_0+1-j}(x)}{j}+D_{n_0}(x)+\sum_{j=2}^{n-n_0+1}\frac{D_{j+n_0-1}(x)}{j}\right). \tag{3}$$

Further from (3) we get

$$\begin{aligned} F_{n,n_0}(x)=&\frac{1}{l(n,n_0)}\left\{\frac{\sin\left(n_0+1+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\sum_{j=2}^{n_0+1}\frac{\cos jx}{j}\right.\\ &-\frac{\cos\left(n_0+1+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\sum_{j=2}^{n_0+1}\frac{\sin jx}{j}\\ &+D_{n_0}(x)+\frac{\sin\left(n_0-1+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\sum_{j=2}^{n-n_0+1}\frac{\cos jx}{j}\\ &\left.+\frac{\cos\left(n_0-1+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\sum_{j=2}^{n-n_0+1}\frac{\sin jx}{j}\right\}\\ =&\frac{1}{l(n,n_0)}\left\{A_1+A_2+A_3+A_4+A_5\right\}. \end{aligned} \tag{4}$$

It is well-known the following estimations (see [9, Ch.5])

$$\left|\sum_{j=1}^{N}\frac{\sin jx}{j}\right|\lesssim 1,\quad x\in\mathbb{T},N\geq 1, \tag{5}$$

$$\left|\sum_{j=1}^{N}\frac{\cos jx}{j}\right|\lesssim \ln\frac{1}{x},\quad \forall x\in(0,\pi], \tag{6}$$

$$\left|\sum_{j=1}^{N}\frac{\cos jx}{j}\right|\lesssim \ln(m+2),\quad x\in\left[\frac{1}{m+2},\pi\right],\quad \forall N\geq 1. \tag{7}$$

We will use the estimation for case $0\leq x<1/n$

$$|F_{n,n_0}(x)|\leq n. \tag{8}$$

Set $x\geq 1/n$. From (5)-(8) we get

$$\begin{aligned} |A_1|&\lesssim\frac{1}{\log n}\left\{n_0\log n_0\mathbb{I}_{\left[\frac{1}{n},\frac{1}{n_0}\right)}(x)+\mathbb{I}_{\left[\frac{1}{n_0},\pi\right]}(x)\frac{1}{x}\log\frac{1}{x}\right\},\\ |A_2|&\lesssim\frac{1}{x\log n}\mathbb{I}_{\left[\frac{1}{n},\pi\right]}(x),\\ |A_3|&\lesssim\frac{1}{\log n}\left\{n_0\mathbb{I}_{\left[\frac{1}{n},\frac{1}{n_0}\right)}(x)+\frac{1}{x}\mathbb{I}_{\left[\frac{1}{n_0},\pi\right]}(x)\right\}, \end{aligned} \tag{9}$$

$$|A_4| \lesssim \frac{1}{\log n}\left\{n_0 \mathbb{I}_{\left[\frac{1}{n},\frac{1}{n_0}\right)}(x)\log\frac{1}{x} + \mathbb{I}_{\left[\frac{1}{n_0},\pi\right]}(x)\frac{1}{x}\log\frac{1}{x}\right\},$$
$$|A_5| \lesssim \frac{1}{x\log n}\mathbb{I}_{\left[\frac{1}{n},\pi\right]}(x).$$

Combining (4) and (9) we complete the proof of Lemma 1. □

Proof of Theorem 1. First, we note that

$$\begin{aligned}(10)\qquad & |t_{n,n_0}(f,x) - f(x)| \\ & \leq \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\frac{1}{n_0}} + \int_{\frac{1}{n_0}}^{\pi}\right)\left|\frac{f(x+t)+f(x-t)}{2} - f(x)\right| |F_{n,n_0}(t)|\,dt \\ & = I_1 + I_2 + I_3\end{aligned}$$

From (2) we can write

$$(11)\qquad I_1 \lesssim n\int_0^{\frac{1}{n}}\left|\frac{f(x+t)+f(x-t)}{2} - f(x)\right| dt = o(1)$$

at every Lebesgue point x of f by virtue of theorem. Again by Lemma 1 we have

$$\begin{aligned}(12)\qquad I_2 & \lesssim \frac{1}{\log n}\int_{\frac{1}{n}}^{\frac{1}{n_0}}\left|\frac{f(x+t)+f(x-t)}{2} - f(x)\right|\frac{dt}{t} \\ & + \frac{n_0}{\log n}\int_{\frac{1}{n}}^{\frac{1}{n_0}}\left|\frac{f(x+t)+f(x-t)}{2} - f(x)\right|\log\frac{1}{t}dt \\ & := I_{21} + I_{22}.\end{aligned}$$

Set

$$F_x(t) = F(t) = \int_0^t\left|\frac{f(x+s)+f(x-s)}{2} - f(x)\right| ds,$$

Since F is absolutely continuous we can write

$$I_{21} \lesssim \frac{1}{\log n}\int_{\frac{1}{n}}^{\frac{1}{n_0}}\frac{F'(t)}{t}dt,$$

Integrating by parts we see that

$$I_{21} \lesssim \frac{1}{\log n} \frac{F(t)}{t} |_{\frac{1}{n}}^{\frac{1}{n_0}} + \frac{1}{\log n} \int_{\frac{1}{n}}^{\frac{1}{n_0}} \frac{F(t)}{t^2} dt. \tag{13}$$

Since, $\frac{F(t)}{t} = o(1)$ at each Lebesgue point x of f, the integrated term in (13) is $o(1)$ as $n \to \infty$. The same is true for the integral in (13), since given $\varepsilon > 0$ we may first choose n large enough so that $\frac{F(t)}{t} < \varepsilon$ in $\left[\frac{1}{n}, \frac{1}{n_0}\right]$. Therefore the integral does not exceed

$$I_{21} \lesssim \frac{\varepsilon}{\log n} \int_{\frac{1}{n}}^{\frac{1}{n_0}} \frac{dt}{t} \lesssim \varepsilon. \tag{14}$$

Analogously, we can prove that

$$I_{22} = o(1) \text{ as } n \to \infty. \tag{15}$$

Combining (12), (14) and (15) we have

$$I_2 = o(1) \text{ as } n \to \infty. \tag{16}$$

We split I_3 into two integrals,

$$I_3 = \int_{\frac{1}{n_0}}^{\eta} + \int_{\eta}^{\pi} = I_{31} + I_{32}, \tag{17}$$

Now, by (2) we have

$$I_{32} \lesssim \frac{1}{\log n} \frac{1}{\eta} \log \frac{1}{\eta} c_f,$$

where c_f depends, of course, on f. So this term can be made small, but there must be a balance between η and n. Choosing $\eta = \frac{1}{\log n_0}$, for instance, we see at once that I_{32} can be made arbitrarily small for sufficiently large n. Therefore,

$$I_{32} = o(1) \text{ as } n \to \infty. \tag{18}$$

Now, by (2) it readily follows that

$$I_{31} \lesssim \frac{1}{\log n} \int_{\frac{1}{n_0}}^{\frac{1}{\log n_0}} \frac{1}{t} \log \frac{1}{t} \left| \frac{f(x+t) + f(x-t)}{2} - f(x) \right| dt. \tag{19}$$

We may rewrite (19) as

$$I_{31} \lesssim \frac{1}{\log n} \int_{\frac{1}{n_0}}^{\frac{1}{\log n_0}} \frac{1}{t} \log \frac{1}{t} F^{'}(t)\, dt,$$

whence integrating by parts we see that

$$I_{31} \lesssim \frac{1}{\log n} \frac{F(t)}{t} \log \frac{1}{t} \Big|_{\frac{1}{n_0}}^{\frac{1}{\log n_0}} + \frac{1}{\log n} \int_{\frac{1}{n_0}}^{\frac{1}{\log n_0}} \frac{F(t)}{t^2} \log \frac{1}{t} dt. \tag{20}$$

Since $\frac{F(t)}{t} = o(1)$ at each Lebesgue point x of f, the integrated term in (20) is $o(1)$ as $n \to \infty$. The same in true for the integral in (20), since given $\varepsilon > 0$ we may first choose n large enough so that $\frac{F(t)}{t} < \varepsilon$ in $\left[\frac{1}{n_0}, \frac{1}{\log n_0}\right]$. Therefore for I_{31} we have

$$I_{31} \lesssim \frac{\varepsilon}{\log n} \int_{\frac{1}{n_0}}^{\frac{1}{\log n_0}} \frac{1}{t} \log \frac{1}{t} dt \lesssim \frac{\varepsilon \log^2 n_0}{\log n} \lesssim \varepsilon. \tag{21}$$

From (17), (18) and (21) we obtain that

$$I_3 = o(1) \text{ as } n \to \infty. \tag{22}$$

Combining (10), (11), (16) and (22) we conclude Theorem 1 □

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