Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
32 (2016), 23–31
www.emis.de/journals
ISSN 1786-0091

A NOTE OF THREE PRIME REPRESENTATION PROBLEMS

SHICHUN YANG AND ALAIN TOGBÉ

Abstract. In this note, we consider a three prime representation problem asked by Sárközy. We give a negative answer to the upper density case of the problem and obtain a conclusion that the Three Primes Theorem still holds for a general thin subset of primes.

1. Introduction

Let $\mathbb{P}$ be the sets of all primes. In 1937, I. M. Vinogradov [28] proved the famous three primes theorem. It states that there exists an absolute constant $N_0 > 0$, such that every odd integer larger than N_0 is the sum of three primes. In 1956, Borodzkin [4] showed that $N_0 < 3^{3^{15}} \approx 10^{7000000}$. Since then, this bound has been significantly improved. Most recently, Chen and Wang [5] have established a bound of 10^{43000}. In [6], Deshouillers, Effinger, Te Riele and Zinoviev outlined a proof that if the Generalized Riemann Hypothesis holds, then every odd number above 5 is a sum of three prime numbers. Recently, Helfgott [10]–[13] completely solve the problem and proved that the ternary Goldbach conjecture is true. The Three Primes Theorem is one of the most important results studied in analytic number theory. For example, Pan [22], Zhan [30] and Jia [15] studied the Three Primes Theorem in short intervals.

Many authors have been looking for thin subsets of primes for which the Three Primes Theorem still holds. In 1986, Wirsing [29] showed that there

2010 *Mathematics Subject Classification.* 11A25, 11A41, 11N05, 11P32.

Key words and phrases. Three primes theorem, additive representation, thin subsets of primes.

The first author was supported by National Natural Science Foundation of China (Grant No.11301363), the Innovative Research Team Project of Sichuan Provincial Universities (No.14TD0040), and the Applied Basic Research Foundation of Sichuan Provincial Department for Science and Technology (No.2013JYZ003), and Department for Education (No.15ZA0337). The second author was supported in part by Purdue University North Central.

exists such a set S with the property that

$$\sum_{p<x, p\in S} 1 \ll (x \log x)^{\frac{1}{3}}.$$

It is also interesting to find more familiar thin sets of primes which serve this purpose. An example is the set of Piatetski-Shapiro primes of type γ which are of the form $\lfloor n^{\frac{1}{\gamma}} \rfloor$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x. Denote this set by P_γ. Piatetski-Shapiro [23] showed that for $\frac{11}{12} < \gamma < 1$,

$$P_\gamma(x) = \sum_{p<x, p=\lfloor n^{\frac{1}{\gamma}} \rfloor} 1 = \big(1+o(1)\big)\frac{x^\gamma}{\log x}.$$

Heath-Brown [9] improved Piatetski-Shapiro's result by extending the range to $\frac{662}{755} < \gamma < 1$. Further improvements were made by Kolesnik [19], Liu and Rivat [21], Balog and Friedlander [2], Jia [17], [16], [18], et. al.

On the other hand, in 2001, P. Sárközy asked the following problem (see Problem 67 in [25]):

Problem A. Is it true that if $Q = \{q_1, q_2, \cdots\}$ is an infinite set of primes such that

$$\lim_{x\to\infty}\inf \frac{\log x}{x} \big|\{q_i : \ q_i \le x, q_i \in Q\}\big| > \frac{1}{2}, \tag{1}$$

then every large odd integer $2n+1$ can be represented in the form

$$q_1 + q_2 + q_3 = 2n+1,$$

with $q_1, q_2, q_3 \in Q$?

Let $\pi(x)$ denote the number of primes $p < x$. From the prime number theorem, we have $\pi(x) \sim \frac{x}{\log x}$. So Problem A is another type of problem of thin subsets of primes related to the three primes theorem. For two non-empty sets X and A of positive integers, we define the upper density and lower density of A relative to X by

$$\overline{d}_X(A) = \lim_{x\to+\infty}\sup \frac{|\,A\cap X\cap[1,x]\,|}{|\,X\cap[1,x]\,|},$$

and

$$\underline{d}_X(A) = \lim_{x\to+\infty}\inf \frac{|\,A\cap X\cap[1,x]\,|}{|\,X\cap[1,x]\,|}.$$

Thus, Problem A is equivalent to the following problem:

If $Q \subset \mathbb{P}$ is a subset of primes with $\underline{d}_{\mathbb{P}}(Q) > \frac{1}{2}$, could every large odd integer be represented as the sum of three positive integers, each integer composed of primes belonging to Q?

In [8], Green obtained a Roth-type generalization of van der Corput's result. Green showed that if P_0 is a subset of $\mathbb{P}$ with $\underline{d}_{\mathbb{P}}(P_0) > 0$ then P_0 contains infinitely many non-trivial 3-term arithmetic progressions. In [20], Li and Pan used Green's idea to extend Vinogradov's theorem as follows.

Theorem 1.1. *Suppose that P_1, P_2, P_3 are three subsets of $\mathbb{P}$ with*

$$\underline{d}_{\mathbb{P}}(P_1) + \underline{d}_{\mathbb{P}}(P_2) + \underline{d}_{\mathbb{P}}(P_3) > 2.$$

Then for every sufficiently large odd integer n, there exist $p_1 \in P_1$, $p_2 \in P_2$ and $p_3 \in P_3$ such that $p_1 + p_2 + p_3 = n$.

By Theorem 1.1, we can derive that if $Q \subset \mathbb{P}$ is a subset of primes such that $\underline{d}_{\mathbb{P}}(Q) > \frac{2}{3}$, then every large odd integer can be represented as the sum of three primes belonging to Q. Recently, Shao [26] obtained that $\frac{2}{3}$ can be improved to $\frac{5}{8}$. And he pointed out that the constant $\frac{5}{8}$ cannot be improved. In fact, we may take $Q = \{p \in \mathbb{P}, p \equiv 1, 2, 4, 7, 13 \pmod{15}\}$, then $\underline{d}_{\mathbb{P}}(Q) = \frac{5}{8}$. It is not hard to see that if $N \equiv 14 \pmod{15}$ then N cannot be written as sum of three elements of Q. Therefore, the answer to Problem A is negative.

In this note, we consider Problem A. First, we give a negative answer to the upper density case of Problem A. Second, we obtain a conclusion that the Three Primes Theorem still holds for a general thin subset of primes.

The organization of this paper is as follows. In Section 2, we give a proof of Theorem 2.1 which is the negative answer to Problem A and we remark that the Problem 68 in [25] has a similar conclusion. In Section 3, we use Vinogradov's Three Primes Theorem and a theorem of Bombieri and Davenport [3] (who showed that a large even number is the sum of two primes), we obtain Theorem 3.2 which states that the Three Primes Theorem still holds for a thin subset of primes. We ask an interesting research question in the last section.

2. The upper density case of problem A

About Problem A, we have the following result.

Theorem 2.1. *For any small positive constant ε, there exists an infinite set of primes $Q = \{q_1, q_2, \ldots\}$ such that*

$$\overline{d}_{\mathbb{P}}(Q) = 1 - \varepsilon, \tag{2}$$

and there are infinitely many odd integers n that cannot be represented in the form $q_1 + q_2 + q_3 = n$, with $q_1, q_2, q_3 \in Q$.

Proof. Let p_i be the i-th prime and s, t two large numbers. We construct an infinite set Q of primes by

$$\begin{aligned} Q = \{&p_1, p_2, \ldots, p_s,\ p_{3s}, p_{3s+1}, \ldots, p_{3ts},\ p_{9ts}, p_{9ts+1}, \ldots, p_{9t^2s}, \ldots, \\ &p_{t^{r-1}3^r s}, p_{t^{r-1}3^r s+1}, \ldots, p_{t^r 3^r s},\ p_{t^r 3^{r+1} s}, p_{t^r 3^{r+1} s+1}, \ldots, p_{t^{r+1}3^{r+1} s}, \ldots\}. \end{aligned}$$

If $x > 355991$, then from the prime number theorem (See Theorem 1.10 in [7]), we have

$$\frac{x}{\log x}\Big(1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}\Big) \le \pi(x) \le \frac{x}{\log x}\Big(1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}\Big). \tag{3}$$

Hence, we have

$$\begin{aligned}
&\lim_{x\to\infty} \sup \frac{\log x}{x} \left|\{q_i :\ q_i \le x, q_i \in Q\}\right| \\
&= \lim_{x\to\infty} \frac{\big((t-1)\,t^r \cdot 3^{r+1}+1\big)+\big((t-1)\,t^{r-1}\cdot 3^r+1\big)+\cdots}{t^{r+1}\cdot 3^{r+1}} \\
&= \lim_{x\to\infty} \frac{t-1}{t}\Big(1+\frac{1}{3t}+\frac{1}{3^2t^2}+\cdots\Big).
\end{aligned}$$

If $t \to \infty$, then

$$\frac{t-1}{t}\Big(1+\frac{1}{3t}+\frac{1}{3^2t^2}+\cdots\Big)=\frac{3(t-1)}{3t-1}=1-\varepsilon,$$

where ε is a small positive constant. So formula (2) holds.

For $k > 2$, we have (see page 11 of [7])

$$k(\log k+\log\log k-1)<p_k<k(\log k+\log\log k). \tag{4}$$

We can take an odd integer n such that

$$3p_{t^r\cdot 3^r s}<n<p_{t^r\cdot 3^{r+1}s}. \tag{5}$$

If n has the form $q_1+q_2+q_3=n$ with $q_1,q_2,q_3\in Q$, then

$$p_1\le q_1,\ q_2,\ q_3\le p_{t^r\cdot 3^r s},$$

and then

$$q_1+q_2+q_3\le 3\cdot p_{t^r\cdot 3^r s}<n.$$

This is impossible. So the integer n cannot have the form $q_1+q_2+q_3=n$. Therefore, from (4) we obtain

$$\begin{aligned}
&p_{t^r\cdot 3^{r+1}s}-3\,p_{t^r\cdot 3^r s} \\
&> t^r\cdot 3^{r+1}s\big(\log(t^r\cdot 3^{r+1}s)+\log\log(t^r\cdot 3^{r+1}s)-1\big) \\
&\quad -3\cdot t^r\cdot 3^r s\big(\log(t^r\cdot 3^r s)+\log\log(t^r\cdot 3^r s)\big) \\
&= t^r\cdot 3^{r+1}s\Big(\log 3-\log\Big(\frac{\log(t^r\cdot 3^{r+1}s)}{\log(t^r\cdot 3^r s)}\Big)-1\Big).
\end{aligned} \tag{6}$$

If $r>10$, $s>10$, and $t>2$, then we have $\frac{\log(t^r\cdot 3^{r+1}s)}{\log(t^r\cdot 3^r s)}<1.01$. So inequality (6) implies

$$p_{2^r\cdot 3^{r+1}s}-3p_{2^r\cdot 3^r s}>t^r\cdot 3^{r+1}s\big(\log 3-\log 1.01-1\big)>0.088\cdot t^r\cdot 3^{r+1}s.$$

Thus, there are infinitely many odd integers n that cannot have the form $q_1+q_2+q_3=n$ with $q_1,q_2,q_3\in Q$. This completes the proof of Theorem 2.1. □

In [25], Sárközy set the following similar problem (see Problem 68 in [25]).

Problem B. Is it true that if Q is an infinite set of primes such that

$$\lim_{x\to\infty}\inf \frac{1}{\log\log x}\sum_{q\in Q,q\le x}\frac{1}{q}>\frac{1}{2}, \tag{7}$$

then every large odd integer can be represented as the sum of three positive integers each composed of primes belonging to Q?

From [26], the answer to Question B is negative also, and $\frac{1}{2}$ must be replaced by $\frac{5}{8}$. Similarly, using the same method, according to Mertens formula (see page 34 of [14]):

$$\prod_{p\le x}\frac{1}{p}=\log\log x+\beta+O\Big(\frac{1}{\log x}\Big),$$

where β is a constant, we get the following Theorem.

Theorem 2.2. *For any small positive constant ε, there exists an infinite set of primes $Q=\{q_1,q_2,\ldots\}$ such that*

$$\limsup_{x\to\infty}\frac{1}{\log\log x}\sum_{q\in Q,q\le x}\frac{1}{q}=1-\varepsilon \tag{8}$$

and there are infinitely many odd integers n that cannot be represented in the form $q_1+q_2+q_3=n$ with $q_1,q_2,q_3\in Q$.

Remark 2.3. The set of primes $Q=\{q_1,q_2,\cdots\}$ in Theorem 2.1 satisfy

$$\underline{d}_{\mathbb{P}}(Q)=\frac{1}{3},\ \overline{d}_{\mathbb{P}}(Q)=1-\varepsilon. \tag{9}$$

A natural question is: Does there exist a set of primes Q such that $\underline{d}_{\mathbb{P}}(Q)=\frac{5}{8}$, $\overline{d}_{\mathbb{P}}(Q)=1-\varepsilon$ and that there are infinitely many odd integers n which cannot be represented in the form $q_1+q_2+q_3=n$, with $q_1,q_2,q_3\in Q$?

3. The main conclusion

In this section, we will proved our main result. First, we state Vinogradov's Three Primes Theorem.

Lemma 3.1. *For every sufficiently large odd integer N, let $r(N)$ denote the number of the solutions of the equation*

$$N=p_1+p_2+p_3,$$

where $p_1<p_2<p_3$ are primes. Then

$$r(N)=\frac{1}{2}\big(1+o(1)\big)C(N)\frac{N^2}{\log^3 N}, \tag{10}$$

with

$$C(N)=\prod_{p|N}\Big(1-\frac{1}{(p-1)^2}\Big)\prod_{p\nmid N}\Big(1+\frac{1}{(p-1)^3}\Big).$$

Proof. See [28] or [27]. □

Now we will prove the following result.

Theorem 3.2. *Let $Q = \{q_1, q_2, \ldots\}$ be an infinite set of primes such that*

$$\big|\{q_i : \ q_i \le x, q_i \in Q\}\big| > \frac{x}{\log x}\Big(1 - \frac{1}{3\sqrt{\log x}}\Big), \tag{11}$$

and $r_1(n)$ denote the number of the solutions of the equation $n = q_1 + q_2 + q_3$ with $q_1, q_2, q_3 \in Q$. Then

$$r_1(n) > \frac{0.067\, n^2}{\log^3 n}. \tag{12}$$

Proof. Let $\mathbb{P} = Q \cup S$ and $Q \cap S = \emptyset$. Assume that x is a large integer. We write

$$S_x = \{q_i : \ q_i \le x, q_i \in S\}.$$

According to the Prime Number Theorem, to calculate the maximum number of elements in the set S_x, we have

$$|S_x| \le \frac{x}{\log x}\Big(1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}\Big) - \frac{x}{\log x}\Big(1 - \frac{1}{3\sqrt{\log x}}\Big) < \frac{1.001x}{3\log^{1.5} x}. \tag{13}$$

On the other hand, a lager odd integer n can be represented by

$$n = q_1' + q_2' + q_3', \ q_1', q_2', q_3' \in \mathbb{P}.$$

From Lemma 3.1, we see that the number of solutions of the above equation is

$$r(n) > \frac{0.999}{2}\, C(n) \frac{n^2}{\log^3 n}. \tag{14}$$

As

$$C(n) = \prod_{p|n}\Big(1 - \frac{1}{(p-1)^2}\Big)\prod_{p\nmid n}\Big(1 + \frac{1}{(p-1)^3}\Big) > \frac{3}{4}\cdot\frac{15}{16}\prod_{n\ge 6}\Big(1 - \frac{1}{n^2}\Big) > 0.585,$$

then

$$r(n) > \frac{0.292\, n^2}{\log^3 n}. \tag{15}$$

If $q_1', q_2', q_3' \in S$ and as $q_1', q_2', q_3' < n$, then using inequalities (3) and (13), one can determine the possible number of the values of q_1'. This is at most $\frac{1.001n}{3\log^{1.5} n}$. Similarly, the possible number of the values of q_2' is also at most $\frac{1.001n}{3\log^{1.5} n}$. Since $q_3' = n - q_1' - q_2'$, then the number of equations $n = q_1' + q_2' + q_3'$ is at most

$$\frac{1.001n}{3\log^{1.5} n}\cdot\frac{1.001n}{3\log^{1.5} n} < \frac{1.003n^2}{9\log^3 n} < \frac{0.112\, n^2}{\log^3 n}.$$

If $q_1', q_2' \in S$, $q_3' \in Q$, using the same method, one can see that the number of equations $n = q_1 + q_2 + q_3$ is at most $\frac{0.112\, n^2}{\log^3 n}$.

If $q_1' \in S$, and $q_2', q_3' \in Q$, then $q_2' + q_3'$ is even. According to Bombieri and Davenport theorem [3] (a large even number is the sum of two primes), if p_1, p_2 are two primes such that $p_1 < p_2$ and N is even, we have

$$\theta(N) = \sum_{p_1+p_2=N} 1 < 8\, D(N)\, \frac{N}{\log^2 N}, \tag{16}$$

where

$$D(N) = \prod_{p>2}\Big(1 - \frac{1}{(p-1)^2}\Big) \prod_{p\,|N,\, p>2} \frac{p-2}{p-1} < 1.$$

Then, we get

$$\sum_{q_2'+q_3'=n-q_1'} 1 \;<\; \frac{8(n-q_1')}{\log^2(n-q_1')} < \frac{8n}{\log^2 n}. \tag{17}$$

Since the possible number of the values of q_1 is at most $\frac{1.001n}{3\log^{1.5} n}$, then the number of equations $n = q_1 + q_2 + q_3$ is at most

$$\frac{8n}{\log^2 n} \cdot \frac{1.001n}{3\log^{1.5} n} = \frac{8.008n^2}{3\log^{3.5} n} < \frac{0.001n^2}{\log^3 n}.$$

So from (15) we obtain

$$r_1(n) > \frac{0.292\, n^2}{\log^3 n} - \frac{0.112\, n^2}{\log^3 n} - \frac{0.112\, n^2}{\log^3 n} - \frac{0.001\, n^2}{\log^3 n} = \frac{0.067\, n^2}{\log^3 n}. \tag{18}$$

Therefore, there are infinitely many odd integers n that can be represented into the form $q_1 + q_2 + q_3 = n$ with $q_1, q_2, q_3 \in Q$. This completes the proof of Theorem 3.2. □

4. A question

A natural question is the following.

Problem C. Let $r_1(n)$ denote the number of the solutions of $n = q_1 + q_2 + q_3$ with $q_1, q_2, q_3 \in Q$. Determine the minimum value of μ such that if Q is an infinite set of primes and

$$\big|\{q :\ q \le x, q \in Q\}\big| > \frac{x}{\log x}\Big(1 - \frac{1}{\log^{\mu} x}\Big), \tag{19}$$

then $r_1(n) \gg \frac{n^2}{\log^3 n}$.

Obviously, Theorem 3.2 shows that $\mu \le \frac{1}{2}$. We conjecture that

$$\mu \to 0.$$

We also conjecture that when Q is an infinite set of primes such that

$$\big|\{q :\ q \le x, q \in Q\}\big| > \frac{x}{\log x}\Big(1 - \frac{1}{\log\log x}\Big),$$

then Vinogradov's Three Primes Theorem holds and $r_1(n) \gg \frac{n^2}{\log^3 n}$.

References

[1] R. C. Baker and G. Harman. The three primes theorem with almost equal summands. *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.*, 356(1738):763–780, 1998.

[2] A. Balog and J. Friedlander. A hybrid of theorems of Vinogradov and Piatetski-Shapiro. *Pacific J. Math.*, 156(1):45–62, 1992.

[3] E. Bombieri and H. Davenport. Small differences between prime numbers. *Proc. Roy. Soc. Ser. A*, 293:1–18, 1966.

[4] K. G. Borodzkin. On I. M. Vinogradov's constant. In *Proc. 3rd All-Union Math. Conf., vol.1.* Izdat. Akad. Nauk SSSR, 1956.

[5] J. R. Chen and T. Z. Wang. On the Goldbach problem. *Acta Math. Sinica*, 32(5):702–718, 1989.

[6] J.-M. Deshouillers, G. Effinger, H. te Riele, and D. Zinoviev. A complete Vinogradov 3-primes theorem under the Riemann hypothesis. *Electron. Res. Announc. Amer. Math. Soc.*, 3:99–104, 1997.

[7] P. Dusart. *Autour de la fonction qui compte le nombre de nombres premiers.* Phd. thesis, Univ. de Limoges, 1998.

[8] B. Green. Roth's theorem in the primes. *Ann. of Math. (2)*, 161(3):1609–1636, 2005.

[9] D. R. Heath-Brown. The Pjateckiĭ-Šapiro prime number theorem. *J. Number Theory*, 16(2):242–266, 1983.

[10] H. A. Helfgott. Major arcs for goldbach's problem. arXiv:1305.2897.

[11] H. A. Helfgott. The ternary Goldbach problem. In *Proceedings of the International Congress of Mathematicians (Seoul, Korea, 2014).*

[12] H. A. Helfgott. The ternary Goldbach conjecture. *Gac. R. Soc. Mat. Esp.*, 16(4):709–726, 2013.

[13] H. A. Helfgott. La conjecture de Goldbach ternaire. *Gaz. Math.*, (140):5–18, 2014. Translated by Margaret Bilu, revised by the author.

[14] H. Iwaniec and E. Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[15] C. Jia. Three primes theorem in a short interval. VII. *Acta Math. Sinica (N.S.)*, 10(4):369–387, 1994. A Chinese summary appears in Acta. Math. Sinica **3**8 (1995), no. 5, 719.

[16] C. Jia. Three primes theorem in a short interval. VII. *Acta Math. Sinica (N.S.)*, 10(4):369–387, 1994. A Chinese summary appears in Acta. Math. Sinica **3**8 (1995), no. 5, 719.

[17] C. H. Jia. On Pjateckiĭ-Šapiro prime number theorem. II. *Sci. China Ser. A*, 36(8):913–926, 1993.

[18] C. H. Jia. On the Piatetski-Shapiro-Vinogradov theorem. *Acta Arith.*, 73(1):1–28, 1995.

[19] G. Kolesnik. Primes of the form $[n^c]$. *Pacific J. Math.*, 118(2):437–447, 1985.

[20] H. Li and H. Pan. A density version of Vinogradov's three primes theorem. *Forum Math.*, 22(4):699–714, 2010.

[21] H. Q. Liu and J. Rivat. On the Pjateckiĭ-Šapiro prime number theorem. *Bull. London Math. Soc.*, 24(2):143–147, 1992.

[22] C. D. Pan and C. B. Pan. On estimations of trigonometric sums over primes in short intervals. III. *Chinese Ann. Math. Ser. B*, 11(2):138–147, 1990. A Chinese summary appears in Chinese Ann. Math. Ser. A **11** (1990), no. 2, 255.

[23] I. I. Pyateckiĭ-Šapiro. On the distribution of prime numbers in sequences of the form $[f(n)]$. *Mat. Sbornik N.S.*, 33(75):559–566, 1953.

[24] B. Rosser. The n-th Prime is Greater than $n \log n$. *Proc. London Math. Soc.*, S2-45(1):21.

[25] A. Sárközy. Unsolved problems in number theory. *Period. Math. Hungar.*, 42(1-2):17–35, 2001.
[26] X. Shao. A density version of the Vinogradov three primes theorem. *Duke Math. J.*, 163(3):489–512, 2014.
[27] I. M. Vinogradov. *The method of trigonometrical sums in the theory of numbers.* Interscience Publishers, London and New York., 1954. Translated, revised and annotated by K. F. Roth and Anne Davenport.
[28] I. M. Vinogradow. Representation of an odd number as a sum of three primes. *C. R. (Dokl.) Acad. Sci. URSS, n. Ser.*, 15:169–172, 1937.
[29] E. Wirsing. Thin subbases. *Analysis*, 6(2-3):285–308, 1986.
[30] T. Zhan. On the representation of large odd integer as a sum of three almost equal primes. *Acta Math. Sinica (N.S.)*, 7(3):259–272, 1991. A Chinese summary appears in Acta Math. Sinica **3**5 (1992), no. 4, 575.

Received December 23, 2014.

Shichun Yang (corresponding author),
Department of Mathematics,
Aba Teachers University,
Wenchuan, Sichuan,
623000, P. R. China
E-mail address: ysc1020@sina.com

Alain Togbé,
Mathematics Department,
Purdue University North Central,
1401 S, U.S. 421, Westville IN 46391 USA
E-mail address: atogbe@pnc.edu