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NON-DEGENERATE HYPERSURFACES OF A SEMI-RIEMANNIAN MANIFOLD WITH A QUARTER-SYMMETRIC NON-METRIC CONNECTION

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Abstract. The object of the present paper is to study a non-degenerate hypersurface of a semi-Riemannian manifold with a quarter-symmetric non-metric connection.

1. Introduction

Let $\tilde{M}$ be an $(n+1)$-dimensional differentiable manifold of class C^∞ and M an n-dimensional differentiable manifold immersed in $\tilde{M}$ by a differentiable immersion

$$i\colon M \to \tilde{M}.$$

$i(M)$ identical to M, is said to be a hypersurface of $\tilde{M}$. The differential di of the immersion i will be denoted by B so that a vector field X in M corresponds to a vector field BX in $\tilde{M}$. We suppose that the manifold $\tilde{M}$ is a semi-Riemannian manifold with the semi-Riemannian metric $\tilde{g}$ of index ν, $0 \leq \nu \leq n+1$. Thus the index of $\tilde{M}$ is the ν, which will be denoted by $ind\tilde{M} = \nu$. If the induced metric tensor $g = \tilde{g}|M$ defined by

$$g(X,Y) = \tilde{g}(BX,BY), \qquad \text{for all } X, Y \text{ in } \chi(M)$$

is non-degenerate, then the hypersurface M is called a non-degenerate hypersurface. Also M is a semi-Riemannian manifold with the induced semi-Riemannian metric g [6]. If the semi-Riemannian manifolds $\tilde{M}$ and M are both orientable and we can choose a unit vector field N defined along M such that

$$\tilde{g}(BX,N) = 0, \quad \tilde{g}(N,N) = \epsilon = \begin{cases} +1, & \text{for spacelike } N, \\ -1, & \text{for spacelike } N, \end{cases}$$

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for all X in $\chi(M)$, where N is called the unit normal vector field to M and $\operatorname{ind} M = \operatorname{ind} \tilde{M}$ if $\epsilon = 1$, $\operatorname{ind} M = \operatorname{ind} \tilde{M} - 1$ if $\epsilon = -1$.

The hypersurface of a manifold have been studied by several authors such as Bucki [1], De and Kamilya [2], De and Mondal [3], O'Neill [6], Yücesan and Yasar [9], Yücesan and Ayyildiz [8], Yano and Kon [7] and many others.

In 1975, Golab [5] defined and studied quarter-symmetric connection in differentiable manifolds with affine connections. A linear connection $\tilde{\nabla}$ on an $(n+1)$-dimensional Riemannian manifold $\tilde{M}$ is called a quarter-symmetric connection [5] if its torsion tensor $\tilde{T}$ satisfies

$$\tilde{T}(\tilde{X}, \tilde{Y}) = \tilde{\eta}(\tilde{Y})\tilde{\phi}\tilde{X} - \tilde{\eta}(\tilde{X})\tilde{\phi}\tilde{Y}, \tag{1.1}$$

where $\tilde{\eta}$ is a 1-form and $\tilde{\xi}$ is a vector field defined by

$$\tilde{\eta}(\tilde{X}) = \tilde{g}(\tilde{X}, \tilde{\xi}), \tag{1.2}$$

for all vector fields $\tilde{X}, \tilde{Y} \in \chi(\tilde{M})$, $\chi(\tilde{M})$ is the set of all differentiable vector fields on $\tilde{M}$.

In particular, if $\tilde{\phi}\tilde{X} = \tilde{X}$, then the quarter-symmetric connection reduces to the semi-symmetric connection [4]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection.

If moreover, a quarter-symmetric connection $\tilde{\nabla}$ satisfies the condition

$$\tilde{\nabla}\tilde{g} \neq 0, \tag{1.3}$$

then $\tilde{\nabla}$ is said to be a quarter-symmetric non-metric connection.

2. Quarter-symmetric non-metric connection

Let $\tilde{M}$ denotes an $(n+1)$-dimensional semi-Riemannian manifold with semi-Riemannian metric $\tilde{g}$ of index ν, $0 \leq \nu \leq n+1$. A linear connection $\tilde{\nabla}$ on $\tilde{M}$ is called a quarter-symmetric non-metric connection if

$$(\tilde{\nabla}_{\tilde{X}}\tilde{g})(\tilde{Y}, \tilde{Z}) = -\tilde{\eta}(\tilde{Y})\tilde{g}(\tilde{\phi}\tilde{X}, \tilde{Z}) - \tilde{\eta}(\tilde{Z})\tilde{g}(\tilde{\phi}\tilde{X}, \tilde{Y}). \tag{2.1}$$

Throughout the paper, we will denote by $\tilde{M}$ the semi-Riemannian manifold admitting a quarter-symmetric non-metric connection given by

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \tilde{\nabla}^{*}_{\tilde{X}}\tilde{Y} + \tilde{\eta}(\tilde{Y})\tilde{\phi}\tilde{X}, \tag{2.2}$$

any vector fields $\tilde{X}$ and $\tilde{Y}$ of $\tilde{M}$. When M is a non-degenerate hypersurface, we have the following orthogonal direct sum :

$$\chi(\tilde{M}) = \chi(M) \oplus \chi(M)^{\perp}. \tag{2.3}$$

According to (2.3), every vector field $\tilde{X}$ on $\tilde{M}$ is decomposed as

$$\tilde{\xi} = B\xi + \mu N, \tag{2.4}$$

where μ is scalar function, a contravariant vector field ξ and a semi-Riemannian metric g on hypersurface M^n respectively.

We denote ∇^* the connection on the non-degenerate hypersurface M induced from the Levi-Civita connection $\tilde{\nabla}^*$ on $\tilde{M}$ with respect to the unit spacelike or timelike normal vector field N. We have the equality

$$\tilde{\nabla}^*_{BX} BY = B(\nabla^*_X Y) + h^*(X, Y)N, \tag{2.5}$$

for arbitrary vector fields X and Y of M, where h^* is the second fundamental form of the non-degenerate hypersurface M. Let us define the connection ∇ on M which is induced by the quarter-symmetric non-metric connection $\tilde{\nabla}$ on $\tilde{M}$ with respect to the unit spacelike or timelike normal vector field N. We obtain the equation

$$\tilde{\nabla}_{BX} BY = B(\nabla_X Y) + h(X, Y)N, \tag{2.6}$$

where h is the second fundamental form of the non-degenerate hypersurface M. We call (2.6) the equation of Gauss with respect to the induced connection ∇.

According to (2.2), we have

$$\tilde{\nabla}_{BX} BY = \tilde{\nabla}^*_{BX} BY + \tilde{\eta}(BY)\tilde{\phi}(BX). \tag{2.7}$$

Using (2.5) and (2.6) in (2.7), we get

$$B(\nabla_X Y) + h(X, Y)N = B(\nabla^*_X Y) + h^*(X, Y)N + \tilde{\eta}(BY)\tilde{\phi}(BX). \tag{2.8}$$

which implies

$$\nabla_X Y = \nabla^*_X Y + \eta(Y)\phi X \tag{2.9}$$

and

$$h(X, Y) = h^*(X, Y),$$

where $\tilde{\eta}(BY) = B\eta(Y)$ and $\tilde{\phi}(BX) = B\phi X$.

From (2.9), we conclude that

$$(\nabla_X g)(Y, Z) = -\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y), \tag{2.10}$$

and

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{2.11}$$

for any X, Y, Z in $\chi(M)$.

From (2.10) and (2.11), we can state the following theorem:

Theorem 2.1. *The connection induced on a non-degenerate hypersurface of a semi-Riemannian manifold with a quarter-symmetric non-metric connection with respect to the unit spacelike or timelike normal vector field is also a quarter-symmetric non-metric connection.*

The equation of Weingarten with respect to the Levi-Civita connection $\tilde{\nabla}^*$ is

$$\tilde{\nabla}^*_{BX} N = -B(A^*_N X), \tag{2.12}$$

for any vector field X in M and where A^*_N is a tensor field of type $(1,1)$ of M which is defined by

$$h^*(X,Y) = \epsilon g(A^*_N X, Y), \tag{2.13}$$

[6].

Using (2.7), we have

$$\tilde{\nabla}_{BX} N = \tilde{\nabla}^*_{BX} N + \epsilon\mu\tilde{\phi}(BX), \tag{2.14}$$

because from (2.4), we obtain

$$\tilde{\eta}(N) = \tilde{g}(\tilde{\xi}, N) = \tilde{g}(B\xi + \mu N, N) = \mu\tilde{g}(N,N) = \epsilon\mu.$$

Combining (2.12) and (2.14), we get

$$\tilde{\nabla}_{BX} N = -B((A^*_N - \epsilon\mu\phi)X), \quad \epsilon \mp 1. \tag{2.15}$$

Applying the tensor field A_N of type $(1,1)$ of M defined

$$A_N = A^*_N - \epsilon\mu\phi, \tag{2.16}$$

From (2.15), we have

$$\tilde{\nabla}_{BX} N = -B(A_N X), \tag{2.17}$$

Combining (2.13) and (2.16), we have

$$h(X,Y) = \epsilon g(A_N X, Y) + \mu g(\phi X, Y). \tag{2.18}$$

From (2.16), we can state the following corollary:

Corollary 2.1. *Let M be a non-degenerate hypersurface of a semi-Riemannian manifold $\tilde{M}$. Then*

(i) if M has a spacelike normal vector field, the shape operator A_N with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$ is

$$A_N = A^*_N - \mu\phi,$$

and

(ii) if M has a timelike normal vector field, the shape operator A_N with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$ is

$$A_N = A^*_N + \mu\phi.$$

We suppose that $e_1, e_2, \ldots, e_\nu, e_{\nu+1}, \ldots, e_n$ are principal vector fields corresponding to unit spacelike or timelike normal vector field N with respect to $\tilde{\nabla}^*$.

From (2.16), we have

$$A_N(e_i) = A^*_N(e_i) - \epsilon\mu\phi(e_i) = k^*_i e_i - \epsilon\mu p_i e_i = (k^*_i - \epsilon\mu p_i)e_i, \quad 1 \leq i \leq n,$$

where k^*_i, $1 \leq i \leq n$, are the principal curvatures corresponding to the unit spacelike or timelike normal vector field N with respect to the Levi-Civita connection $\tilde{\nabla}^*$. If we take

$$k_i = k^*_i - \epsilon\mu p_i \text{ and } \phi(e_i) = p_i e_i, \quad 1 \leq i \leq n, \tag{2.19}$$

then we obtain

$$A_N^*(e_i) = k_i^* e_i \text{ and } A_N(e_i) = k_i e_i, \quad 1 \le i \le n, \tag{2.20}$$

where k_i, $1 \le i \le n$, are the principal curvatures corresponding to the unit spacelike or timelike normal vector field N with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$.

Therefore, we can give the following corollary:

Corollary 2.2. *Let M be a non-degenerate hypersurface of a semi-Riemannian manifold $\tilde{M}$. Then*

(i) if M has a spacelike normal vector field, the principal curvatures corresponding to the unit spacelike normal vector field N with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$ are $k_i = k_i^* - \mu p_i$, $1 \le i \le n$,

and

(ii) if M has a timelike normal vector field, the principal curvatures corresponding to the unit timelike normal vector field N with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$ are $k_i = k_i^* + \mu p_i$, $1 \le i \le n$.

3. Equations of Gauss curvature and Codazzi-Mainardi

We denote the curvature tensor of $\tilde{M}$ with respect to the Levi-Civita connection $\tilde{\nabla}^*$ by

$$\tilde{R}^*(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}^*_{\tilde{X}} \tilde{\nabla}^*_{\tilde{Y}} \tilde{Z} - \tilde{\nabla}^*_{\tilde{Y}} \tilde{\nabla}^*_{\tilde{X}} \tilde{Z} - \tilde{\nabla}^*_{[\tilde{X},\tilde{Y}]} \tilde{Z}$$

and that of M with respect to the Levi-Civita connection ∇^* by

$$R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z.$$

Then the equation of Gauss curvature is given by

$$\begin{aligned} R^*(X,Y,Z,U) = {} & \tilde{R}^*(BX,BY,BZ,BU) \\ & + \epsilon\{h^*(X,U)h^*(Y,Z) - h^*(Y,U)h^*(X,Z)\}, \end{aligned}$$

where

$$\begin{aligned} \tilde{R}^*(BX,BY,BZ,BU) &= \tilde{g}(\tilde{R}^*(BX,BY)BZ,BU), \\ R^*(X,Y,Z,U) &= g(R^*(X,Y)Z,U) \end{aligned}$$

and the equation of Codazzi-Mainardi [6] is given by

$$\tilde{R}^*(BX,BY,BZ,N) = \epsilon\{(\nabla^*_X h^*)(Y,Z) - (\nabla^*_Y h^*)(X,Z)\}.$$

We find the equation of Gauss curvature and Codazzi-Mainardi with respect to the quarter-symmetric non-metric connection. The curvature tensor of the quarter-symmetric non-metric connection $\tilde{\nabla}$ of $\tilde{M}$ is

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}} \tilde{\nabla}_{\tilde{Y}} \tilde{Z} - \tilde{\nabla}_{\tilde{Y}} \tilde{\nabla}_{\tilde{X}} \tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]} \tilde{Z}. \tag{3.1}$$

Putting $\tilde{X} = BX$, $\tilde{Y} = BY$, $\tilde{Z} = BZ$ in (3.1) and using (2.6) and (2.17), we get

$$\tilde{R}(BX,BY)BZ = B[R(X,Y)Z + h(X,Z)A_N Y - h(Y,Z)A_N X]$$

$$+[(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) + h\{\eta(Y)\phi X - \eta(X)\phi Y, Z\}]N, \tag{3.2}$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

is the curvature tensor of the quarter-symmetric non-metric connection ∇.

Combining (2.18) and (3.2), we obtain

$$\begin{aligned}\tilde{R}(BX,BY,BZ,BU) = R(X,Y,Z,U) + \epsilon[h(X,Z)h(Y,U) - h(Y,Z)h(X,U)\\ +\mu h(Y,Z)g(\phi X,U) - \mu h(X,Z)g(\phi Y,U)]\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\tilde{R}(BX,BY,BZ,N) = \epsilon[(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z)\\ +h\{\eta(Y)\phi X - \eta(X)\phi Y, Z\}],\end{aligned} \tag{3.4}$$

where $\tilde{R}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{U}) = \tilde{g}(\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{U})$ and $R(X,Y,Z,U) = g(R(X,Y)Z,U)$.

The above equations (3.3) and (3.4) are Gauss curvature and Codazzi-Mainardi with respect to the quarter-symmetric non-metric connection respectively.

4. The Ricci and scalar curvatures

Suppose that $\{Be_1,\dots,Be_\nu,Be_{\nu+1},\dots,Be_n,N\}$ is an orthonormal basis of $\chi(\tilde{M})$. Then the Ricci tensor of $\tilde{M}$ with respect to the quarter-symmetric non-metric connection is

$$\begin{aligned}\tilde{\mathrm{Ric}}(BY,BZ) = \sum_{i=1}^{n}\epsilon_i\tilde{g}(\tilde{R}(Be_i,BY)BZ,Be_i)\\ +\epsilon\tilde{g}(\tilde{R}(N,BY)BZ,N),\end{aligned} \tag{4.1}$$

for all X,Y in $\chi(M)$.

Using (2.18) in (3.3), we get

$$\begin{aligned}\tilde{R}(BX,BY,BZ,BU) = R(X,Y,Z,U) + \epsilon g(A_N X,Z)g(A_N Y,U)\\ +\mu g(\phi X,Z)g(A_N Y,U)\\ -\epsilon g(A_N Y,Z)g(A_N X,U) - \mu g(A_N X,U)g(\phi Y,U).\end{aligned} \tag{4.2}$$

Putting $X = e_i$ and $U = e_i$ in (4.2) and using (2.19) and (2.20), we have

$$\begin{aligned}\sum_{i=1}^{n}\epsilon_i\tilde{R}(Be_i,BY,BZ,Be_i) = \sum_{i=1}^{n}\epsilon_i\tilde{g}(\tilde{R}(Be_i,BX)BY,Be_i) = \mathrm{Ric}(Y,Z)\\ +\sum_{i=1}^{n}[\epsilon k_i + \mu p_i - \epsilon f]g(A_N Y,Z) - \mu f g(\phi Y,Z),\end{aligned} \tag{4.3}$$

where

$$f = \sum_{i=1}^{n}\epsilon_i g(A_N e_i,e_i).$$

Using (4.3) in (4.1), we obtain

$$\begin{aligned}\tilde{\mathrm{Ric}}(BY,BZ) &= \mathrm{Ric}(Y,Z) + \sum_{i=1}^{n}[\epsilon k_i + \mu p_i - \epsilon f]g(A_N Y,Z) - \mu f g(\phi Y,Z)\\ &\quad + \epsilon\tilde{g}(\tilde{R}(N,BY)BZ,N),\end{aligned} \tag{4.4}$$

where $\tilde{\mathrm{Ric}}$ and Ric are the Ricci tensor with respect to $\tilde{\nabla}$ and ∇ respectively.

The scalar curvature of $\tilde{M}$ with respect to the quarter-symmetric non-metric connection is

$$\tilde{r} = \sum_{i=1}^{n}\epsilon_i\tilde{\mathrm{Ric}}(e_i,e_i) + \epsilon\tilde{\mathrm{Ric}}(N,N). \tag{4.5}$$

Putting $Y = e_i$ and $Z = e_i$ in (4.4), we get

$$\tilde{r} = r + \sum_{i=1}^{n}[\epsilon k_i + \mu p_i - \epsilon f]f - \mu f\alpha + 2\epsilon\tilde{\mathrm{Ric}}(N,N), \tag{4.6}$$

where $\alpha = g(\phi e_i, e_i)$ and $\tilde{r}$ and r are the scalar curvature with respect to $\tilde{\nabla}$ and ∇ respectively.

From the above discussion, we can state the following theorem:

Theorem 4.1. *Let M be non-degenerate hypersurface of a semi-Riemannian manifold $\tilde{M}$ with respect to the quarter-symmetric non-metric connection, then*
(i) the Ricci tensor is given by (4.4)
and
(ii) the scalar curvature is given by (4.6).

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