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ON THE COHOMOLOGY OF SOME CR–FOLIATIONS ON THE TANGENT BUNDLE OF A FINSLER SPACE

CRISTIAN IDA AND PAUL POPESCU

Dedicated to Professor Lajos Tamássy on the occasion of his 90th birthday

Abstract. In this paper we identify some natural CR-foliations on the tangent bundle of a Finsler space and next, some cohomological aspects of these CR-foliations in relation with corresponding results for CR-submanifolds of almost Kähler manifolds are studied.

1. Introduction and preliminaries

1.1. **Introduction.** The study of CR-submanifolds of Kähler manifolds was initiated in [2] and [9]. Some aspects concerning to cohomology of CR-submanifolds of Kähler manifolds was studied in [8]. In this direction the cohomology of such submanifolds in a locally conformal Kähler manifolds was studied in [10], [14] and in the case of locally product manifolds a similar study is given in [19].

On the other hand in the paper [4] Bejancu and Farran have initiated a study of interrelations between the geometry of foliations on the tangent manifold of a Finsler manifold and the geometry of the Finsler manifold itself. The main idea of their paper is to emphasize the importance of some foliations which exist on the tangent bundle of a Finsler manifold (M, F), in studying the differential geometry of (M, F) itself. Other generalizations are studied in [18].

From the other point of view the study of lifted foliation to the tangent bundle of a foliated manifold in relation with Lagrange or Finsler metrics was initiated in [20]. In this direction, recently in [15] is given an identification of Riemannian foliations on the tangent bundle which are compatible with SODE structure and some geometric properties of such foliations are studied. Also, in this direction, a cohomology of foliated Finsler manifolds is studied in [16]

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and some vertical tangential invariants for some foliated Lagrange spaces are introduced in [12].

Taking into account that the tangent manifold of a Finsler space has a model of an almost Kähler manifold (TM^0, G, J), the aim of this paper is to identify some natural CR-foliations on the tangent manifold TM^0 and to study some cohomological properties of these CR-foliations in relation with corresponding results for CR-submanifolds of Kähler manifolds.

The paper is organized as follows: In the preliminary subsection we recall some basic facts on Finsler manifolds and we present the almost Kähler model of the tangent manifold TM^0 of a Finsler space (M, F). In the second section we present the vertical Liouville distribution and we briefly recall some natural foliations on the tangent manifold TM^0. In the third section using the vertical Liouville vector field and the natural almost complex structure on TM^0 we give an adapted basis in $T(TM^0)$ and we identify a CR-foliation on TM^0 given by the orthogonal complement in $T(TM^0)$ of the line distribution spanned by the vertical Liouville vector field. In fact we have that the c–indicatrix $I(M,F)(c)$ of a Finsler space (M, F) is a CR–submanifold of (TM^0, G, J). Then, by applying the general theory for cohomology of CR-submanifolds of an (almost) Kähler manifold, [8], we obtain some cohomological properties of the c–indicatrix $I(M,F)(c)$. In the last section we consider the case when the Finsler space (M, F) is endowed with a regular foliation $\mathcal{F}$ which is compatible with the Finsler structure in a certain sense. Then, we identify a canonical CR-foliation on TM^0 produced by the lifted foliation $\mathcal{F}^*$ on the tangent manifold TM^0 of the foliation $\mathcal{F}$ on (M, F) and we also present some cohomological aspects of this CR–foliation.

1.2. **Preliminaries.** Let (M, F) be a n–dimensional Finsler manifold with (x^i, y^i), $i = 1, \ldots, n$ the local coordinates on TM (for necessary definitions see for instance [1, 6, 17]).

The vertical bundle $V(TM^0)$ of $TM^0 = TM - \{\text{zero section}\}$ is the tangent (structural) bundle to vertical foliation $\mathcal{F}_V$ determined by the fibers of $\pi\colon TM \to M$ and characterized by $x^k = \text{const.}$ on the leaves. Also, we locally have $V(TM^0) = \text{span}\left\{\frac{\partial}{\partial y^i}\right\}$, $i = 1, \ldots, n$.

A canonical transversal (also called horizontal) distribution to $V(TM^0)$ is constructed by Bejancu and Farran in [5] pag. 225 or [4] as follows:

Let $(g^{ji}(x, y))_{n\times n}$ be the inverse matrix of $(g_{ij}(x, y))_{n\times n}$, where

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x,y), \tag{1.1}$$

and F is the fundamental function of the Finsler space.

If consider the local functions

$$G^i = \frac{1}{4}g^{ik}\left(\frac{\partial^2 F^2}{\partial y^k \partial x^h}y^h - \frac{\partial F^2}{\partial x^k}\right),\ G_i^j = \frac{\partial G^j}{\partial y^i}, \tag{1.2}$$

then, there exists on TM^0 a n–distribution $H(TM^0)$ locally spanned by the vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j \frac{\partial}{\partial y^j},\ i = 1, \dots, n. \tag{1.3}$$

The local basis $\left\{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right\}$, $i = 1, \dots, n$ is called *adapted* to vertical foliation $\mathcal{F}_V$ and we have the decomposition

$$T(TM^0) = H(TM^0) \oplus V(TM^0). \tag{1.4}$$

If we consider the dual adapted bases $\{dx^i, \delta y^i = dy^i + G_j^i dx^j\}$, then the Riemannian metric G on TM^0 given by the Sasaki lift of the fundamental metric tensor g_{ij} from (1.1) satisfies

$$G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g_{ij}\,,\ G\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = 0,\ i, j = 1, \dots, n. \tag{1.5}$$

We also notice that there is a natural almost complex structure on TM^0 which is compatible with G and locally given by

$$J = \frac{\delta}{\delta x^i} \otimes \delta y^i - \frac{\partial}{\partial y^i} \otimes dx^i\,,\ J\left(\frac{\delta}{\delta x^i}\right) = -\frac{\partial}{\partial y^i}\,,\ J\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta}{\delta x^i}. \tag{1.6}$$

According to [6, 17] we have that (TM^0, G, J) is an almost Kählerian manifold with the almost Kähler form given by $\Omega(X, Y) = G(JX, Y)$, $\forall\, X, Y \in \mathcal{X}(TM^0)$, and, locally expressed as

$$\Omega = g_{ij} \delta y^i \wedge dx^j. \tag{1.7}$$

2. A vertical Liouville distribution on TM^0

In this section, following [3, 4], we present the vertical Liouville distribution on TM^0 as the complementary orthogonal distribution in $V(TM^0)$ to the line distribution spanned by the vertical Liouville vector field $\Gamma = y^i \frac{\partial}{\partial y^i}$, and we also discuss about some natural foliations on TM^0.

According to [5, 6, 17], from the homogeneity condition of the fundamental function of the Finsler manifold (M, F) we have

$$F^2 = g_{ij} y^i y^j,\ \frac{\partial F}{\partial y^k} = \frac{1}{F} g_{ki} y^i,\ \frac{\partial g_{ij}}{\partial y^k} y^i = 0,\ k = 1, \dots, n. \tag{2.1}$$

Hence it results

$$G(\Gamma, \Gamma) = F^2. \tag{2.2}$$

By means of G and Γ, we define the vertical one form ζ by

$$\zeta(X) = \frac{1}{F} G(X, \Gamma)\,,\ \forall\, X \in \Gamma(V(TM^0)). \tag{2.3}$$

Denote by $\{\Gamma\}$ the line vector bundle over TM^0 spanned by Γ and consider the *vertical Liouville distribution* as the complementary orthogonal distribution V_Γ

to $\{\Gamma\}$ in $V(TM^0)$ with respect to G. Hence, V_Γ is defined by ζ, that is we have

$$\Gamma(V_\Gamma) = \{X \in \Gamma(V(TM^0)) \, : \, \zeta(X) = 0\}. \tag{2.4}$$

Thus, any vertical vector field $X = X^i \frac{\partial}{\partial y^i}$ can be expressed as follows:

$$X = PX + \frac{1}{F}\zeta(X)\Gamma, \tag{2.5}$$

where P is the projection morphism of $V(TM^0)$ on V_Γ.

Then the local components of ζ and P with respect to the basis $\{\delta y^i\}$ and $\left\{\delta y^i \otimes \frac{\partial}{\partial y^j}\right\}$, respectively, are given by

$$\zeta_i = \frac{\partial F}{\partial y^i}\,,\; P_i^j = \delta_i^j - \frac{1}{F}\zeta_i y^j, \tag{2.6}$$

where δ_j^i are the components of the Kronecker delta.

Remark 2.1. We notice that the projector P can be related in terms of the *angular metric tensor* of the Finsler space (M, F), which is defined by

$$h_{ij} = g_{ij} - \zeta_i\zeta_j. \tag{2.7}$$

More exactly, by (2.6) and (2.7) it is easy to see that $P_i^j = g^{jk}h_{ki}$.

Now, by (2.6) the rank of the projector P is $n-1$ and taking into account that $P_i^j y^i = 0$ it follows that V_Γ is an $(n-1)$–dimensional vertical sub-distribution, orthogonal to Γ, locally spanned by the vertical vector fields $\left\{\frac{\overline{\partial}}{\overline{\partial} y^i}\right\}$, $i = 1, \ldots, n$, where

$$\frac{\overline{\partial}}{\overline{\partial} y^i} = \frac{\partial}{\partial y^i} - \frac{\zeta_i}{F}\Gamma = P_i^j \frac{\partial}{\partial y^j}. \tag{2.8}$$

Taking into account that $\Gamma(F) = F$ and $\frac{\partial F}{\partial y^i} = \zeta_i$ (which easily follows from the homogeneity of the Finsler structure F), we obtain that an important property of the vertical Liouville sub-distribution V_Γ is the following:

For any $Y \in \Gamma(V_\Gamma)$ we have $Y(F) = 0$.

Theorem 2.1 ([3, 4])**.** *The vertical Liouville distribution V_Γ is integrable and hence it defines a foliation on TM^0 denoted by $\mathcal{F}_{V_\Gamma}$.*

Remark 2.2. The Theorem 2.1 can be also obtained using an argument similar to [7]. More exactly, if $\frac{\overline{\partial}}{\overline{\partial} y^i}, \frac{\overline{\partial}}{\overline{\partial} y^i} \in \Gamma(V_\Gamma) \subset \Gamma(V(TM^0))$, then

$$\left[\frac{\overline{\partial}}{\overline{\partial} y^i}, \frac{\overline{\partial}}{\overline{\partial} y^j}\right] = A_{ij}^k \frac{\overline{\partial}}{\overline{\partial} y^k} + B_{ij}\Gamma, \tag{2.9}$$

for some locally defined functions A_{ij}^k and B_{ij}, since $V(TM^0) = V_\Gamma \oplus \{\Gamma\}$ is integrable. Now, if we apply the vector fields in both sides of formula (2.9) to the Finsler function F and using the fact that $\Gamma(F) = F$ and $\frac{\overline{\partial} F}{\overline{\partial} y^i} = 0$, we obtain $B_{ij}F = 0$. This implies that $B_{ij} = 0$, and then the formula (2.9) says that the vertical Liouville distribution V_Γ is integrable.

By direct caculations, we obtain the following relations for the Lie brakets of vertical vector fields adapted to the decomposition $V(TM^0) = V_\Gamma \oplus \{\Gamma\}$,

$$\left[\frac{\overline{\partial}}{\partial y^i}, \frac{\overline{\partial}}{\partial y^j}\right] = \frac{1}{F}\left(\zeta_i \frac{\overline{\partial}}{\partial y^j} - \zeta_j \frac{\overline{\partial}}{\partial y^i}\right), \; \left[\frac{\overline{\partial}}{\partial y^i}, \Gamma\right] = \frac{\overline{\partial}}{\partial y^i}, \tag{2.10}$$

for all $i, j = 1, \ldots, n$, and the first relation of (2.10) says also that V_Γ is integrable.

Let us consider now the following complementary orthogonal distribution to $\{\Gamma\}$ in $T(TM^0)$:

$$\{\Gamma\}^\perp = \{X \in \Gamma(T(TM^0)) \, : \, G(X, \Gamma) = 0\}. \tag{2.11}$$

According to [4], the distribution $\{\Gamma\}^\perp$ is integrable and we also have the decomposition

$$\{\Gamma\}^\perp = H(TM^0) \oplus V_\Gamma. \tag{2.12}$$

Proposition 2.1 ([4]). *i) The foliation $\mathcal{F}_{\{\Gamma\}^\perp}$ determined by the distribution $\{\Gamma\}^\perp$ is just the foliation determined by the level hypersurfaces of the fundamental function F of the Finsler manifold, denoted by $\mathcal{F}_F$ and called* the fundamental foliation *on (TM^0, G).*

ii) For every fixed point $x_0 \in M$, the leaves of the vertical Liouville foliation $\mathcal{F}_{V_\Gamma}$ determined by the distribution V_Γ on $T_{x_0}M$ are just the c–indicatrices of (M, F):

$$I_{x_0}(M, F)(c) = \{y \in T_{x_0}M \, : \, F(x_0, y) = c\}. \tag{2.13}$$

iii) The foliation $\mathcal{F}_{V_\Gamma}$ is a subfoliation of the vertical foliation $\mathcal{F}_V$.

3. A fundamental CR–foliation on (TM^0, G, J) and the cohomology of the c–indicatrix bundle

In this section, using the vertical Liouville vector field Γ and the natural almost complex structure J on TM^0, we give an adapted basis in $T(TM^0)$. Next we prove that the c–indicatrix bundle $I(M, F)(c)$ of (M, F) is a CR–submanifold of the almost Kählerian manifold (TM^0, G, J) and we study some cohomological properties of $I(M, F)(c)$ in relation with classical cohomology of CR–submanifolds, [8].

For the natural almost complex structure J on $T(TM^0)$, we consider now the new local vector field frame in $T(TM^0)$ as $\left\{\frac{\delta}{\delta x^i}, \xi, \frac{\overline{\partial}}{\partial y^i}, \Gamma\right\}$, where

$$\xi = J(\Gamma) = y^i \frac{\delta}{\delta x^i} \tag{3.1}$$

and

$$\frac{\overline{\delta}}{\overline{\delta} x^i} = J\left(\frac{\overline{\partial}}{\overline{\partial} y^i}\right) = \frac{\delta}{\delta x^i} - \frac{\zeta_i}{F}\xi = P_i^j \frac{\delta}{\delta x^j}. \tag{3.2}$$

As in the previous section it follows that $H_\xi := \operatorname{span}\left\{\frac{\overline{\delta}}{\overline{\delta} x^1}, \ldots, \frac{\overline{\delta}}{\overline{\delta} x^n}\right\}$ is an $(n-1)$–dimensional horizontal sub-distribution, orthogonal to $\{\xi\}$ in $H(TM^0)$, where $\{\xi\}$ is the line distribution spanned by the horizontal Liouville vector field ξ.

Since the vertical Liouville vector field Γ is orthogonal to the level hypersurfaces of the fundamental function F, the vector fields $\left\{\frac{\overline{\delta}}{\overline{\delta} x^i}, \xi, \frac{\overline{\partial}}{\overline{\partial} y^i}\right\}$ are tangent to these hypersurfaces in TM^0, so they generate the distribution $\{\Gamma\}^\perp$. The vertical indicatrix (Liouville) distribution V_Γ is locally generated by $\left\{\frac{\overline{\partial}}{\overline{\partial} y^i}\right\}$, and the vertical foliation has the structural bundle locally generated by $\left\{\frac{\overline{\partial}}{\overline{\partial} y^i}, \Gamma\right\}$, $i = 1, \ldots, n$. Also, we have the decomposition

$$\{\Gamma\}^\perp = \{\xi\} \oplus H_\xi \oplus V_\Gamma. \tag{3.3}$$

For any $c > 0$, let us consider now the *c–indicatrix bundle* over M, given by $I(M,F)(c) = \bigcup_{x\in M} I_x(M,F)(c)$ and we briefly recall the CR–submanifold notion.

According to [2, 5], if $(\widetilde{N}, \widetilde{g}, \widetilde{J})$ is an (almost) Kähler manifold, where $\widetilde{g}$ is the Riemannian metric and $\widetilde{J}$ is the (almost) complex structure on $\widetilde{N}$, then N is a *CR–submanifold* of $\widetilde{N}$ if N admits two complementary orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ such that

i) $\mathcal{D}$ is $\widetilde{J}$-invariant, i.e., $\widetilde{J}(\mathcal{D}) \subset \mathcal{D}$;
ii) $\mathcal{D}^\perp$ is $\widetilde{J}$-anti-invariant, i.e., $\widetilde{J}\left(\mathcal{D}^\perp\right) \subset (TN)^\perp$.

$\mathcal{D}$ is called *maximal complex (holomorphic)* distribution of N and $\mathcal{D}^\perp$ is called *totally real* distribution of N.

We have

Proposition 3.1. *Let $i\colon I(M,F)(c) \hookrightarrow TM^0$ be the immersion of $I(M,F)(c)$ in TM^0. Then $I(M,F)(c)$ is a CR–submanifold of TM^0 with holomorphic distribution given by $\mathcal{D} = H_\xi \oplus V_\Gamma$ and the totally real distribution given by $\mathcal{D}^\perp = \{\xi\}$.*

Proof. We have that $\{\Gamma\}^\perp = \{\xi\} \oplus H_\xi \oplus V_\Gamma$ is the tangent bundle of $I(M,F)(c)$. Taking into account the behaviour of the almost complex structure J of (TM^0, G) we have

$$J\left(H_\xi \oplus V_\Gamma\right) = V_\Gamma \oplus H_\xi\,,\ J\left(\{\xi\}\right) = \{\Gamma\} = \left(\{\Gamma\}^\perp\right)^\perp$$

which end's the proof. $\square$

We recall that a r–dimensional distribution $\mathcal{D}$ on a Riemannian manifold (M,g) is minimal if the mean-curvature vector field H on $\mathcal{D}$ vanishes identically, where

$$H=\frac{1}{r}\sum_{i=1}^{r}(\nabla_{X_i}X_i)^{\perp},$$

where ∇ is the Levi-Civita connection on (M,g), $\{X_1,\ldots,X_r\}$ is an orthonormal frame of $\mathcal{D}$, and $(\nabla_XY)^{\perp}$ denotes the component of ∇_XY in the orthogonal complementary distribution $\mathcal{D}^{\perp}$ of $\mathcal{D}$ in TM.

It is well known, see [9, 8], that the totally real distribution of a CR-submanifold of an (almost) Kähler manifold is integrable and its maximal complex (holomorphic) distribution is minimal. Then we obviously have that the line distribution $\{\xi\}$ is integrable, and the distribution $H_\xi\oplus V_\Gamma$ is minimal.

Let ω^i be the dual 1–forms of the vertical vector fields $\frac{\overline{\partial}}{\overline{\partial} y^i}$ and θ^i be the dual 1–forms of the horizontal vector fields $\frac{\overline{\delta}}{\overline{\delta} x^i}$, that is $\omega^i\left(\frac{\overline{\partial}}{\overline{\partial} y^j}\right)=\delta^i_j$ and $\theta^i\left(\frac{\overline{\delta}}{\overline{\delta} x^j}\right)=\delta^i_j$, respectively. It is easy to see that we have the following relations:

$$\delta y^j=P^j_i\omega^i \ \text{ and } \ dx^j=P^j_i\theta^i. \tag{3.4}$$

As we already noticed the vertical vector fields $\left\{\frac{\overline{\partial}}{\overline{\partial} y^i}\right\}$, $i=1,\ldots,n$ are linear dependent and we consider the linear independent system $\left\{\frac{\overline{\partial}}{\overline{\partial} y^1},\ldots,\frac{\overline{\partial}}{\overline{\partial} y^{i-1}},\frac{\overline{\partial}}{\overline{\partial} y^{i+1}},\ldots,\frac{\overline{\partial}}{\overline{\partial} y^n}\right\}$ that generates V_Γ. Consequently, by means of J, we get the linear independent system of horizontal vector fields $\left\{\frac{\overline{\delta}}{\overline{\delta} x^1},\ldots,\frac{\overline{\delta}}{\overline{\delta} x^{i-1}},\frac{\overline{\delta}}{\overline{\delta} x^{i+1}},\ldots,\frac{\overline{\delta}}{\overline{\delta} x^n}\right\}$ that generates H_ξ.

Then, the general theory for cohomology of CR–submanifolds of an (almost) Kähler manifolds, [8], leads to

Theorem 3.1. *The differential form*

$$\nu=\omega^1\wedge\ldots\wedge\widehat{\omega^i}\wedge\ldots\wedge\omega^n\wedge\theta^1\wedge\ldots\wedge\widehat{\theta^i}\wedge\ldots\wedge\theta^n$$

is closed and it defines a cohomology class

$$[\nu]\in H^{2n-2}\left(I(M,F)(c)\right). \tag{3.5}$$

Definition 3.1. The cohomology class $[\nu]$ is called the canonical class of the c-indicatrix bundle $I(M,F)(c)$ of a Finsler space (M,F).

Remark 3.1. The form ν which defines the canonical class can be expressed in the form

$$\nu=\frac{(-1)^{n-1}}{(n-1)!}\left(i^*\Omega\right)^{n-1},$$

where Ω is the fundamental form given in (1.7).

Since $I(M,F)(c)$ is compact when M is compact, according to [8] we have

Corollary 3.1. *If the cohomology groups $H^{2k}(I(M,F)(c)) = 0$ for some $k < n$ then either holomorphic distribution $H_\xi \oplus V_\Gamma$ is not integrable or its totally real distribution $\{\xi\}$ is not minimal.*

We notice that the Poincaré-Cartan 2–form associated to a Finsler function has rank $2n-2$ and it play an important role in projective metrizability problem in Finsler geometry, see [7, 11]. In the end of this section, as in the case of Cartan manifolds [13], we prove that the differential form ν that represents the canonical class of the c–indicatrix bundle $I(M,F)(c)$ of a Finsler space (M,F) can be related in terms of the Poincaré-Cartan 2–form associated to the Finsler function F which is defined as follows:

Using the Frölicher-Nijenhuis formalism, for the tangent structure $\mathcal{J}$ on TM^0 locally given by $\mathcal{J} = \frac{\partial}{\partial y^i} \otimes dx^i$ we consider the differential $d_\mathcal{J} = i_\mathcal{J} \circ d - d \circ i_\mathcal{J}$, where $\mathcal{J}$ is considered as a vector valued 1–differential form. Then, using the horizontal projector $h = \frac{\delta}{\delta x^j} \otimes dx^j$ and the vertical projector $v = \frac{\partial}{\partial y^j} \otimes \delta y^j$, by direct calculus we obtain

$$d_\mathcal{J} F = i_\mathcal{J} dF = i_\mathcal{J}(d_h F + d_v F) = i_\mathcal{J} \zeta = \zeta_i dx^i,$$

where we have used $d_h F = 0$ and $d_v F = \zeta$. Now, taking into account that $h_{ij} = F \frac{\partial^2 F}{\partial y^i \partial y^j}$ and using again $d_h F = 0$, we obtain the following Poincaré-Cartan 2–form associated to the Finsler function F

$$dd_\mathcal{J} F = \frac{1}{F} h_{ij} \delta y^j \wedge dx^i. \tag{3.6}$$

Using the relations (3.4) and the fact that $h_{ij} P^j_l = h_{il}$, we have the following expression of the Poincaré-Cartan 2–form

$$dd_\mathcal{J} F = \frac{1}{F} h_{lk} \omega^l \wedge \theta^k. \tag{3.7}$$

Finally, taking the $n-1$ power of the above 2–form, we get

$$\nu = \alpha (dd_\mathcal{J} F)^{n-1}, \tag{3.8}$$

where

$$\alpha = \frac{(-1)^{\frac{(n-1)(n-2)}{2}} F^{n-1}}{(n-1)! \det \widetilde{H}}.$$

and $\widetilde{H} = (h_{kl})$, $l,k \in \{1,\dots,i-1,i+1,\dots,n\}$.

Remark 3.2. Let us consider a conformal change of the Finsler structure F given by $\widetilde{F} = e^{\sigma(x)} F$, where $\sigma \in C^\infty(M)$. Taking into account that $d_\mathcal{J} \widetilde{F} = e^\sigma d_\mathcal{J} F$ we obtain $dd_\mathcal{J} \widetilde{F} = e^\sigma (dd_\mathcal{J} F + d\sigma \wedge d_\mathcal{J} F)$ which leads to

$$\widetilde{\nu} = \nu + (n-1)\alpha (dd_\mathcal{J} F)^{n-2} \wedge d\sigma \wedge d_\mathcal{J} F. \tag{3.9}$$

The relation (3.9) says that if $\sigma = $ const., that is $\widetilde{F}$ and F are homothetic, then $\widetilde{\nu} = \nu$.

4. CR–foliations on the tangent manifold of a foliated Finsler space

In this section we consider the case when the Finsler space (M, F) is endowed with a regular foliation $\mathcal{F}$ which will be compatible with the Finsler structure F in a certain sense. Then, as in the previous section, we identify and we study from topological point of view a canonical CR-foliation on (TM^0, G, J) produced by the lifted foliation $\mathcal{F}^*$ on the tangent manifold TM^0 of the foliation $\mathcal{F}$ on (M, F).

4.1. **Foliated Finsler spaces.** Let us consider (M, F) a Finsler space endowed with a m-codimensional foliation $\mathcal{F}$. It follows that there is a partition of M into $(n-m)$–dimensional submanifolds, called leaves. In the following, the indices take the values $u, v, \ldots = m+1, \ldots, n$ and $a, b, \ldots = 1, \ldots, m$. There is an atlas on M adapted to this foliation with local adapted charts $(U, (x^a, x^u))$ such that the leaves are locally defined by $x^a = \text{const.}$, for all $a = 1, \ldots, m$.

The local coordinates on the tangent manifold TM^0 are (x^a, x^u, y^a, y^u). Generally, for two local charts $(U, (x^i))$ and $(\widetilde{U}, (\widetilde{x}^j))$, whose domains overlap, on TM^0, in $U \cap \widetilde{U}$ we have

$$\widetilde{y}^j = \frac{\partial \widetilde{x}^j}{\partial x^i} y^i. \tag{4.1}$$

Now, the above relations give the following changing coordinates rules on TM^0:

$$\widetilde{x}^b = \widetilde{x}^b(x^a)\,,\ \widetilde{x}^v = \widetilde{x}^v(x^a, x^u),$$

$$\widetilde{y}^b = \frac{\partial \widetilde{x}^b}{\partial x^a} y^a\,,\ \widetilde{y}^v = \frac{\partial \widetilde{x}^v}{\partial x^a} y^a + \frac{\partial \widetilde{x}^v}{\partial x^u} y^u.$$

According to [20], [15] the foliation $\mathcal{F}$ on M determine a $2m$-codimensional foliation $\mathcal{F}^*$ on TM^0, called the *natural lift of $\mathcal{F}$ to TM^0*, whose leaves are locally defined by $x^a = \text{const.}$ and $y^a = \text{const.}$

Taking into account decomposition (1.4), the local base $\left\{\frac{\delta}{\delta x^a}, \frac{\delta}{\delta x^u}, \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^u}\right\}$ adapted to vertical foliation satisfies the following relations in $U \cap \widetilde{U}$:

$$\frac{\delta}{\delta \widetilde{x}^b} = \frac{\partial x^a}{\partial \widetilde{x}^b}\frac{\delta}{\delta x^a} + \frac{\partial x^u}{\partial \widetilde{x}^b}\frac{\delta}{\delta x^u}\,,\ \frac{\delta}{\delta \widetilde{x}^v} = \frac{\partial x^u}{\partial \widetilde{x}^v}\frac{\delta}{\delta x^u},$$

$$\frac{\partial}{\partial \widetilde{y}^b} = \frac{\partial x^a}{\partial \widetilde{x}^b}\frac{\partial}{\partial y^a} + \frac{\partial x^u}{\partial \widetilde{x}^b}\frac{\partial}{\partial y^u}\,,\ \frac{\partial}{\partial \widetilde{y}^v} = \frac{\partial x^u}{\partial \widetilde{x}^v}\frac{\partial}{\partial y^u}.$$

Returning now to the foliation $\mathcal{F}^*$, the tangent bundle $T\mathcal{F}^*$ to leaves, the structural bundle of this foliation, is locally spanned by $\left\{\frac{\delta}{\delta x^u}, \frac{\partial}{\partial y^u}\right\}$ and it is a subbundle of $T(TM^0)$.

Definition 4.1 ([16])**.** We say that the foliation $\mathcal{F}$ on M is *compatible with the Finsler structure* F on M if, in every local chart around a point $(x, y) \in$

TM^0, the matrix $(g_{uv})_{(n-m)\times(n-m)}$ is nondegenerate and the functions G^a are satisfying the relation

$$G^a_u = \frac{\partial G^a}{\partial y^u} = 0, \ \forall\, a = 1, \ldots, m, \ u = m+1, \ldots, n. \tag{4.2}$$

Proposition 4.1 ([16]). *If the foliation $\mathcal{F}$ on M is compatible with the Finsler structure F on M, then the vector fields on TM^0 locally given by*

$$\xi_a = \frac{\delta}{\delta x^a} - t^u_a \frac{\delta}{\delta x^u}, \ \zeta_a = \frac{\partial}{\partial y^a} - t^u_a \frac{\partial}{\partial y^u}, \tag{4.3}$$

are orthogonal to $\left\{\frac{\delta}{\delta x^u}\right\}$, $\left\{\frac{\partial}{\partial y^u}\right\}$, *with respect to Sasaki-Finsler metric G from* (1.5), *where $\{t^u_a\}$ are solutions of the system $g_{av} - t^u_a g_{uv} = 0$.*

As a consequence of the above proposition, for every vector field $X \in \mathcal{X}(TM^0)$ we have the following decomposition:

$$\begin{aligned} X &= X^i \frac{\delta}{\delta x^i} + Y^i \frac{\partial}{\partial y^i} \\ &= X^a \frac{\delta}{\delta x^a} + X^u \frac{\delta}{\delta x^u} + Y^a \frac{\partial}{\partial y^a} + Y^u \frac{\partial}{\partial y^u} \\ &= X^a \xi_a + (X^u + X^a t^u_a) \frac{\delta}{\delta x^u} + Y^a \zeta_a + (Y^u + Y^a t^u_a) \frac{\partial}{\partial y^u}. \end{aligned}$$

The basis

$$\left\{\xi_a, \frac{\delta}{\delta x^u}, \zeta_a, \frac{\partial}{\partial y^u}\right\} \tag{4.4}$$

is adapted to foliation $\mathcal{F}^*$ and to vertical foliation $\mathcal{F}_V$, too.

The relation (4.2) guarantees that the tangent bundle of the lifted foliation $\mathcal{F}^*$ admits the decomposition:

$$T\mathcal{F}^* = H\mathcal{F}^* \oplus V\mathcal{F}^*, \tag{4.5}$$

where $H\mathcal{F}^* = \operatorname{span}\left\{\frac{\delta}{\delta x^u} = \frac{\partial}{\partial x^u} - G^v_u \frac{\partial}{\partial y^v}\right\}$ and $V\mathcal{F}^* = \operatorname{span}\left\{\frac{\partial}{\partial y^u}\right\}$, see [20], [15].

Also, if we consider $T^\perp\mathcal{F}^*$ the orthogonal complement of $T\mathcal{F}^*$ in $T(TM^0)$ with respect to metric G from (1.5), then by Proposition 4.1 we have the orthogonal decomposition

$$T^\perp\mathcal{F}^* = H^\perp\mathcal{F}^* \oplus V^\perp\mathcal{F}^*, \tag{4.6}$$

where $H^\perp\mathcal{F}^* = \operatorname{span}\{\xi_a\}$ and $V^\perp\mathcal{F}^* = \operatorname{span}\{\zeta_a\}$. Finally, we obtain the following orthogonal decomposition:

$$T(TM^0) = H^\perp\mathcal{F}^* \oplus V^\perp\mathcal{F}^* \oplus H\mathcal{F}^* \oplus V\mathcal{F}^*. \tag{4.7}$$

4.2. **A canonical CR–foliation on** $(TM^0, G, J, \mathcal{F}^*)$**.** Let us suppose that the Finsler space (M, F) is endowed with a foliation $\mathcal{F}$ compatible with the Finsler structure F. Then, we have

Proposition 4.2. *The distribution* $\mathcal{D} = V^\perp\mathcal{F}^* \oplus H\mathcal{F}^* \oplus V\mathcal{F}^*$ *on* TM^0 *is integrable and its foliation* $\mathcal{F}_\mathcal{D}$ *is a CR–foliation on* (TM^0, G, J).

Proof. If $X, Y \in \Gamma\left(H\mathcal{F}^* \oplus V\mathcal{F}^*\right)$ then we have $[X, Y] \in \Gamma\left(H\mathcal{F}^* \oplus V\mathcal{F}^*\right)$, since $T\mathcal{F}^* = H\mathcal{F}^* \oplus V\mathcal{F}^*$ is integrable. Also, by direct calculus we have

$$\left[\zeta_a, \frac{\delta}{\delta x^u}\right] = \left(\frac{\delta t_a^v}{\delta x^u} - \zeta_a G_u^v\right)\frac{\partial}{\partial y^v} \in \Gamma(\mathcal{D}),$$

$$\left[\zeta_a, \frac{\partial}{\partial y^u}\right] = \frac{\partial t_a^v}{\partial y^u}\frac{\partial}{\partial y^v} \in \Gamma(\mathcal{D}),$$

$$[\zeta_a, \zeta_b] = \left(\zeta_b t_a^v - \zeta_a t_b^v\right)\frac{\partial}{\partial y^v} \in \Gamma(\mathcal{D})$$

which say that $\mathcal{D}$ is integrable.

Now, taking into account the behaviour of the almost complex structure J on $\frac{\delta}{\delta x^u}$ and $\frac{\partial}{\partial y^u}$, respectively we easily deduce that

$$J\left(T\mathcal{F}^*\right) = J\left(H\mathcal{F}^* \oplus V\mathcal{F}^*\right) = V\mathcal{F}^* \oplus H\mathcal{F}^* = T\mathcal{F}^*. \tag{4.8}$$

Also by (4.3) we have $J(\zeta_a) = \xi_a$ which say that

$$J\left(V^\perp\mathcal{F}^*\right) = H^\perp\mathcal{F}^* = \mathcal{D}^\perp. \tag{4.9}$$

Thus, the relations (4.8) and (4.9) say that the foliation $\mathcal{F}_\mathcal{D}$ given by the integrable distribution $\mathcal{D}$ is a CR–foliation on the almost Kähler manifold (TM^0, G, J) with the maximal complex (holomorphic) subbundle given by $T\mathcal{F}^*$ and the totally real subbundle given by $V^\perp\mathcal{F}^*$. $\square$

Definition 4.2. The foliation $\mathcal{F}_\mathcal{D}$ given by the distribution $\mathcal{D} = V^\perp\mathcal{F}^* \oplus H\mathcal{F}^* \oplus V\mathcal{F}^*$ on (TM^0, G, J) is called the *canonical CR-foliation* on the tangent manifold of a Finsler space (M, F) endowed with a compatible foliation $\mathcal{F}$.

Similarly to the previous section, we have

Proposition 4.3. *The distribution* $V^\perp\mathcal{F}^*$ *is integrable.*

Proposition 4.4. *The distribution* $T\mathcal{F}^*$ *is minimal.*

Consequently, we have

$$\zeta_b t_a^v - \zeta_a t_b^v = 0. \tag{4.10}$$

Definition 4.3. The foliation given by the integrable distribution $V^\perp\mathcal{F}^*$ is called the *canonical subfoliation* of the canonical CR-foliation $\mathcal{F}_\mathcal{D}$ on the tangent manifold of a Finsler space (M, F) endowed with a compatible foliation $\mathcal{F}$. We denote this foliation by $\mathcal{V}^\perp\mathcal{F}^*$.

Let us consider now $\{dx^a, \theta^u, \delta y^a, \eta^u\}$ the cobasis dual to basis (4.4), where

$$\theta^u = dx^u + t^u_a dx^a \, , \, \eta^u = \delta y^u + t^u_a \delta y^a .$$

Theorem 4.1. *The differential form*

$$\mu = \theta^{m+1} \wedge \ldots \wedge \theta^n \wedge \eta^{m+1} \wedge \ldots \wedge \eta^n$$

is closed and it defines a basic cohomology class (with respect to the foliation $\mathcal{V}^\perp\mathcal{F}^*$)

$$c\left(\mathcal{V}^\perp\mathcal{F}^*\right) := [\mu] \in H^{2n-2m}_b\left(\mathcal{V}^\perp\mathcal{F}^*\right), \tag{4.11}$$

where $H^\bullet_b(\mathcal{F})$ denotes the basic cohomology of a foliation $\mathcal{F}$.

Definition 4.4. The cohomology class $c\left(\mathcal{V}^\perp\mathcal{F}^*\right)$ is called the canonical class of the subfoliation $\mathcal{V}^\perp\mathcal{F}^*$.

Remark 4.1. The form μ which defines the canonical class $c\left(\mathcal{V}^\perp\mathcal{F}^*\right)$ can be expressed in the form

$$\mu = \frac{(-1)^{n-m}}{(n-m)!}\left(\Omega|_{\mathcal{D}}\right)^{n-m} \tag{4.12}$$

where Ω is the fundamental form given in (1.7).

Remark 4.2. The form μ can be considered as a closed leafwise (foliated) form with respect to the foliation $\mathcal{F}_\mathcal{D}$, and so

$$c(\mathcal{V}^\perp\mathcal{F}^*) = c(\mathcal{F}_\mathcal{D}) \in H^{2n-2m}(\mathcal{F}_\mathcal{D}).$$

Since the maximal complex subbundle $T\mathcal{F}^*$ is integrable, then according to [8] we obtain

Proposition 4.5. *If the distribution $V^\perp\mathcal{F}^*$ is minimal then the canonical class $c\left(\mathcal{V}^\perp\mathcal{F}^*\right)$ is nontrivial in $H^{2n-2m}\left(\mathcal{F}_\mathcal{D}\right)$ and*

$$H^{2k}(\mathcal{F}_\mathcal{D}) \neq 0, \ \forall\, k = 1, \ldots, n-m. \tag{4.13}$$

Corollary 4.1. *If the foliated cohomology spaces $H^{2k}\left(\mathcal{F}_\mathcal{D}\right) = 0$ for some $k \leq n-m$ then $V^\perp\mathcal{F}^*$ is not minimal.*

Theorem 4.2. *Let $\mathcal{V}^\perp\mathcal{F}^*$ be the canonical subfoliation of the CR-foliation $\mathcal{F}_\mathcal{D}$ on the almost Kähler manifold (TM^0, G, J). Then the Godbillon-Vey class $GV\left(\mathcal{V}^\perp\mathcal{F}^*\right)$ vanishes.*

Proof. Since every almost Kähler manifold is in particular an almost locally conformal Kähler manifold with vanishing Lee form, the result follows by [10], [14]. □

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Department of Mathematics and Computer Science,
University Transilvania of Braşov,
Braşov 500091, Str. Iuliu Maniu 50, România
E-mail address: cristian.ida@unitbv.ro

Department of Applied Mathematics,
University of Craiova,
Craiova, 200585, Str. Al. Cuza, No. 13, România
E-mail address: paul_p_popescu@yahoo.com