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THE INTERSECTION GRAPH OF FINITE COMMUTATIVE PRINCIPAL IDEAL RINGS

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Abstract. In this article we consider the intersection graph $G(R)$ of non-trivial proper ideals of a finite commutative principal ideal ring R with unity 1. Two distinct ideals are adjacent if they have non-trivial intersection. We characterize when the intersection graph is complete, bipartite, planar, Eulerian or Hamiltonian. We also find a formula to calculate the number of ideals in each ring and the degree of each ideal. We apply our results to the intersection graph of the ring of Gaussian integers modulo n.

1. Introduction

All rings are assumed to be finite commutative principal ideal rings with unity 1.

Let $F = \{A_j : j \in J\}$ be a family of non-empty sets. The intersection graph $G(F)$ defined on F is a simple graph whose vertices are A_j, $j \in J$ and two vertices A_j and A_i are adjacent if $A_j \neq A_i$ and $A_j \cap A_i \neq \phi$. It is interesting to study the intersection graphs $G(F)$ when the members of F have an algebraic structure. There were studies for the intersection graph for semigroups, subgroups of a finite group, and ideals of a ring, see [5]. In [2], the intersection graph $G(R)$ of ideals of a ring R was studied. In this graph two proper non-trivial ideals I and J are adjacent if $I \neq J$ and $I \cap J \neq \{0\}$. The authors studied when $G(R)$ is disconnected and when it is complete for arbitrary ring R. They characterized when $G(\mathbb{Z}_n)$ is connected, complete, planar and when it has cycles, they also characterized when $G(\mathbb{Z}_n)$ is Eulerian or Hamiltonian. In [5], the authors continued the investigation of the intersection graph for ideals, and gave characterization for $G(R)$ to be planar, where R is a commutative ring with unity 1.

In this article we consider the intersection graph $G(R)$ of non-trivial proper ideals of a finite commutative principal ideal ring R with unity 1. We generalize

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all results of [2], and we also calculate the number of ideals in each such rings, and find the degree of each ideal. Finally, we apply our results to the intersection graph $G(\mathbb{Z}_n[i])$.

For distinct vertices x and y of a graph Γ, let $d(x, y)$ be the length of a shortest path from x to y, and let $\text{diam}(\Gamma) = \sup\{d(x, y) : x, y$ are distinct vertices in $\Gamma\}$. For each vertex x in Γ, let $\deg(x)$ be the number of vertices adjacent to x and let $N(x)$ be the set of vertices adjacent to x in Γ. For any undefined terms, the reader may contact [6].

Let R be a finite local ring with maximal ideal M. Then there exists $n \in \mathbb{N}$ such that $M^n = \{0\}$ but $M^{n-1} \neq \{0\}$. In this case we say that $\text{Nilpotency}(R) = n$. Note that if R is a field, then $\text{Nilpotency}(R) = 1$. If $R = \prod_{j=1}^{m} R_j$, where R_j is a ring with identity for each j, then any ideal I in R is of the form $\prod_{j=1}^{m} I_j$ with I_j is an ideal in R_j. Hence if $\{R_j : j = 1, 2, \ldots, m\}$ is a family of finite commutative principal ideal rings, then so is $\prod_{j=1}^{m} R_j$.

The following lemma is well known and it can be found in many classical books. For instance see the proof of Proposition 8.8 in [1].

Lemma 1. *Let R be a finite local principal ideal ring with Nilpotency$(R) = n$. Then R has $n - 1$ non-trivial proper ideals.*

Theorem 2. *Let R be a finite non-local ring such that $R = \prod_{j=1}^{m} R_j$, where R_j is a local principal ideal ring with Nilpotency$(R_j) = n_j$ for $j = 1, \ldots, m$. Then R has $(\prod_{j=1}^{m}(n_j + 1)) - 2$ non-trivial proper ideals.*

Proof. The result follows immediately, since any ideal I in R is of the form $\prod_{j=1}^{m} I_j$, where I_j is an ideal in R_j, and we subtract 2 to eliminate the ideals $\{0\}$ and R. $\square$

2. When is $G(R)$ complete?

Definition 3. A graph Γ is called *complete* if any two vertices in Γ are adjacent. If a complete graph Γ has n vertices, then it is denoted by K_n.

It is clear that in a local principal ideal ring R with $\text{Nilpotency}(R) = n$, the intersection of any non-trivial ideals is non-trivial, and so $G(R)$ is the complete graph K_{n-1}. If R is non-local principal ideal ring, then $R = R_1 \times R_2$ with $R_j \neq \{0\}$ for $j = 1, 2$. Let $I_1 = R_1 \times \{0\}$ and $I_2 = \{0\} \times R_2$. Then $I_1 \cap I_2 = \{0\}$ and $G(R)$ is not complete. Hence $G(R)$ is complete if and only if R is local.

We now calculate the diameter of the intersection graph. If R is local, then $\text{diam}(G(R)) = 1$. If R is a product of two fields, then $\text{diam}(G(R)) = \infty$, since

$G(R) = 2K_1$. If $R = F \times R_2$, where F is a field and R_2 is not, then the possible forms of the ideals are: $F \times \{0\}, F \times J, \{0\} \times R_2$ and $\{0\} \times J$, where J is a non-trivial ideal of R_2, and a simple argument shows that $\operatorname{diam}(G(R)) = 2$. If $R = R_1 \times R_2$, where R_1 and R_2 are non-fields, then the possible forms of the ideals in R are: $R_1 \times \{0\}, R_1 \times J, \{0\} \times R_2, \{0\} \times J, I \times R_2, I \times \{0\}$, and $I \times J$, where I and J are non-trivial proper ideals in R_1 and R_2, respectively. Again $\operatorname{diam}(G(R)) = 2$.

3. When is $G(R)$ Eulerian?

Definition 4. A graph Γ is called *Eulerian* if there exists a closed trail containing every edge of Γ.

The following is an easier criterion for a graph to be Eulerian, which was proved by Euler.

Proposition 5 (Euler). *A connected graph Γ is Eulerian if and only if the degree of each vertex of Γ is even.*

In this article we will consider a discrete graph to be Eulerian, and if a graph Γ is the disjoint union of the subgraphs A and B with A is discrete, then Γ is Eulerian if and only if B is.

Let R be a local principal ideal ring with $\text{Nilpotency}(R) = n$. Then $G(R) = K_{n-1}$ and for each vertex I in $G(R)$, $\deg(I) = n - 2$. So, we have the following result:

Theorem 6. *Let R be a finite local principal ideal ring. Then $G(R)$ is Eulerian if and only if Nilpotency(R) is even.*

Let R be a non-local ring with $R = \prod_{j=1}^{m} R_j$, where R_j is a local ring with $\text{Nilpotency}(R_j) = n_j$ for $j = 1, \ldots, m$. Any ideal I in R is of the form $I = \prod_{j=1}^{m} I_j$, where I_j is an ideal in R_j. Define the function $\pi_j \colon G(R) \to \{0, 1\}$ such that

$$\pi_j(I) = \begin{cases} 0, & I_j \neq \{0\}, \\ 1, & I_j = \{0\}. \end{cases}$$

Then number of proper non-trivial ideals J such that $I \cap J = \{0\}$ is $\prod_{j=1}^{m} (n_j + 1)^{\pi_j(I)} - 1$, and hence

$$\deg(I) = \prod_{j=1}^{m} (n_j + 1) - \prod_{j=1}^{m} (n_j + 1)^{\pi_j(I)} - 2.$$

Theorem 7. *If R is a finite non-local principal ideal ring, then $G(R)$ is Eulerian if and only if R is a product of fields or a product of local rings each of which has even nilpotency number.*

Proof. It is clear that $G(R)$ is Eulerian if R is a product of fields or a product of local rings each of which has even nilpotency number, since in these cases the degree of each vertex is even. Now, let $R = \prod_{j=1}^{m} R_j$, where R_j is a local ring with Nilpotency$(R_j) = n_j = 2m_j + 1$, for each j, and Nilpotency$(R_1) > 1$. Consider the ideal $I = \prod_{j=1}^{m} I_j$, with $I_1 \notin \{\{0\}, R_1\}$ and $I_j = R_j$ for $j = 2, \ldots, m$. Then I is adjacent to every other vertex in $G(R)$ and $\deg(I) = \prod_{j=1}^{m} (n_j + 1) - 3$ which is an odd number, hence $G(R)$ is not Eulerian. Finally, assume that $R = \prod_{j=1}^{m} R_j$, where R_j is a local ring with Nilpotency$(R_j) = n_j$, n_1 is even, while n_2 is odd. Consider the ideal, $I = \prod_{j=1}^{m} I_j$ with $I_1 = \{0\}$ and $I_j = R_j$ otherwise. Then $\deg(I) = \prod_{j=1}^{m} (n_j + 1) - n_1 - 3$, which is an odd number, hence $G(R)$ is not Eulerian. □

4. When is $G(R)$ Hamiltonian?

Definition 8. A *Hamiltonian cycle* of a graph Γ is a cycle that contains every vertex of Γ. A graph is *Hamiltonian* if it contains a Hamiltonian cycle.

The following proposition is a well known fact, see for example [6, page 36].

Proposition 9. *If Γ is a graph with n (≥ 3) vertices, and $\deg(v) \geq \frac{n}{2}$ for each vertex v, then Γ is Hamiltonian.*

Let R be a finite local principal ideal ring. Then it is clear that $G(R)$ is Hamiltonian if and only if Nilpotency$(R) > 3$.

The following two lemmas are easy to prove and are left as an exercise for the reader.

Lemma 10. *If $R = F_1 \times F_2$, where F_1 and F_2 are fields, or $R = F \times R_2$, where F is a field and R_2 is a finite local principal ideal ring with Nilpotency$(R_2) = 2$, then $G(R)$ is not Hamiltonian.*

Lemma 11. *If $R = F_1 \times F_2 \times F_3$, where F_1, F_2 and F_3 are finite fields or $R = F \times R_2$, where F is a finite field and R_2 is a finite local principal ideal ring with Nilpotency$(R_2) > 2$, or $R = F_1 \times F_2 \times R_3$, where F_1 and F_2 are finite fields, and R_3 is a finite local principal ideal ring with Nilpotency$(R_3) = 2$, then $G(R)$ is Hamiltonian.*

Theorem 12. *If $R = \prod_{j=1}^{m} R_j$ with R_j is a finite local principal ideal ring which is not a field for each j, then $G(R)$ is Hamiltonian.*

Proof. Assume that $1 < \text{Nilpotency}(R_1) = n_1 \leq n_j = \text{Nilpotency}(R_j)$ for each j. Let $I = R_1 \times \{0\} \times \{0\} \times \cdots \times \{0\}$. Then

$$\deg(I) = \prod_{j=1}^{m}(n_j + 1) - \prod_{j=2}^{m}(n_j + 1) - 2,$$

and so I is a vertex with least degree in $G(R)$. So for each proper non-trivial ideal J of R,

$$\begin{aligned}\deg(J) \geq \deg(I) &= \left(\prod_{j=2}^{m}(n_j + 1)\right)(n_1 + 1 - 1) - 2 \\ &= n_1 \prod_{j=2}^{m}(n_j + 1) - 2 \geq \frac{(n_1 + 1)}{2} \prod_{j=2}^{m}(n_j + 1) - 1 \\ &= \frac{\prod_{j=1}^{m}(n_j + 1) - 2}{2} = \frac{|V(G(R))|}{2},\end{aligned}$$

where $V(G(R))$ is the set of vertices in $G(R)$, so it follows by Proposition 9 that $G(R)$ is Hamiltonian. □

Theorem 13. *If $R = F \times K$, where F is a finite field and K is a finite principal ideal ring such that $G(K)$ is Hamiltonian, then $G(R)$ is Hamiltonian.*

Proof. Let $I_1, \ldots, I_n$ be all distinct proper non-trivial ideals of K and assume that $I_1 - I_2 - \cdots - I_n - I_1$ is a Hamiltonian cycle in $G(R)$. The ideals of R are $F \times \{0\}, \{0) \times K, F \times I_j, \{0\} \times I_j$ for $j = 1, 2, \ldots, n$. Thus we have the following Hamiltonian cycle: $(F \times I_1) - (\{0\} \times K) - (F \times I_2) - (F \times \{0\}) - (F \times I_3) - \cdots - (F \times I_n) - (\{0\} \times I_1) - (\{0\} \times I_2) - \cdots - (\{0\} \times I_n) - (F \times I_1)$. Thus $G(R)$ is Hamiltonian. □

Theorem 14. *Let R be a finite non-local principal ideal ring. Then $G(R)$ is Hamiltonian if and only if R has more than 4 proper non-trivial ideals.*

Proof. Assume $4 < n =$ number of proper non-trivial ideals of R. Assume $R = \prod_{j=1}^{m} R_j$, where R_j is a local ring with $\text{Nilpotency}(R_j) = n_j$. Then $n = \prod_{j=1}^{m}(n_j + 1) - 2$, and so $6 < n + 2 = \prod_{j=1}^{m}(n_j + 1)$. Now, the result follows by Lemma 11 and Theorems 12 and 13. □

5. When is $G(R)$ bipartite or planar?

Definition 15. A graph Γ is called *bipartite* if the vertex set of Γ can be split into two disjoint sets A and B so that each edge of Γ joins a vertex of A to a vertex of B.

It is known that a simple graph Γ is bipartite if and only if it has no odd cycles, see [2].

Lemma 16. *Let R be a finite local principal ideal ring. Then $G(R)$ is bipartite if and only if*

$$\textit{Nilpotency}(R) = 2 \text{ or } 3.$$

Proof. In these cases $G(R)$ is K_1 or K_2, which are bipartite, otherwise $G(R)$ has a triangle and so it is not bipartite. □

Lemma 17. *Let R be a finite non-local principal ideal ring. Then $G(R)$ is never a bipartite graph*

Proof. If R is a product of two fields, then it is an edgeless graph. So assume that $R = R_1 \times R_2$ with R_1 is not a field and Nilpotency$(R_1) = n \geq 2$. The ideals are $R_1 \times \{0\}$, $M \times \{0\}$, and $M \times R_2$ form a triangle, where M is a maximal ideal in R_1, and so $G(R)$ is not bipartite. □

According to the previous work one can calculate the girth for $G(R)$. Recall that the *girth* of a graph Γ, denoted by $g(\Gamma)$ is the length of a shortest cycle in Γ, ($g(\Gamma) = \infty$ if Γ contains no cycles). If R is a finite local principal ideal ring, with Nilpotency$(R) = n \geq 4$, then $G(R) = K_{n-1}$and so, $g(G(R)) = 3$. If Nilpotency$(R) \leq 3$, then $G(R)$ has no cycles. So assume now that R is a non-local principal ideal ring. If R is a product of two fields, then $G(R)$ has no cycles. If $R = R_1 \times R_2$ with R_1 is not a field and Nilpotency$(R_1) \geq 2$, then the ideals $R_1 \times \{0\}$, $M \times \{0\}$, and $M \times R_2$ form a triangle, where M is a maximal ideal in R_1, and so $g(G(R)) = 3$. Hence we have the following theorem.

Theorem 18. *If R is finite local principal ideal ring with Nilpotency$(R) \geq 4$, or R is a product of two finite non-trivial principal ideal rings with at least one of them is not a field, then $g(G(R)) = 3$.*

A graph is called *planar* if it can be drawn in the plane so that its edges intersect only at their ends. Two graphs are said to be *homeomorphic* if both can be obtained from the same graph by inserting new vertices of degree 2 into its edges.

Proposition 19 (Kuratowski)**.** *A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.*

Theorem 20. *If R is a finite local principal ideal ring, then $G(R)$ is planar if and only if $1 <$ Nilpotency$(R) < 6$.*

Proof. The result follows immediately since if Nilpotency$(R) \geq 6$, then $G(R)$ contains the subgraph K_5, while if $1 <$ Nilpotency$(R) < 6$, then $G(R)$ has at most 4 vertices, and so it contains on subgraph homeomorphic to K_5 or $K_{3,3}$. □

Theorem 21. *Let $R = R_1 \times R_2$ be a finite non-local principal ideal ring. Then $G(R)$ is planar if and only if R_1 and R_2 are fields, or R_1 is a field and Nilpotency$(R_2) = 2$.*

Proof. If R is a product of two fields, then $G(R) = 2K_1$ is planar. If R_1 is a field and Nilpotency$(R_2) = 2$, then the vertices of $G(R)$ are $\{R_1 \times \{0\}, R_1 \times M, \{0\} \times R_2, \{0\} \times M\}$ which form a planar graph. If R_1 and R_2 are non-fields, then the subgraph consisting of $\{R_1 \times M_2, R_1 \times \{0\}, M_1 \times R_2, M_1 \times M_2, M_1 \times \{0\}\}$ is the graph K_5, so $G(R)$ cannot be planar. If R_1 is a field and Nilpotency$(R_2) > 2$, then the subgraph $\{R_1 \times M, R_1 \times M^2, \{0\} \times R_2, \{0\} \times M, \{0\} \times M^2\}$ is K_5, so $G(R)$ cannot be planar. □

6. Gaussian Integers Modulo n

The set of all complex numbers $a + ib$, where a and b are integers, forms a Euclidean domain with the usual complex number operations. This domain will be denoted by $\mathbb{Z}[i]$ and well be called the ring of Gaussian integers. If t is a prime integer, then $t = 2$ or $t \equiv 1 \pmod 4$ or $t \equiv 3 \pmod 4$. In this article p and p_i will denote prime integers that are congruent to $1 \pmod 4$, while q and q_i will denote prime integers that are congruent to $3 \pmod 4$. If $p \equiv 1 \pmod 4$, then there are integers a, b such that $p = a^2 + b^2$.

Let n be a natural number and let $< n >$ be the principal ideal generated by n in $\mathbb{Z}[i]$ and let $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ be the ring of integers modulo n. Then the factor ring $\mathbb{Z}[i]/< n >$ is isomorphic to $\mathbb{Z}_n[i] = \{\overline{a} + i\,\overline{b} : \overline{a}, \overline{b} \in \mathbb{Z}_n\}$, which implies that $\mathbb{Z}_n[i]$ is a principal ideal ring. The ring $\mathbb{Z}_n[i]$ is called the ring of *Gaussian integers modulo n*.

If $n = \prod_{j=1}^{s} t_j^{k_j}$, the prime power decomposition of the positive integer n, then $\mathbb{Z}_n[i]$ is the direct product of the rings $\mathbb{Z}_{t_j^{k_j}}[i]$. Also if $m = t^k$ for some prime t and positive integer k, then $\mathbb{Z}_m[i]$ is local if and only if $t = 2$ or $t \equiv 3 \pmod 4$. In $\mathbb{Z}_{2^m}[i]$, the ideal $< \overline{1} + i >$ is the maximal ideal, and $< \overline{1} + i >^{2m} = \{0\}$. In $\mathbb{Z}_{q^m}[i]$, $\mathbb{Z}_q[i]$ is a field, while $\mathbb{Z}_{q^m}[i]$ is local with maximal ideal $< \overline{q} >$, in this case $< \overline{q} >^m = \{\overline{0}\}$. If $p \equiv 1 \pmod 4$, then $p = a^2 + b^2$, and $\mathbb{Z}_{p^m}[i] \simeq \mathbb{Z}[i]/< a + ib >^m \times \mathbb{Z}[i]/< a - ib >^m$, in this case $\mathbb{Z}[i]/< a + ib >^m$ and $\mathbb{Z}[i]/< a - ib >^m$ are local rings with maximal ideals $< a + ib > /< a + ib >^m$ and $< a - ib > /< a - ib >^m$, respectively. For more details on Gaussian integers and Gaussian integers modulo n, the reader may refer to [3] and [4].

It was shown in [2], Corollary 2.7, that for a commutative ring R with unity, the graph $G(R)$ is disconnected if and only if R is a product of two fields. Hence we conclude immediately that $G(\mathbb{Z}_n[i])$ is disconnected if and only if $n = p$ or $n = q_1 q_2$.

Theorem 22. *The intersection graph $G(\mathbb{Z}_n[i])$ is complete if and only if $n = q^m$, where $m > 1$ or $n = 2^m$ for any $m \geq 1$.*

Since Nilpotency($\mathbb{Z}_{2^m}[i]$) $= 2m$, we have $G(\mathbb{Z}_{2^m}[i])$ is Eulerian for any m and $G(\mathbb{Z}_{q^m}[i])$ is Eulerian if and only if m is even.

Theorem 23. *The intersection graph $G(\mathbb{Z}_n[i])$ is Eulerian if and only if $n = 2^m,\ m \geq 1, n = q^{2m},\ m \geq 1, n = \prod_{j=1}^{m} t_j$, where the t_j's are distinct odd primes, or $n = 2^k \prod_{j=1}^{m} t_j^{2m_j}$, where the t_j's are distinct odd primes and k is any non-negative integer.*

Theorem 24. *The intersection graph $G(\mathbb{Z}_n[i])$ is Hamiltonian if and only if $n \neq p, q, q_1q_2, q_1q_2^2, 2q$.*

Theorem 25. *The intersection graph $G(\mathbb{Z}_n[i])$ is bipartite if and only if $n = q^3$.*

Theorem 26. *If $n \neq 1, 2, p, q, q^2, q^3, q_1q_2$, then $g(G(\mathbb{Z}_n[i]) = 3$.*

Theorem 27. *The intersection graph $G(\mathbb{Z}_n[i])$ is planar if and only if $n = 2, 4, p, 2q, qp, q_1q_2, q_1q_2^2, q_1q_2q_3, q^m$ with $1 < m < 6$.*

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