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SOME RECENT RESULTS ON CONVERGENCE AND DIVERGENCE WITH RESPECT TO WALSH-FOURIER SERIES

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Dedicated to Professor Ferenc Schipp on the occasion of his 75th birthday, to Professor William Wade on the occasion of his 70th birthday and to Professor Péter Simon on the occasion of his 65th birthday.

Abstract. It is of main interest in the theory of Fourier series the reconstruction of a function from the partial sums of its Fourier series. Just to mention two examples: Billard proved [2] the theorem of Carleson for the Walsh-Paley system, that is, for each function in L^2 we have the almost everywhere convergence $S_n f \to f$ and Fine proved [4] the Fejér-Lebesgue theorem, that is for each integrable function in L^1 we have the almost everywhere convergence of Fejér means $\sigma_n f \to f$. In 1992 Móricz, Schipp and Wade proved [18], that for each two-variable function in the space $L \log^+ L$ the Fejér means of the two-dimensional Walsh-Fourier series converge to the function almost everywhere. In this paper we summarize some results with respect to this issue concerning convergence and also divergence.

Introduction

Let the numbers $n \in \mathbb{N}$ and $x \in I := [0,1)$ be expanded with respect to the binary number system:

$$n = \sum_{k=0}^{\infty} n_k 2^k, \ x = \sum_{k=0}^{\infty} x_k 2^{-k-1},$$

where if x is a dyadic rational, that is an element of the set $\{k/2^n : k, n \in \mathbb{N}\}$, then we choose the finite expansion. For $n \in \mathbb{P}$ let $|n| = \lfloor \log_2 n \rfloor$. That is,

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$2^{|n|} \le n < 2^{|n|+1}$. Let $(\omega_n, n \in \mathbb{N})$ represent the Walsh-Paley system. That is, the nth Walsh-Paley function is

$$\omega_n(x) := \prod_{k=0}^{\infty} (-1)^{n_k x_k}.$$

The nth Walsh-Fourier coefficient of the integrable function $f \in L^1(I)$ is

$$\hat{f}(n) := \int_I f(x)\omega_n(x)dx.$$

The nth partial sum of the Walsh-Fourier series of the integrable function $f \in L^1(I)$:

$$S_n f(y) := \sum_{k=0}^{n-1} \hat{f}(k)\omega_k(y).$$

The Cesàro means

The nth Fejér or $(C, 1)$ mean of the function f is

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f.$$

In 1955 Fine proved [4] for the Walsh-Paley system the well known Fejér-Lebesgue theorem. Namely, for every integrable function f we have the a.e. relation

$$\sigma_n f \to f.$$

Let have a look for the situation with the (C, α) means. What are they? Let $A_n^\alpha := \frac{(1+\alpha)...(n+\alpha)}{n!}$, where $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ $(-\alpha \notin \mathbb{N})$. It is known, that $A_n^\alpha \sim n^\alpha$.

The nth (C, α) mean of the function $f \in L^1(I)$:

$$\sigma_{n+1}^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f.$$

For the proof that $\sigma_n^\alpha f \to f$ a.e. for each $f \in L^1(I)$ and $\alpha > 0$ see the papers of Fine [4], Yano [26] and Schipp [20] with different methods. The method of Schipp based on the investigation of maximal operator $\sup_n |\sigma_n|$ brought a new and widely used approach to this issue.

In other words, the maximal convergence space of the (C, α) means is the L^1 Lebesgue space. That is, the largest possible.

It is also of prior interested that what can be said – with respect to this reconstruction issue – if we have only a subsequence of the partial sums. In

1936 Zalcwasser [27] asked how "rare" can the sequence of integers $a(n)$ be such that

$$\frac{1}{N}\sum_{n=1}^{N} S_{a(n)}f \to f \tag{1}$$

for functions belonging to some space. This problem with respect to the trigonometric system is completely solved [19, 3] for continuous functions (uniform convergence). That is, if the sequence a is convex, then the condition $\sup_n n^{-1/2}\log a(n) < +\infty$ is necessary and sufficient for the uniform convergence for every continuous function. For the time being, this issue with respect to the Walsh-Paley system has not been solved. Only, a sufficient condition is known, which is the same as in the trigonometric case. The paper about this was written by Glukhov [12]. See also the more dimensional case also by Glukhov [13].

With respect to convergence almost everywhere, and integrable functions the situation is more complicated. Belinksky proved [1] for the trigonometric system the existence of a sequence $a(n) \sim \exp(\sqrt[3]{k})$ such that the relation (1) holds a.e. for every integrable function. In his paper Belinksky also conjectured that if the sequence a is convex, then the condition $\sup_n n^{-1/2}\log a(n) < +\infty$ is necessary and sufficient again. So, that would be an answer for the problem of Zalcwasser [27] in this point of view (trigonometric system, a.e. convergence and L^1 functions). This is not the case for the Walsh-Paley system. See below Theorem 1 in [8].

If the sequence a is lacunary, then the a.e. relation $S_{a(n)}f \to f$ holds for all functions f in the Hardy space H. The trigonometric and the Walsh-Paley case can be found in [29] (trigonometric case) and [15] (Walsh-Paley case). But, the space H is a proper subspace of L^1. Therefore, it is of interest to investigate relation (1) for L^1 functions and lacunary sequences a. The following two a.e. convergence theorems with respect to the Fejér and logarithmic means of subsequences of the partial sums of the Walsh-Fourier series of integrable functions can be found in [8]:

Theorem 1 ([8])**.** *Let $a\colon \mathbb{N} \to \mathbb{N}$ be a sequence with property $\frac{a(n+1)}{a(n)} \geq q > 1\,(n \in \mathbb{N})$. Then for all integrable function $f \in L^1(I)$ we have the a.e. relation*

$$\frac{1}{N}\sum_{n=1}^{N} S_{a(n)}f \to f.$$

Theorem 2 ([8])**.** *Let $a\colon \mathbb{N} \to \mathbb{N}$ be a convex sequence with property $a(+\infty) = +\infty$. Then for each integrable function f we have the a.e. relation*

$$\frac{1}{\log N}\sum_{n=1}^{N} \frac{S_{a(n)}f}{n} \to f.$$

That is, if one have any subsequence (with convex sequence of indices) of the partial sums of the Walsh-Fourier series of an integrable function, then its Riesz logarithmic means reconstruct the function a.e. It is worth to mention that these two results have no trigonometric analogue.

Two dimensional Cesàro means

What can be said in the two dimensional situation? This is quite a different story. Define the two-dimensional Walsh-Paley functions in the following way:

$$\omega_n(x) := \omega_{n_1}(x^1)\omega_{n_2}(x^2),$$

where $n := (n_1, n_2) \in \mathbb{N}^2$, $x := (x^1, x^2) \in I^2$. Let f be an integrable function. Its Fourier coefficients, rectangular partial sums of its Fourier series:

$$\hat{f}(n) := \int_{I^2} f(x)\omega_n(x)dx, \ S_{n_1,n_2}f := \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} \hat{f}(k_1, k_2)\omega_k.$$

Moreover, the two-dimensional Fejér or $(C, 1)$ means of the function $f \in L^1(I^2)$:

$$\sigma_{n_1,n_2}f := \frac{1}{n_1 n_2}\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} S_{k_1,k_2}f \quad (n \in \mathbb{P}^2).$$

In 1931 Marczinkiewicz and Zygmund proved for the two-dimensional trigonometric system [16], and in 1992 Móricz, Schipp and Wade verified for the two-dimensional Walsh-Paley system, that for every $f \in L\log^+ L(I^2)$

$$\sigma_{n_1,n_2}f \to f$$

a.e. as $\min\{n_1, n_2\} \to \infty$, that is, in the Pringsheim sense.

Since $L\log^+ L(I^2) \subsetneqq L^1(I^2)$, then it would be interesting to "enlarge" the convergence space, if possible. In 2000 Gát proved [6], that it is impossible. That is:

Theorem 3 ([6])**.** *For each measurable function $\delta\colon [0, +\infty) \to [0, +\infty)$, $\delta(\infty) = 0$, (that is vanishing at plus infinity) there exists an $f \in L\log^+ L\delta(L)$ such that for the Walsh-Fejér means $\sigma_{n_1,n_2}f \not\to f$ a.e. (in the Pringsheim sense).*

However, what "positive" can be said for the functions in $L^1(I^2)$ as if the a.e. convergence of the two-dimensional Fejér means in the Pringsheim sense can not be said? That could be the so called restricted convergence. For the two-dimensional trigonometric system Marcinkiewicz and Zygmund proved [17] in 1939, that

$$\sigma_{n_1,n_2}f \to f$$

a.e. for every $f \in L^1(I^2)$ as if $\min\{n_1, n_2\} \to \infty$, provided that

$$2^{-\alpha} \le \frac{n_1}{n_2} \le 2^{\alpha}$$

for some $\alpha \geq 0$. In other words, the set of admissible indices (n_1, n_2) remains in some cone. This theorem for the two-dimensional Walsh-Paley system was verified by Móricz, Schipp and Wade in 1992 in the case when n_1, n_2 both are powers of two.

$$\sigma_{2^{n_1},2^{n_2}} f \to f$$

a.e. for every $f \in L^1(I^2)$ as if $\min\{n_1, n_2\} \to \infty$, provided that $|n_1 - n_2| \leq \alpha$ for some $\alpha \geq 0$.

The proof of the Marcinkiewicz-Zygmund theorem [17] (with respect to the Walsh-Paley system) for arbitrary set of indices remaining in some cone is due to Gát and Weisz [5, 21], separately in 1996.

It is an interesting question that is it possible to weaken somehow the "cone restriction" in a way that a.e. convergence remains for each function in L^1. Maybe for some "interim space" if not for space L^1. The answer is negative both in the point of view of space and in the point of view of restriction. Namely, in 2001 Gát proved [7] the theorem below:

Theorem 4 ([7])**.** *Let $\delta\colon [0, +\infty) \to [0, +\infty)$ measurable, $\delta(+\infty) = 0$ and let $w\colon \mathbb{N} \to [1, +\infty)$ be an arbitrary increasing function such that*

$$\sup_{x\in\mathbb{N}} w(x) = +\infty.$$

Moreover, $\vee n := \max(n_1, n_2)$, $\wedge n := \min(n_1, n_2)$. Then, there exists a function $f \in L\log^+ L\delta(L)$ such that for the two dimensional Walsh-Fejér means

$$\sigma_{n_1,n_2} f \not\to f$$

a.e. as $\wedge n \to \infty$ provided that the restriction condition $\frac{\vee n}{\wedge n} \leq w(\wedge n)$ is fulfilled.

That is there is no "interim" space. Either we have space $L\log^+ L$ and "no restriction at all", or the "cone restriction" and then the maximal convergence space is L^1. As a consequence of this we have that

$$\sigma_{n_1,n_2} f \to f$$

a.e. for each $f \in L(I^2)$ as $\min\{n_1, n_2\} \to \infty$, provided that

$$\frac{\vee n}{\wedge n} \leq w(\wedge n)$$

if and only if

$$\sup w(x) < \infty.$$

Another question. What is the situation with the (C, α) summation methods of 2-dimensional Walsh-Fourier series? What are they?

$$\sigma^\alpha_{n_1+1,n_2+1} f := \frac{1}{A^\alpha_{n_1} A^\alpha_{n_2}} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} A^{\alpha-1}_{n_1-k_1} A^{\alpha-1}_{n_2-k_2} S_{k_1,k_2} f.$$

In 1999 Weisz proved [22], that

$$\sigma^\alpha_{n_1,n_2} f \to f$$

a.e. as $\min\{n_1, n_2\} \to \infty$ for each $f \in L\log^+ L(I^2)$ and $\alpha > 0$.

The question is that is it possible to give a "larger" convergence space for the (C,α) summability method ($\alpha > 0$)? Is there such an α? If $\alpha \le 1$, then not. Because for the $(C,1)$ method one can not give such a "larger" space. Karagulyan and the author of this paper answered this question (among others) in the negative [11].

In the sequel consider also the Riesz means of the Walsh-Fourier means of f. The Riesz means of the integrable function f are defined as follows:

$$\sigma_n^{\alpha,\gamma} f := \frac{1}{\prod_{i=1}^2 n_i^{\alpha_i \gamma_i}} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \left(\prod_{i=1}^2 (n_i^{\gamma_i} - k_i^{\gamma_i})^{\alpha_i} \right) \hat{f}(k)\omega_k,$$

where $n = (n_1, n_2), \gamma = (\gamma_1, \gamma_2)$ and $0 < \alpha_j \le 1 \le \gamma_j$ $(j = 1, 2)$. In special case, $\alpha_j = \gamma_j = 1$ $(j = 1, 2)$, the Riesz means and the Fejér means coincide. The proof of the almost everywhere convergence of the restricted Riesz means of Walsh-Fourier series (moreover, in the more general d-dimensional case) is due to Weisz (see [25] or [24, page 132]). In [10] there are some generalized Cesàro and Riesz means in a way that the set of indices of the means need not necessarily be in a positive cone. (In [10] the general d dimensional situation is discussed.)

Theorem 5 ([10])**.** *Let $a = (a_1, a_2)\colon \mathbb{N} \to \mathbb{N}^2$ be a sequence with property $a_j(+\infty) = +\infty$ $(j = 1, 2)$. Suppose that there exists a $\delta > 0$ such that $a_j(n + 1) \ge \delta \sup_{k \le n} a_j(k)$ $(j = 1, 2,\ n \in \mathbb{N})$. Then for all $0 < \alpha_j \le 1 \le \gamma_j$ $(j = 1, 2)$ we have for the 2-dimensional Cesàro and Riesz means the following a.e. relation for each integrable function $f \in L^1(I^2)$*

$$\lim_{n\to\infty} \sigma_{a(n)}^{\alpha} f = f, \quad \lim_{n\to\infty} \sigma_{a(n)}^{\alpha,\gamma} f = f.$$

From Theorem 5 it follows the a.e. convergence of the restricted Cesàro ([5], [21]) and Riesz ([25]) means of integrable functions. For the proof of this consequence see also [10].

The Marcinkiewicz means

This is another and also very interesting story the investigation of the almost everywhere convergence of the Marcinkiewicz means

$$t_n f := \frac{1}{n} \sum_{j=0}^{n-1} S_{j,j} f$$

of integrable functions with respect to orthonormal systems. Although, this means are defined for two-variable functions, in the view of almost everywhere convergence there are similarities with the one-dimensional case. On the one side, the maximal convergence space for two dimensional Fejér means (no restriction on the set of indices other than they have to converge to $+\infty$) is $L\log^+ L$ and on the other side, for the Marcinkiewicz means we have a.e.

convergence for every integrable functions (for the trigonometric, Walsh Paley systems).

We mention that the first result is due to Marcinkiewicz [16]. But he proved "only" for functions in the space $L\log^+ L$ the a.e. relation $t_n f \to f$ with respect to the trigonometric system. For the "L^1 result" for the trigonometric and Walsh-Paley systems see the papers of Zhizhiasvili [28] (trigonometric system) and Weisz [23] (Walsh system).

After then, we turn our attention to a generalization of Walsh-Marcinkiewicz means. Let $\alpha := (\alpha_1, \alpha_2)\colon \mathbb{N}^2 \to \mathbb{N}^2$ be a function. Define the following Marcinkiewicz-like kernels and means [9]:

$$M_n^\alpha(x) := \frac{1}{n}\sum_{k=0}^{n-1} D_{\alpha_1(|n|,k)}(x^1) D_{\alpha_2(|n|,k)}(x^2),$$

$$t_n^\alpha f(y) := \int_{I^2} f(x) M_n^\alpha(y+x)dx \quad (f \in L^1(I^2), y \in I^2, n \in \mathbb{P}).$$

The following properties play a prominent role in the behaviour of Marcinkiewicz-like means. ($\#B$ denotes the cardinality of set B.) Roughly speaking they will be necessary and sufficient conditions in order to have almost everywhere condition for each integrable function.

$$\#\{l \in \mathbb{N} : \alpha_j(|n|,l) = \alpha_j(|n|,k), l < n\} \le C \quad (k < n, n \in \mathbb{P}, j = 1,2) \tag{2}$$

$$\max\{\alpha_j(|n|,k) : k < n\} \le Cn \quad (n \in \mathbb{P}, j = 1,2). \tag{3}$$

The "theorem of convergence" can be found in [9].

Theorem 6 ([9])**.** *Let α satisfy (2) and (3). Then we have $t_n^\alpha f \to f$ for each $f \in L^1(I^2)$.*

Condition (2) is clearly a necessary one in the following sense. Let $\alpha_1(|n|,k) = 0$, $\alpha_2(|n|,k) = k$ for every $n, k \in \mathbb{N}$. Then (3) is satisfied and (2) is not. It is very simple to give a function $f \in L^1(I^2)$ such as $t_n^\alpha f \to f$ fails to hold a.e. To construct an α with (2) which fails to satisfy (3) and a $f \in L^1(I^2)$ such that $t_n^\alpha f$ does not converge to f a.e. is more complicated.

The "theorem of divergence" aims to show that (3) is also a necessary condition in certain sense. That is,

Theorem 7 ([9])**.** *Let $\gamma\colon \mathbb{N} \to \mathbb{N}$ be any function with property $\gamma(+\infty) = +\infty$. Then there exists a function α satisfying (2),*

$$\max\{\alpha_1(|n|,k) : k < n\} \le Cn, \ \max\{\alpha_2(|n|,k) : k < n\} \le Cn\gamma(n) \quad (n \in \mathbb{P})$$

and $f \in L^1(I^2)$ such that $\limsup_{n\in\mathbb{N}} |t_n^\alpha f| = +\infty$ *almost everywhere.*

A corollary of Theorem 6 also can be found in [9]

Corollary 1 ([9])**.** *Let (a_n) be a lacunary sequence of natural numbers, i.e. $a_{n+1} \ge a_n q$ for some $q > 1$ ($n \in \mathbb{N}$) and α satisfy conditions (2) and $\alpha_j(n,k) \le$*

Ca_n $(k < a_n,\ j = 1, 2)$ (modified version of condition (3)). Then for every integrable function $f \in L^1(I^2)$ we have

$$\frac{1}{a_n}\sum_{k=0}^{a_n-1} S_{\alpha_1(n,k),\alpha_2(n,k)} f(x) \to f(x)$$

for a.e. $x \in I^2$.

A straightforward consequence of Corollary 1 is the a.e. convergence of lacunary subsequences of the triangular means of two dimensional Walsh-Fourier series. In other words, let $\alpha_1(n,k) = a_n - k, \alpha_2(n,k) = k$. Then we have

$$\frac{1}{a_n}\sum_{k=0}^{a_n-1} S_{a_n-k,k} f(x) \to f(x)$$

for a.e. $x \in I^2$. This last result in the case of $a_n = 2^n$ is due to Goginava and Weisz [14]. The general case $a_n = n$ is still open.

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