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# ON THE GROWTH OF SOLUTIONS OF SOME NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate the growth of solutions to the non-homogeneous linear differential equation

$$f^{(k)} + A_{k-1}e^{b_{k-1}z}f^{(k-1)} + \cdots + A_1 e^{b_1 z}f' + A_0 e^{b_0 z}f = Fe^{az},$$

where $A_j(z) \not\equiv 0$ $(j = 0, 1, \ldots, k-1)$, $F(z) \not\equiv 0$ are entire functions and $a \neq 0$, $b_j \neq 0$ $(j = 0, 1, \ldots, k-1)$ are complex numbers.

## 1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory [7, 9, 14]. In addition, we will use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and distinct zeros of a meromorphic function $f$, $\rho(f)$ to denote the order of growth of $f$. In order to estimate the growth of infinite order solutions, we recall the definition of the hyper-order.

**Definition 1.1** ([14]). Let $f$ be a meromorphic function. Then the hyper-order $\rho_2(f)$ of $f(z)$ is defined by

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of $f$. If $f$ is an entire function, then the hyper-order $\rho_2(f)$ of $f(z)$ is defined by

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log\log M(r,f)}{\log r},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

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**Definition 1.2.** [14] Let $f$ be a meromorphic function. Then the hyper-exponent of convergence of the sequence of zeros of $f(z)$ is defined by

$$\lambda_2(f) = \limsup_{r\to+\infty} \frac{\log\log N\left(r, \frac{1}{f}\right)}{\log r},$$

where $N\left(r, \frac{1}{f}\right)$ is the integrated counting function of zeros of $f(z)$ in $\{z : |z| \leq r\}$. Similarly, the hyper-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r},$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the integrated counting function of distinct zeros of $f(z)$ in $\{z : |z| \leq r\}$.

The complex oscillatory problems of the non-homogeneous linear differential equations are a very important aspect of the complex oscillation theory of differential equations, which has a large number of potential applications. For growth estimates of solutions of a non-homogeneous linear differential equation, in general there exist exceptional solutions that are not easy to discuss, see [9, Chapter 8].

In [11] Wang and Laine investigated the growth of solutions of some second order nonhomogeneous linear differential equation and obtained.

**Theorem A** ([11]). *Let $A_j(z) \not\equiv 0$ $(j = 0, 1)$ and $F(z)$ be entire functions with $\max\{\rho(A_j)\ (j = 0, 1), \rho(F)\} < 1$, and let $a$, $b$ be complex constants that satisfy $ab \neq 0$ and $a \neq b$. Then every nontrivial solution $f$ of the differential equation*

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = F,$$

*is of infinite order.*

*Remark* 1.1. The result of Theorem A was also obtained by the author and El Farissi in [2] by another proof.

Later in [12] Wang and Laine extend the result of Theorem A to higher order non-homogeneous linear differential as follows.

**Theorem B** ([12]). *Suppose that $A_j(z) = h_j(z)e^{P_j(z)}$ $(j = 0, 1, \dots, k-1)$ where $P_j(z) = a_{jn}z^n + \cdots + a_{j0}$ are polynomials with degree $n \geq 1$, $h_j(z) \not\equiv 0$ $(j = 0, 1, \dots, k-1)$ are entire functions of order less than $n$, and that $H(z) \not\equiv 0$ is an entire function of order less than $n$. If $a_{jn}$ $(j = 0, 1, \dots, k-1)$ are distinct complex numbers, then every solution $f$ of the differential equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = H(z)$$

*is of infinite order.*

Recently, the authors investigated the growth of some nonhomogeneous higher order linear differential equations and obtained the following result.

**Theorem C** ([3]). *Let $k \geq 2$ be an integer, $I_j \subset \mathbb{N}$ $(j = 0, 1, \ldots, k-1)$ be finite sets such that $I_j \cap I_m = \emptyset$ $(j \neq m)$ and $I = \bigcup_{j=0}^{k-1} I_j$. Suppose that $B_j = \sum_{i \in I_j} A_i e^{P_i(z)}$ $(j = 0, 1, \ldots, k-1)$, where $A_i(z) \not\equiv 0$ $(i \in I)$ are entire functions with $\max\{\rho(A_i), i \in I\} < n$, $P_i(z) = a_{in} z^n + \cdots + a_{i1} z + a_{i0}$ $(i \in I)$ are polynomials with degree $n \geq 1$ and that $F(z) \not\equiv 0$ is an entire function with $\rho(F) < n$. If $a_{in}$ $(i \in I)$ are distinct complex numbers, then every solution $f$ of the differential equation*

$$f^{(k)} + B_{k-1} f^{(k-1)} + \cdots + B_l f^{(l)} + \cdots + B_1 f' + B_0 f = F$$

*satisfies $\rho(f) = +\infty$.*

The main purpose of this paper is to investigate the order and the hyper-order of growth to some higher order linear differential equations. In fact we will prove the following results.

Suppose that

$$\begin{aligned} I &= \{0, 1, 2, \ldots, k-1\}, \\ I_1 &= \{i \in I : c_i > 1\} \neq \emptyset, \\ I_2 &= \{i \in I : 0 < c_i < 1\} \neq \emptyset, \\ I_3 &= \{i \in I : c_i < 0\} \neq \emptyset, \\ I_4 &= \{i \in I : c_i = 1\} \neq \emptyset, \end{aligned}$$

where $I_1 \cup I_2 \cup I_3 \cup I_4 = I$ and $c_i$ $(i \in I)$ are real numbers.

**Theorem 1.1.** *Let $A_j(z) \not\equiv 0$ $(j \in I)$, $F(z) \not\equiv 0$ be entire functions with $\max\{\rho(A_j)(j \in I), \rho(F)\} < 1$, $a \neq 0$ and $b_i \neq 0$ $(i \in I)$ be complex numbers such that $b_i = c_i a$ $(i \in I)$. Suppose that there is one $s \in I_1$ such that $c_s > c_j$ for all $j \in I_1 \setminus \{s\}$, suppose that there is one $l \in I_3$ such that $c_l < c_j$ for all $j \in I_3 \setminus \{l\}$, and suppose that $c_0 \neq 1$ and $c_0 \neq c_j$ for all $j \in I \setminus \{0\}$. Then every solution $f$ of the differential equation*

$$f^{(k)} + A_{k-1} e^{b_{k-1} z} f^{(k-1)} + \cdots + A_1 e^{b_1 z} f' + A_0 e^{b_0 z} f = F e^{az} \tag{1.1}$$

*has infinite order and the hyper-order satisfies $\rho_2(f) \leq 1$.*

**Theorem 1.2.** *Under the hypotheses of Theorem 1.1, suppose further that $\varphi(z) \not\equiv 0$ is an entire function with finite order. Then every solution $f$ of (1.1) satisfies*

$$\lambda(f) = \overline{\lambda}(f) = \lambda(f - \varphi) = \overline{\lambda}(f - \varphi) = \rho(f) = \infty$$

*and*

$$\lambda_2(f) = \overline{\lambda}_2(f) = \lambda_2(f - \varphi) = \overline{\lambda}_2(f - \varphi) = \rho_2(f) \leq 1.$$

## 2. Auxiliary lemmas

**Lemma 2.1** ([13]). *Suppose that $f_1(z), f_2(z), \ldots, f_n(z)$ $(n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), \ldots, g_n(z)$ are entire functions satisfying the following conditions:*

*(i)* $\sum_{j=1}^{n} f_j(z)\, e^{g_j(z)} \equiv f_{n+1}$.

*(ii) If $1 \leq j \leq n+1$ and $1 \leq k \leq n$, then the order of $f_j$ is less than the order of $e^{g_k(z)}$. If $n \geq 2$, $1 \leq j \leq n+1$ and $1 \leq h < k \leq n$, then the order of $f_j$ is less than the order of $e^{g_h - g_k}$.*

*Then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n+1)$.*

**Lemma 2.2** ([4]). *Suppose that $P(z) = (\alpha + i\beta)\, z^n + \cdots$ $(\alpha, \beta$ are real numbers, $|\alpha| + |\beta| \neq 0)$ is a polynomial with degree $n \geq 1$, that $A(z) \not\equiv 0$ is an entire function with $\rho(A) < n$. Set $g(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then for any given $\varepsilon > 0$, there is a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus (E_1 \cup E_2)$, there is $R > 0$, such that for $|z| = r > R$, we have*

*(i) If $\delta(P, \theta) > 0$, then*

$$\exp\{(1-\varepsilon)\, \delta(P, \theta)\, r^n\} \leq \left|g\left(re^{i\theta}\right)\right| \leq \exp\{(1+\varepsilon)\, \delta(P, \theta)\, r^n\}.$$

*(ii) If $\delta(P, \theta) < 0$, then*

$$\exp\{(1+\varepsilon)\, \delta(P, \theta)\, r^n\} \leq \left|g\left(re^{i\theta}\right)\right| \leq \exp\{(1-\varepsilon)\, \delta(P, \theta)\, r^n\},$$

*where $E_2 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ is a finite set.*

**Lemma 2.3** ([6]). *Let $f$ be a transcendental meromorphic function of finite order $\rho$. Let $\varepsilon > 0$ be a constant, $k$ and $j$ be integers satisfying $k > j \geq 0$. Then the following two statements hold:*

*(i) There exists a set $E_3 \subset (1, +\infty)$ which has finite logarithmic measure, such that for all $z$ satisfying $|z| \notin E_3 \cup [0, 1]$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.1}$$

*(ii) There exists a set $E_4 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) \setminus E_4$, then there is a constant $R = R(\theta) > 0$ such that (2.1) holds for all $z$ satisfying $\arg z = \theta$ and $|z| \geq R$.*

**Lemma 2.4** ([12]). *Let $f(z)$ be an entire function and suppose that*

$$G(z) := \frac{\log^+ \left|f^{(k)}(z)\right|}{|z|^{\rho}}$$

*is unbounded on some ray $\arg z = \theta$ with constant $\rho > 0$. Then there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ $(n = 1, 2, \ldots)$, where $r_n \to +\infty$, such* *that* $G(z_n) \to \infty$ *and*

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right| \leq \frac{1}{(k-j)!}(1+o(1))\, r_n^{k-j},\ j=0,1,\ldots,k-1$$

*as* $n \to +\infty$.

**Lemma 2.5** ([12])**.** *Let* $f(z)$ *be an entire function with* $\rho(f) = \rho < +\infty$. *Suppose that there exists a set* $E_5 \subset [0, 2\pi)$ *which has linear measure zero, such that* $\log^+ \left|f\left(re^{i\theta}\right)\right| \leq Mr^{\sigma}$ *for any ray* $\arg z = \theta \in [0, 2\pi) \setminus E_5$, *where* $M$ *is a positive constant depending on* $\theta$, *while* $\sigma$ *is a positive constant independent of* $\theta$. *Then* $\rho(f) \leq \sigma$.

**Lemma 2.6** ([1, 5])**.** *Let* $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ *be finite order meromorphic functions.*

*(i) If* $f$ *is a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = F \tag{2.2}$$

*with* $\rho(f) = +\infty$, *then* $f$ *satisfies*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty.$$

*(ii) If* $f$ *is a meromorphic solution of equation* (2.2) *with* $\rho(f) = +\infty$ *and* $\rho_2(f) = \rho$, *then*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty,\ \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho.$$

By using Wiman–Valiron theory [8], we easily obtain the following result which we omit the proof.

**Lemma 2.7.** *Let* $B_0(z), \ldots, B_{k-1}(z)$, $H(z)$ *be entire functions of finite order. If* $f$ *is a solution of the equation*

$$f^{(k)} + B_{k-1}(z)f^{(k-1)} + \cdots + B_1(z)f' + B_0(z)f = H,$$

*then* $\rho_2(f) \leq \max\{\rho(B_0), \ldots, \rho(B_{k-1}), \rho(H)\}$.

## 3. Proof of Theorem 1.1

Since $b_i = c_i a$ ($i \in I$), then by (1.1) we get

$$e^{-az}f^{(k)} + \sum_{i\in I_1} A_i e^{(c_i-1)az} f^{(i)} + \sum_{i\in I_2} A_i e^{(c_i-1)az} f^{(i)} + \sum_{i\in I_3} A_i e^{(c_i-1)az} f^{(i)} + \sum_{i\in I_4} A_i f^{(i)} = F. \tag{3.1}$$

First we prove that every solution $f$ of (1.1) satisfies $\rho(f) \geq 1$. We assume that $\rho(f) < 1$. It is clear that $f \not\equiv 0$. We can rewrite (3.1) in the form

$$f^{(k)}e^{-az} + \sum_{i\in\Gamma} B_i e^{(c_i-1)az} + A_0 f e^{(c_0-1)az} = F - \sum_{i\in I_4} A_i f^{(i)}, \tag{3.2}$$

where $\Gamma \subseteq I \setminus (I_4 \cup \{0\})$ such that $(c_i - 1)a$ $(i \in \Gamma)$ are distinct numbers and $B_i$ $(i \in \Gamma)$ are entire functions with order less than 1. We can see that $-a$, $(c_0 - 1)\,a$, $(c_i - 1)\,a$ $(i \in \Gamma)$ are distinct numbers. Obviously, $\rho\left(f^{(k)}\right) < 1$ and $\rho\left(A_i f^{(i)}\right) < 1$ $(i \in I)$. Then by (3.2) and the Lemma 2.1, we have $A_0 f \equiv 0$. This is a contradiction. Hence, $\rho(f) \geq 1$. Therefore $f$ is a transcendental solution of equation (1.1).

Now we prove that $\rho(f) = +\infty$. Suppose that $\rho(f) = \rho < +\infty$. Set $\alpha = \max\{\rho(A_j)\,(j = 0, 1, \ldots, k-1), \rho(F)\}$. Then $\alpha < 1$. For any given $\varepsilon$ $(0 < 2\varepsilon < 1 - \alpha)$ and for sufficiently large $r$, we have

$$|F(z)| \leq \exp\left\{r^{\alpha+\varepsilon}\right\}, \tag{3.3}$$

$$|A_i(z)| \leq \exp\left\{r^{\alpha+\varepsilon}\right\}, \ i \in I_4. \tag{3.4}$$

By Lemma 2.2, there exists a set $E \subset [0, 2\pi)$ of linear measure zero, such that whenever $\theta \in [0, 2\pi) \setminus E$, then $\delta(az, \theta) \neq 0$. By Lemma 2.3, there exists a set $E_4 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) \setminus E_4$, then there is a constant $R = R(\theta) > 1$ such that for all $z$ satisfying $\arg z = \theta$ and $|z| \geq R$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq |z|^{2\rho}, \ 0 \leq i < j \leq k. \tag{3.5}$$

For any fixed $\theta \in [0, 2\pi) \setminus (E \cup E_4)$, set

$$\delta_s = \delta\left((c_s - 1)\,az, \theta\right), \ \delta_l = \delta\left((c_l - 1)\,az, \theta\right),$$
$$\delta_1 = \max\left\{\delta\left((c_i - 1)\,az, \theta\right) : i \in I_1 \setminus \{s\}\right\}$$

and

$$\delta_3 = \max\left\{\delta\left((c_i - 1)\,az, \theta\right) : i \in I_3 \setminus \{l\}\right\}.$$

Then $\delta_s \neq 0$, $\delta_l \neq 0$, $\delta_1 \neq 0$ and $\delta_3 \neq 0$. We now discuss two cases separately.

**Case 1.** $\delta(-az, \theta) > 0$. We know that $\delta\left((c_i - 1)\,az, \theta\right) = (1 - c_i)\,\delta(-az, \theta)$, hence:

If $i \in I_1$, then $\delta\left((c_i - 1)\,az, \theta\right) < 0$.

If $i \in I_2$, then $0 < \delta\left((c_i - 1)\,az, \theta\right) < \delta(-az, \theta)$.

If $i \in I_3 \setminus \{l\}$, then $0 < \delta(-az, \theta) < \delta\left((c_i - 1)\,az, \theta\right) \leq \delta_3 < \delta_l$.

By Lemma 2.2, for any given $\varepsilon$ with $0 < 2\varepsilon < \min\left\{\frac{\delta_l - \delta_3}{\delta_l}, 1 - \alpha\right\}$, we obtain

$$\left|A_l e^{(c_l - 1)az}\right| \geq \exp\left\{(1 - \varepsilon)\,\delta_l r\right\}, \tag{3.6}$$

$$\left|e^{-az}\right| \leq \exp\left\{(1 + \varepsilon)\,\delta(-az, \theta)\,r\right\}, \tag{3.7}$$

$$\left|A_i e^{(c_i - 1)az}\right| \leq \exp\left\{(1 - \varepsilon)\,\delta\left((c_i - 1)\,az, \theta\right) r\right\} < 1, \ i \in I_1, \tag{3.8}$$

$$\begin{aligned} \left|A_i e^{(c_i - 1)az}\right| &\leq \exp\left\{(1 + \varepsilon)\,\delta\left((c_i - 1)\,az, \theta\right) r\right\} \\ &< \exp\left\{(1 + \varepsilon)\,\delta(-az, \theta)\,r\right\}, \ i \in I_2, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \left|A_i e^{(c_i - 1)az}\right| &\leq \exp\left\{(1 + \varepsilon)\,\delta\left((c_i - 1)\,az, \theta\right) r\right\} \\ &\leq \exp\left\{(1 + \varepsilon)\,\delta_3 r\right\}, \ i \in I_3 \setminus \{l\} \end{aligned} \tag{3.10}$$

for sufficiently large $r$. We now prove that $\log^+ \left|f^{(l)}(z)\right| / |z|^{\alpha+\varepsilon}$ is bounded on the ray $\arg z = \theta$. We assume that $\log^+ \left|f^{(l)}(z)\right| / |z|^{\alpha+\varepsilon}$ is unbounded on the ray $\arg z = \theta$. Then by Lemma 2.4, there is a sequence of points $z_m = r_m e^{i\theta}$, such that $r_m \to +\infty$, and that

$$\frac{\log^+ \left|f^{(l)}(z_m)\right|}{r_m^{\alpha+\varepsilon}} \to +\infty, \tag{3.11}$$

$$\left|\frac{f^{(j)}(z_m)}{f^{(l)}(z_m)}\right| \le \frac{1}{(l-j)!}(1+o(1))\, r_m^{l-j},\ (j = 0, 1, \dots, l-1), \tag{3.12}$$

for $m$ is large enough. From (3.3) and (3.11), we get

$$\left|\frac{F(z_m)}{f^{(l)}(z_m)}\right| \to 0, \tag{3.13}$$

for $m$ is large enough. From (3.1), we obtain

$$\begin{aligned} \left|A_l e^{(c_l-1)az}\right| \le{}& \left|e^{-az}\right| \left|\frac{f^{(k)}}{f^{(l)}}\right| + \sum_{i\in I_1} \left|A_i e^{(c_i-1)az}\right| \left|\frac{f^{(i)}}{f^{(l)}}\right| \\ &+ \sum_{i\in I_2} \left|A_i e^{(c_i-1)az}\right| \left|\frac{f^{(i)}}{f^{(l)}}\right| + \sum_{i\in I_3\setminus\{l\}} \left|A_i e^{(c_i-1)az}\right| \left|\frac{f^{(i)}}{f^{(l)}}\right| \\ &+ \sum_{i\in I_4} |A_i| \left|\frac{f^{(i)}}{f^{(l)}}\right| + \left|\frac{F}{f^{(l)}}\right|. \end{aligned} \tag{3.14}$$

Substituting (3.4)–(3.10), (3.12) and (3.13) into (3.14), we have

$$\begin{aligned} \exp\{(1-\varepsilon)\delta_l r_m\} &\le \left|A_l(z_m) e^{(c_l-1)az_m}\right| \\ &\le \left|e^{-az_m}\right| \left|\frac{f^{(k)}(z_m)}{f^{(l)}(z_m)}\right| + \sum_{i\in I_1} \left|A_i(z_m) e^{(c_i-1)az_m}\right| \left|\frac{f^{(i)}(z_m)}{f^{(l)}(z_m)}\right| \\ &\quad + \sum_{i\in I_2} \left|A_i(z_m) e^{(c_i-1)az_m}\right| \left|\frac{f^{(i)}(z_m)}{f^{(l)}(z_m)}\right| \\ &\quad + \sum_{i\in I_3\setminus\{l\}} \left|A_i(z_m) e^{(c_i-1)az_m}\right| \left|\frac{f^{(i)}(z_m)}{f^{(l)}(z_m)}\right| \\ &\quad + \sum_{i\in I_4} |A_i(z_m)| \left|\frac{f^{(i)}(z_m)}{f^{(l)}(z_m)}\right| + \left|\frac{F(z_m)}{f^{(l)}(z_m)}\right| \\ &\le M_0 r_m^{M_1} \exp\{(1+\varepsilon)\delta_3 r_m\} \exp\left\{r_m^{\alpha+\varepsilon}\right\}, \end{aligned} \tag{3.15}$$

where $M_0 > 0$ and $M_1 > 0$ are some constants. By $0 < \varepsilon < \frac{\delta_l - \delta_3}{2\delta_l}$ and (3.15), we can get

$$\exp\left\{\frac{(\delta_l - \delta_3)^2}{2\delta_l} r_m\right\} \le M_0 r_m^{M_1} \exp\left\{r_m^{\alpha+\varepsilon}\right\},$$

which is a contradiction because $\alpha+\varepsilon<1$. Therefore, $\log^{+}\left|f^{(l)}(z)\right|/|z|^{\alpha+\varepsilon}$ is bounded and we have

$$\left|f^{(l)}(z)\right| \leq M\exp\left\{r^{\alpha+\varepsilon}\right\}$$

on the ray $\arg z=\theta$. By the same reasoning as in the proof of [10, Lemma 3.1], we immediately conclude that

$$\begin{aligned}|f(z)| &\leq \frac{1}{l!}\left(1+o\left(1\right)\right)r^{l}\left|f^{(l)}(z)\right| \leq \frac{1}{l!}\left(1+o\left(1\right)\right)Mr^{l}\exp\left\{r^{\alpha+\varepsilon}\right\}\\ &\leq M\exp\left\{r^{\alpha+2\varepsilon}\right\}\end{aligned}$$

on the ray $\arg z=\theta$.

**Case 2.** $\delta\left(-az,\theta\right)<0$. We know that $\delta\left(\left(c_i-1\right)az,\theta\right)=\left(1-c_i\right)\delta\left(-az,\theta\right)$, hence:

If $i\in I_1\setminus\{s\}$, then $0<\delta\left(\left(c_i-1\right)az,\theta\right)\leq\delta_1<\delta_s$.

If $i\in I_2\cup I_3$, then $\delta\left(\left(c_i-1\right)az,\theta\right)<0$.

By Lemma 2.2, for any given $\varepsilon$ with $0<2\varepsilon<\min\left\{\frac{\delta_s-\delta_1}{\delta_s},1-\alpha\right\}$, we obtain

$$\left|A_s e^{(c_s-1)az}\right| \geq \exp\left\{\left(1-\varepsilon\right)\delta_s r\right\}, \tag{3.16}$$

$$\left|e^{-az}\right| \leq \exp\left\{\left(1-\varepsilon\right)\delta\left(-az,\theta\right)r\right\}<1, \tag{3.17}$$

$$\begin{aligned}\left|A_i e^{(c_i-1)az}\right| &\leq \exp\left\{\left(1+\varepsilon\right)\delta\left(\left(c_i-1\right)az,\theta\right)r\right\}\\ &\leq \exp\left\{\left(1+\varepsilon\right)\delta_1 r\right\},\ i\in I_1\setminus\{s\},\end{aligned} \tag{3.18}$$

$$\left|A_i e^{(c_i-1)az}\right| \leq \exp\left\{\left(1-\varepsilon\right)\delta\left(\left(c_i-1\right)az,\theta\right)r\right\}<1,\ i\in I_2\cup I_3 \tag{3.19}$$

for sufficiently large $r$. We now prove that $\log^{+}\left|f^{(s)}(z)\right|/|z|^{\alpha+\varepsilon}$ is bounded on the ray $\arg z=\theta$. We assume that $\log^{+}\left|f^{(s)}(z)\right|/|z|^{\alpha+\varepsilon}$ is unbounded on the ray $\arg z=\theta$. Then by Lemma 2.4, there is a sequence of points $z_m=r_m e^{i\theta}$, such that $r_m\to+\infty$, and that

$$\frac{\log^{+}\left|f^{(s)}\left(z_m\right)\right|}{r_m^{\alpha+\varepsilon}}\to+\infty, \tag{3.20}$$

$$\left|\frac{f^{(j)}\left(z_m\right)}{f^{(s)}\left(z_m\right)}\right| \leq \frac{1}{(s-j)!}\left(1+o\left(1\right)\right)r_m^{s-j},\ \left(j=0,1,\ldots,s-1\right), \tag{3.21}$$

for $m$ is large enough. From (3.3) and (3.20) we get

$$\left|\frac{F\left(z_m\right)}{f^{(s)}\left(z_m\right)}\right|\to 0, \tag{3.22}$$

for $m$ is large enough. From (3.1) we obtain

$$\begin{aligned}&\left|A_s e^{(c_s-1)az}\right|\\ &\leq \left|e^{-az}\right|\left|\frac{f^{(k)}}{f^{(s)}}\right|+\sum_{i\in I_1\setminus\{s\}}\left|A_i e^{(c_i-1)az}\right|\left|\frac{f^{(i)}}{f^{(s)}}\right|+\sum_{i\in I_2}\left|A_i e^{(c_i-1)az}\right|\left|\frac{f^{(i)}}{f^{(s)}}\right|\end{aligned} \tag{3.23}$$

$$+\sum_{i\in I_3}\left|A_i e^{(c_i-1)az}\right|\left|\frac{f^{(i)}}{f^{(s)}}\right|+\sum_{i\in I_4}|A_i|\left|\frac{f^{(i)}}{f^{(s)}}\right|+\left|\frac{F}{f^{(s)}}\right|.$$

Substituting (3.4), (3.5), (3.16)–(3.19), (3.21) and (3.22) into (3.23), we have

$$\exp\{(1-\varepsilon)\delta_s r_m\}\le\left|A_s(z_m)e^{(c_s-1)az_m}\right| \tag{3.24}$$

$$\le\left|e^{-az_m}\right|\left|\frac{f^{(k)}(z_m)}{f^{(s)}(z_m)}\right|+\sum_{i\in I_1\setminus\{s\}}\left|A_i(z_m)e^{(c_i-1)az_m}\right|\left|\frac{f^{(i)}(z_m)}{f^{(s)}(z_m)}\right|$$

$$+\sum_{i\in I_2}\left|A_i(z_m)e^{(c_i-1)az_m}\right|\left|\frac{f^{(i)}(z_m)}{f^{(s)}(z_m)}\right|$$

$$+\sum_{i\in I_3}\left|A_i(z_m)e^{(c_i-1)az_m}\right|\left|\frac{f^{(i)}(z_m)}{f^{(s)}(z_m)}\right|$$

$$+\sum_{i\in I_4}|A_i(z_m)|\left|\frac{f^{(i)}(z_m)}{f^{(s)}(z_m)}\right|+\left|\frac{F(z_m)}{f^{(s)}(z_m)}\right|$$

$$\le M_2 r_m^{M_3}\exp\{(1+\varepsilon)\delta_1 r_m\}\exp\left\{r_m^{\alpha+\varepsilon}\right\},$$

where $M_2>0$ and $M_3>0$ are some constants. By $0<\varepsilon<\frac{\delta_s-\delta_1}{2\delta_s}$ and (3.24), we get

$$\exp\left\{\frac{(\delta_s-\delta_1)^2}{2\delta_s}r_m\right\}\le M_2 r_m^{M_3}\exp\left\{r_m^{\alpha+\varepsilon}\right\},$$

which is a contradiction because $\alpha+\varepsilon<1$. Therefore, $\log^+\left|f^{(s)}(z)\right|/|z|^{\alpha+\varepsilon}$ is bounded and we have

$$\left|f^{(s)}(z)\right|\le M\exp\left\{r^{\alpha+\varepsilon}\right\}$$

on the ray $\arg z=\theta$. This implies, as in Case 1, that

$$|f(z)|\le M\exp\left\{r^{\alpha+2\varepsilon}\right\}. \tag{3.25}$$

Therefore, for any given $\theta\in[0,2\pi)\setminus(E\cup E_4)$, we have got (3.25) on the ray $\arg z=\theta$, provided that $r$ is large enough. Then by Lemma 2.5, we have $\rho(f)\le\alpha+2\varepsilon<1$, which is a contradiction. Hence every transcendental solution $f$ of (1.1) must be of infinite order. Since

$$\max\left\{\rho\left(A_j e^{b_j z}\right)\ (j=0,\dots,k-1),\ \rho\left(Fe^{az}\right)\right\}=1,$$

then by Lemma 2.7 we have $\rho_2(f)\le 1$.

## 4. Proof of Theorem 1.2

Suppose that $f$ is a solution of equation (1.1). Then, by Theorem 1.1 we have $\rho(f)=+\infty$ and $\rho_2(f)\le 1$. Set $g(z)=f(z)-\varphi(z)$. Then $g(z)$ is an entire function with $\rho(g)=\rho(f)=+\infty$ and $\rho_2(g)=\rho_2(f)\le 1$. Substituting $f=g+\varphi$ into (1.1) , we have

$$g^{(k)}+A_{k-1}e^{b_{k-1}z}g^{(k-1)}+\cdots+A_1e^{b_1z}g'+A_0e^{b_0z}g=D, \tag{4.1}$$

where

$$D = Fe^{az} - \left[\varphi^{(k)} + A_{k-1}e^{b_{k-1}z}\varphi^{(k-1)} + \cdots + A_1 e^{b_1 z}\varphi' + A_0 e^{b_0 z}\varphi\right].$$

We prove that $D \not\equiv 0$. In fact, if $D \equiv 0$, then

$$\varphi^{(k)} + A_{k-1}e^{b_{k-1}z}\varphi^{(k-1)} + \cdots + A_1 e^{b_1 z}\varphi' + A_0 e^{b_0 z}\varphi = Fe^{az}.$$

Hence $\rho(\varphi) = +\infty$, which is a contradiction. Therefore $D \not\equiv 0$. We know that the functions $A_j e^{b_j z}$ $(j = 0, 1, \ldots, k-1)$, $D$ are of finite order. By Lemma 2.6 and (4.1) we have

$$\overline{\lambda}(f-\varphi) = \lambda(f-\varphi) = \rho(f-\varphi) = \rho(f) = \infty,$$

$$\overline{\lambda}_2(f-\varphi) = \lambda_2(f-\varphi) = \rho_2(f-\varphi) = \rho_2(f) \leq 1.$$

Then, by $f$ is infinite order solution of equation (1.1) and Lemma 2.6 we obtain

$$\lambda(f) = \overline{\lambda}(f) = \overline{\lambda}(f-\varphi) = \lambda(f-\varphi) = \rho(f) = \infty$$

$$\lambda_2(f) = \overline{\lambda}_2(f) = \overline{\lambda}_2(f-\varphi) = \lambda_2(f-\varphi) = \rho_2(f) \leq 1.$$

which completes the proof.

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