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A SHARP GENERAL L_2 INEQUALITY OF OSTROWSKI TYPE

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Abstract. A sharp general L_2 inequality of Ostrowski type is established, which provides a generalization of some previous results and gives some other interesting results as special cases.

1. Introduction

In [1] and [2], we may find the following interesting sharp trapezoid type inequality and midpoint type inequality:

Theorem 1. *Let $f\colon [a,b] \to \mathbf{R}$ be such that f' is absolutely continuous on $[a,b]$ and $f'' \in L_2[a,b]$. Then we have sharp inequality*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \leq \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}}\sqrt{\sigma(f'')}, \tag{1}$$

where $\sigma(\cdot)$ is defined by

$$\sigma(f) = \|f\|_2^2 - \frac{1}{b-a}\left(\int_a^b f(t)\,dt\right)^2 \tag{2}$$

and $\|f\|_2 := [\int_a^b f^2(t)\,dt]^{\frac{1}{2}}$.

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Theorem 2. *Under the assumptions of Theorem 1, we have sharp inequality*

$$\left|\int_a^b f(t)\,dt - (b-a)f(\frac{a+b}{2}) - \frac{(b-a)^2}{24}[f'(b)-f'(a)]\right| \le \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}}\sqrt{\sigma(f'')}. \tag{3}$$

In [5], the author provided a Sharp L_2 inequality of Ostrowski type as follows:

Theorem 3. *Let the assumptions of Theorem 1 hold. Then for any $\theta \in [0,1]$ and $x \in [a,b]$ we have sharp inequality*

$$\begin{aligned} &\left|\int_a^b f(t)dt - (b-a)\left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right]\right. \\ &\quad +(1-\theta)(b-a)\left(x-\frac{a+b}{2}\right)f'(x) \\ &\quad \left. - \left[\frac{1-\theta}{2}\left(x-\frac{a+b}{2}\right)^2 + \frac{1-3\theta}{24}(b-a)^2\right][f'(b)-f'(a)]\right| \\ &\le \left[\frac{\theta(1-\theta)}{4}(b-a)\left(x-\frac{a+b}{2}\right)^4\right. \\ &\quad +\frac{3\theta^2-5\theta+2}{24}(b-a)^3\left(x-\frac{a+b}{2}\right)^2 \\ &\quad \left. + \frac{15\theta^2-15\theta+4}{2880}(b-a)^5\right]^{\frac{1}{2}}\sqrt{\sigma(f'')}. \end{aligned} \tag{4}$$

The inequality (4) not only provide a generalization of inequalities (1) and (3), but also give some other interesting sharp inequalities as special cases. Moreover, it has been shown that the corrected Simpson rule (see [8],[11] and [10]) gives better result than Simpson rule and, in particular, the corrected averaged midpoint-trapezoid quadrature rule is optimal.

In this work, we will derive a new sharp general inequality of Ostrowski type for functions whose $(n-1)$th derivatives are absolutely continuous and whose nth derivatives belong to $L_2(a,b)$. This will not only provide a generalization of the inequality (4), but also give some other interesting sharp inequalities as special cases.

2. The results

In [6], we may find the identity

$$
\begin{aligned}
(5)\qquad & (-1)^n \int_a^b K_n(t,x,\theta) f^{(n)}(t)\,dt \\
&= \int_a^b f(t)\,dt - \frac{b-a}{2}[\theta f(a) + 2(1-\theta)f(x) + \theta f(b)] \\
&\quad - \sum_{k=1}^{n-1} \left\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} \right. \\
&\quad \left. - \frac{\theta(b-a)[(-1)^k (x-a)^k + (b-x)^k]}{2k!} \right\} f^{(k)}(x)
\end{aligned}
$$

where $K_n(t,x,\theta)$ is the kernel given by

$$
(6)\qquad K_n(t,x,\theta) = \begin{cases} \frac{(t-a)^n}{n!} - \frac{\theta(b-a)(t-a)^{n-1}}{2(n-1)!}, & \text{if } t \in [a,x], \\ \frac{(t-b)^n}{n!} + \frac{(b-a)(t-b)^{n-1}}{2(n-1)!}, & \text{if } t \in (x,b]. \end{cases}
$$

By elementary calculus, it is not difficult to get

$$
(7)\qquad \int_a^b K_n(t,x,\theta)\,dt = \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!} - \frac{\theta(b-a)[(x-a)^n + (-1)^n (b-x)^n]}{2n!}
$$

and

$$
\begin{aligned}
(8)\qquad \int_a^b K_n^2(t,x,\theta)dt &= \frac{(x-a)^{2n+1} + (b-x)^{2n+1}}{(2n+1)(n!)^2} \\
&\quad - \frac{\theta(b-a)[(x-a)^{2n} + (b-x)^{2n}]}{2(n!)^2} + \frac{\theta^2 (b-a)^2 [(x-a)^{2n-1} + (b-x)^{2n-1}]}{4(2n-1)[(n-1)!]^2}.
\end{aligned}
$$

For brevity, in what follows, we will use the notations

$$
\begin{aligned}
(9)\qquad G_n(x,\theta) := \sum_{k=1}^{n-1} & \left\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} \right. \\
& \left. - \frac{\theta(b-a)[(-1)^k (x-a)^k + (b-x)^k]}{2k!} \right\} f^{(k)}(x),
\end{aligned}
$$

$$
(10)\qquad H_n(x,\theta) := \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!} - \frac{\theta(b-a)[(x-a)^n + (-1)^n (b-x)^n]}{2n!},
$$

$$I_n(x,\theta) := \frac{(x-a)^{2n+1}+(b-x)^{2n+1}}{(2n+1)(n!)^2} - \frac{\theta(b-a)[(x-a)^{2n}+(b-x)^{2n}]}{2(n!)^2} + \frac{\theta^2(b-a)^2[(x-a)^{2n-1}+(b-x)^{2n-1}]}{4(2n-1)[(n-1)!]^2} \tag{11}$$

and

$$D_n := \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}. \tag{12}$$

Theorem 4. *Let $f\colon [a,b]\to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)}\in L_2[a,b]$. Then for any $\theta\in[0,1]$ and $x\in[a,b]$ we have*

$$\left|(-1)^n\left\{\int_a^b f(t)dt-(b-a)\left[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}\right]-G_n(x,\theta)\right\} - D_nH_n(x,\theta)\right| \le \left[I_n(x,\theta)-\frac{H_n^2(x,\theta)}{b-a}\right]^{\frac{1}{2}}\sqrt{\sigma(f^{(n)})}. \tag{13}$$

The inequality (13) is sharp.

Proof. From (5),(7)-(10),(12) we get

$$\int_a^b\left[K_n(t,x,\theta)-\frac{1}{b-a}\int_a^b K_n(s,x,\theta)ds\right]\left[f^{(n)}(t)-\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right]dt = (-1)^n\left\{\int_a^b f(t)\,dt-(b-a)\left[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}\right]-G_n(\theta,x)\right\} -D_nH_n(\theta,x). \tag{14}$$

By using the Cauchy-Schwarz-Bunyakowski inequality, we have

$$\left|\int_a^b\left[K_n(t,x,\theta)-\frac{1}{b-a}\int_a^b K_n(s,x,\theta)ds\right]\left[f^{(n)}(t)-\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right]dt\right| \le \left\|K(\cdot,x,\theta)-\frac{1}{b-a}\int_a^b K_n(s,x,\theta)ds\right\|_2\left\|f^{(n)}-\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right\|_2. \tag{15}$$

From (7),(8),(10),(11) we also have

$$\left\|K(\cdot,x,\theta)-\frac{1}{b-a}\int_a^b K_n(t,s,\theta)\,ds\right\|_2^2 = I_n(\theta,x)-\frac{H_n^2(\theta,x)}{b-a}. \tag{16}$$

and by (2),

$$\left\|f^{(n)}-\frac{1}{b-a}\int_a^b f^{(n)}(s)\,ds\right\|_2^2 = \|f^{(n)}\|_2^2-\frac{(f^{(n-1)}(b)-f^{(n-1)}(a))^2}{b-a}=\sigma(f^{(n)}). \tag{17}$$

Consequently, the inequality (13) follows from (14)-(17).

It is clear that the inequality (13) is sharp, since in its proof only Cauchy-Schwarz-Bunyakowski inequality is used, and so the equality condition follows from the well-known inequality. □

Remark 1. If we take $n = 1$ in Theorem 4, then we have

$$G_1(x,\theta) = 0,$$
$$H_1(x,\theta) = (1-\theta)(b-a)\left(x - \frac{a+b}{2}\right),$$
$$I_1(x,\theta) = \frac{1-3\theta+3\theta^2}{12}(b-a)^3 + (1-\theta)(b-a)\left(x-\frac{a+b}{2}\right)^2,$$

and

$$I_1(x,\theta) - \frac{H_1^2(x,\theta)}{b-a} = \theta(1-\theta)(b-a)\left(x-\frac{a+b}{2}\right)^2 + \frac{1-3\theta+3\theta^2}{12}(b-a)^3.$$

Thus we can derived a sharp L_2 inequality of Ostrowski type as

$$\begin{aligned}&\left|(b-a)\left[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}\right] - (1-\theta)\left(x-\frac{a+b}{2}\right)[f(b)-f(a)] \right.\\ &\quad \left. - \int_a^b f(t)dt\right| \\ &\le \left[\theta(1-\theta)(b-a)\left(x-\frac{a+b}{2}\right)^2 + \frac{1-3\theta+3\theta^2}{12}(b-a)^3\right]^{\frac{1}{2}}\sqrt{\sigma(f')},\end{aligned}$$

which has been first proved in Theorem 1 of [4].

Remark 2. If we take $n = 2$ in Theorem 4, then we have

$$G_2(x,\theta) = (1-\theta)(b-a)\left(\frac{a+b}{2} - x\right)f'(x),$$
$$H_2(x,\theta) = \frac{(1-3\theta)(b-a)^3}{24} + \frac{(1-\theta)(b-a)}{2}\left(x-\frac{a+b}{2}\right)^2,$$

$$\begin{aligned}I_2(x,\theta) = {}&\frac{3-15\theta+20\theta^2}{960}(b-a)^5 + \frac{1-3\theta+2\theta^2}{8}(b-a)^3\left(x-\frac{a+b}{2}\right)^2\\ &- \frac{1-\theta}{4}(b-a)\left(x-\frac{a+b}{2}\right)^4,\end{aligned}$$

and

$$I_2(x,\theta) - \frac{H_2^2(x,\theta)}{b-a} = \frac{\theta(1-\theta)}{4}(b-a)\left(x-\frac{a+b}{2}\right)^4 + \frac{3\theta^2-5\theta+2}{24}(b-a)^3\left(x-\frac{a+b}{2}\right)^2 + \frac{15\theta^2-15\theta+4}{2880}(b-a)^5$$

Thus we can derived a sharp L_2 inequality of Ostrowski type as

$$\begin{aligned}&\bigg|\int_a^b f(t)dt - (b-a)\left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right]\\ &+(1-\theta)(b-a)\left(x-\frac{a+b}{2}\right)f'(x)\\ &-\left[\frac{1-\theta}{2}\left(x-\frac{a+b}{2}\right)^2 + \frac{1-3\theta}{24}(b-a)^2\right][f'(b)-f'(a)]\bigg|\\ &\le \left[\frac{\theta(1-\theta)}{4}(b-a)\left(x-\frac{a+b}{2}\right)^4 + \frac{3\theta^2-5\theta+2}{24}(b-a)^3\left(x-\frac{a+b}{2}\right)^2\right.\\ &\left.+\frac{15\theta^2-15\theta+4}{2880}(b-a)^5\right]^{\frac{1}{2}}\sqrt{\sigma(f'')}.\end{aligned}$$

which is just the inequality (4).

Corollary 1. *Let the assumptions of Theorem 4 hold. Then for any $\theta \in [0,1]$, we get a sharp inequality*

$$\begin{aligned}&\bigg|(-1)^n\left\{\int_a^b f(t)dt - (b-a)\left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right]\right.\\ &-\sum_{k=1}^{n-1}\frac{[(-1)^k+1][1-(k+1)\theta](b-a)^{k+1}}{2^{k+1}(k+1)!}f^{(k)}(x)\\ &-\frac{[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{2^{n+1}(n+1)!}D_n\bigg|\\ &\le \frac{c(n,\theta)}{(n+1)!}\left(\frac{b-a}{2}\right)^{n+\frac{1}{2}}\sqrt{\sigma(f^{(n)})},\end{aligned} \tag{18}$$

where $c(n,\theta) = [\frac{2(n+1)^2[(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2]-[1+(-1)^n](4n^2-1)[1-(n+1)\theta]^2}{4n^2-1}]^{\frac{1}{2}}$.

Proof. We set $x = \frac{a+b}{2}$ in (4) and observe that

$$\begin{aligned}G_n\left(\frac{a+b}{2},\theta\right) &= \sum_{k=1}^{n-1}\left\{\frac{[(-1)^k+1](b-a)^{k+1}}{2^{k+1}(k+1)!} - \frac{\theta[(-1)^k+1](b-a)^{k+1}}{2^{k+1}k!}\right\}f^{(k)}(x)\\ &= \sum_{k=1}^{n-1}\frac{[(-1)^k+1][1-(k+1)\theta](b-a)^{k+1}}{2^{k+1}(k+1)!}f^{(k)}(x),\end{aligned}$$

$$H_n\left(\frac{a+b}{2},\theta\right) = \frac{[1+(-1)^n](b-a)^{n+1}}{2^{n+1}(n+1)!} - \frac{[1+(-1)^n]\theta(b-a)^{n+1}}{2^{n+1}n!}$$
$$= \frac{[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{2^{n+1}(n+1)!},$$

$$I_n\left(\frac{a+b}{2},\theta\right) = \frac{(b-a)^{2n+1}}{2^{2n}(2n+1)(n!)^2} - \frac{\theta(b-a)^{2n+1}}{2^{2n}(n!)^2} + \frac{\theta^2(b-a)^{2n+1}}{2^{2n}(2n-1)[(n-1)!]^2}$$
$$= \frac{(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2}{2^{2n}(4n^2-1)(n!)^2}(b-a)^{2n+1},$$

and

$$I_n\left(\frac{a+b}{2},\theta\right) - \frac{H_n^2(\frac{a+b}{2},\theta)}{b-a} = \frac{c^2(n,\theta)}{2^{2n+1}[(n+1)!]^2}(b-a)^{2n+1},$$

then the inequality (18) follows. □

If n is an odd integer, then for any $\theta \in [0,1]$ we have

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}\left[\theta f(a) + 2(1-\theta)f\left(\frac{a+b}{2}\right) + \theta f(b)\right] - \sum_{k=1}^{n-1}\frac{[(-1)^k+1][1-(k+1)\theta](b-a)^{k+1}}{2^{k+1}(k+1)!}f^{(k)}(x)\right| \le \frac{(b-a)^{n+\frac{1}{2}}}{2^n n!}\sqrt{\frac{(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2}{4n^2-1}}\sqrt{\sigma(f^{(n)})}, \tag{19}$$

which also has been proved in Theorem 8 of [3] as well as Theorem 7 of [7], simultaneously.

If n is an even integer, then for any $\theta \in [0,1]$ we have

$$\left|\int_a^b f(x)dx - \frac{b-a}{2}\left[\theta f(a) + 2(1-\theta)f\left(\frac{a+b}{2}\right) + \theta f(b)\right] - \sum_{k=1}^{n-1}\frac{[(-1)^k+1][1-(k+1)\theta](b-a)^{k+1}}{2^{k+1}(k+1)!}f^{(k)}(x) - \frac{[1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^n}D_n\right| \le \frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!}\sqrt{\sigma(f^{(n)})} \times \times\sqrt{\frac{2n^3-n^2-(4n^4-5n^2+1)\theta+(2n^5+n^4-4n^3-2n^2+2n+1)\theta^2}{4n^2-1}}. \tag{20}$$

which also has been proved in Theorem 9 of [3] as well as Theorem 8 of [7], simultaneously.

Remark 3. If we take $\theta = 0$ in (19) and (20) then we get sharp midpoint type inequalities

$$\left|\int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right) + \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(b-a)^{2k+1}}{(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^{n+\frac{1}{2}}}{2^n n!\sqrt{2n+1}}\sqrt{\sigma(f^{(n)})} \tag{21}$$

for an odd n and

$$\left|\int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right) + \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(b-a)^{2k+1}}{(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right) - \frac{(b-a)^n}{(n+1)!2^n}[f^{(n-1)}(b) - f^{(n-1)}(a)]\right| \leq \frac{n(b-a)^{n+\frac{1}{2}}}{2^n (n+1)!\sqrt{2n+1}}\sqrt{\sigma(f^{(n)})} \tag{22}$$

for an even n.

In particular, for $n = 1$ in (21), we have

$$\left|\int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}}\sqrt{\sigma(f')},$$

and for $n = 2$ in (22), we have

$$\left|\int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24}[f'(b) - f'(a)]\right| \leq \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}}\sqrt{\sigma(f'')}.$$

Remark 4. If we take $\theta = 1$ in (19) and (20) then we get sharp trapezoid type inequalities

$$\left|\int_a^b f(x)dx - \frac{(b-a)}{2}[f(a) + f(b)] - \sum_{k=1}^{[\frac{n-1}{2}]} \frac{k(b-a)^{2k+1}}{(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^{n+\frac{1}{2}}}{2^n n!}\sqrt{\frac{2n^3 - 3n^2 + 2n}{4n^2 - 1}}\sqrt{\sigma(f^{(n)})} \tag{23}$$

for an odd n and

$$\begin{aligned}(24)\quad &\left|\int_a^b f(x)dx - \frac{(b-a)}{2}[f(a)+f(b)] - \sum_{k=1}^{[\frac{n-1}{2}]} \frac{k(b-a)^{2k+1}}{(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right.\\ &\quad \left. + \frac{n(b-a)^n}{(n+1)!2^n}[f^{(n-1)}(b) - f^{(n-1)}(a)]\right|\\ &\quad \le \frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!}\sqrt{\frac{2n^5-3n^4-2n^3+2n^2+2n}{4n^2-1}}\sqrt{\sigma(f^{(n)})}\end{aligned}$$

for an even n.

In particular, for $n=1$ in (23), we have

$$\left|\int_a^b f(x)dx - \frac{b-a}{2}[f(a)+f(b)]\right| \le \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}}\sqrt{\sigma(f')},$$

and for $n=2$ in (24), we have

$$\left|\int_a^b f(x)dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \le \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}}\sqrt{\sigma(f'')}.$$

Remark 5. If we take $\theta = \frac{1}{3}$ then we get sharp Simpson type inequalities

$$\begin{aligned}(25)\quad &\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right.\\ &\quad \left. + \sum_{k=2}^{[\frac{n-1}{2}]} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right|\\ &\quad \le \frac{1}{3}\frac{(b-a)^{n+\frac{1}{2}}}{2^n n!}\sqrt{\frac{2n^3-11n^2+18n-6}{4n^2-1}}\sqrt{\sigma(f^{(n)})}\end{aligned}$$

for an odd n and

$$\begin{aligned}(26)\quad &\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right.\\ &\quad + \sum_{k=2}^{[\frac{n-1}{2}]} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right)\\ &\quad \left. + \frac{(n-2)(b-a)^n}{3(n+1)!2^n}[f^{(n-1)}(b) - f^{(n-1)}(a)]\right|\\ &\quad \le \frac{1}{3}\frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!}\sqrt{\frac{2n^5-11n^4+14n^3+4n^2+2n-2}{4n^2-1}}\sqrt{\sigma(f^{(n)})}\end{aligned}$$

for an even n, which have been first proved in Theorem 5 and Theorem 6 of [9] in a more direct way.

In particular, for $n = 1, 3$ in (25) and $n = 2$ in (26), we have

$$\left|\int_a^b f(x)dx - \frac{b-a}{6}[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)]\right| \le C_n(b-a)^{n+\frac{1}{2}}\sqrt{\sigma(f^{(n)})},$$

where $C_1 = \frac{1}{6}$, $C_2 = \frac{1}{12\sqrt{30}}$, $C_3 = \frac{1}{48\sqrt{105}}$.

Remark 6. If we take $\theta = \frac{1}{2}$ then we have sharp averaged midpoint-trapezoid inequalities

$$\begin{aligned}\left|\int_a^b f(x)dx - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\right.\\ \left. - \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(2k-1)(b-a)^{2k+1}}{(2k+1)!2^{2k+1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right|\\ \le \frac{(b-a)^{n+\frac{1}{2}}}{2^{n+1}n!}\sqrt{\frac{2n^3 - 7n^2 + 8n - 2}{4n^2 - 1}}\sqrt{\sigma(f^{(n)})}\end{aligned} \tag{27}$$

for an odd n and

$$\begin{aligned}\left|\int_a^b f(x)dx - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\right.\\ - \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(2k-1)(b-a)^{2k+1}}{(2k+1)!2^{2k+1}} f^{(2k)}\left(\frac{a+b}{2}\right)\\ \left. + \frac{(n-1)(b-a)^n}{(n+1)!2^{n+1}}[f^{(n-1)}(b) - f^{(n-1)}(a)]\right|\\ \le \frac{(b-a)^{n+\frac{1}{2}}}{2^{n+1}(n+1)!}\sqrt{\frac{2n^5 - 7n^4 + 4n^3 + 4n^2 + 2n - 1}{4n^2 - 1}}\sqrt{\sigma(f^{(n)})}\end{aligned} \tag{28}$$

for an even n.

In particular, for $n = 1$ in (27), we have

$$\left|\int_a^b f(x)dx - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \le \frac{(b-a)^{\frac{3}{2}}}{4\sqrt{3}}\sqrt{\sigma(f')},$$

and for $n = 2$, in (28), we have

$$\begin{aligned}\left|\int_a^b f(x)dx - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] + \frac{(b-a)^2}{48}[f'(b) - f'(a)]\right|\\ \le \frac{(b-a)^{\frac{5}{2}}}{48\sqrt{5}}\sqrt{\sigma(f'')},\end{aligned}$$

Remark 7. If we take $\theta = \frac{7}{15}$ then we have sharp corrected Simpson type inequalities

$$\begin{aligned} &\left|\int_a^b f(x)dx - \frac{b-a}{30}\left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b)\right]\right. \\ &\quad \left. - \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(1-14k)(b-a)^{2k+1}}{15(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{(b-a)^{n+\frac{1}{2}}}{n!2^n}\sqrt{\frac{98n^3 - 371n^2 + 450n - 120}{225(4n^2-1)}}\sqrt{\sigma(f^{(n)})} \end{aligned} \tag{29}$$

for an odd n and

$$\begin{aligned} &\left|\int_a^b f(x)dx - \frac{b-a}{30}\left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b)\right]\right. \\ &\quad - \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(1-14k)(b-a)^{2k+1}}{15(2k+1)!2^{2k}} f^{(2k)}(\frac{a+b}{2}) \\ &\quad \left. + \frac{(7n-8)(b-a)^n}{15(n+1)!2^{n+1}}[f^{(n-1)}(b) - f^{(n-1)}(a)]\right| \\ &\leq \frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!}\sqrt{\frac{98n^5 - 371n^4 + 254n^3 + 202n^2 + 98n - 56}{225(4n^2-1)}}\sqrt{\sigma(f^{(n)})} \end{aligned} \tag{30}$$

for an even n.

In particular, for $n = 1$ in (29), we have

$$\left|\int_a^b f(x)dx - \frac{b-a}{30}\left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b)\right]\right| \leq \frac{\sqrt{19}(b-a)^{\frac{3}{2}}}{30}\sqrt{\sigma(f')},$$

and for $n = 2$, in (30), we have

$$\begin{aligned} &\left|\int_a^b f(x)dx - \frac{b-a}{30}\left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b)\right]\right. \\ &\qquad\qquad \left. + \frac{(b-a)^2}{60}[f'(b) - f'(a)]\right| \leq \frac{(b-a)^{\frac{5}{2}}}{60\sqrt{3}}\sqrt{\sigma(f'')}, \end{aligned}$$

References

[1] N. S. Barnett, P. Cerone, and S. S. Dragomir. A sharp bound for the error in the corrected trapezoid rule and application. *Tamkang J. Math.*, 33(3):253–258, 2002.

[2] P. Cerone. On perturbed trapezoidal and midpoint rules. *Korean J. Comput. Appl. Math.*, 9(2):423–435, 2002.

[3] W. Liu, Y. Jiang, and A. Tuna. A unified generalization of some quadrature rules and error bounds. *Appl. Math. Comput.*, 219(9):4765–4774, 2013.

[4] Z. Liu. Note on a paper by N. Ujević: "Sharp inequalities of Simpson type and Ostrowski type" [Comput. Math. Appl. **48** (2004), no. 1-2, 145–151; mr2086792]. *Appl. Math. Lett.*, 20(6):659–663, 2007.
[5] Z. Liu. A sharp L_2 inequality of Ostrowski type. *ANZIAM J.*, 49(3):423–429, 2008.
[6] Z. Liu. A sharp general Ostrowski type inequality. *Bull. Aust. Math. Soc.*, 83(2):189–209, 2011.
[7] Z. Liu. On generalizations of some classical integral inequalities. *J. Math. Inequal.*, 7(2):255–269, 2013.
[8] J. Pečarić and I. Franjić. Generalisation of a corrected Simpson's formula. *ANZIAM J.*, 47(3):367–385, 2006.
[9] Y. Shi and Z. Liu. Some sharp Simpson type inequalities and applications. *Appl. Math. E-Notes*, 9:205–215, 2009.
[10] N. Ujević. A generalization of the modified Simpson's rule and error bounds. *ANZIAM J.*, 47((E)):E1–E13, 2005.
[11] N. Ujević and A. J. Roberts. A corrected quadrature formula and applications. *ANZIAM J.*, 45((E)):E41–E56, 2003.

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