Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
26 (2010), 377–382
www.emis.de/journals
ISSN 1786-0091

FINSLER STRUCTURE IN THERMODYNAMICS AND STATISTICAL MECHANICS

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Abstract. We consider the Finsler structure and contact structure in variational principle of equilibrium thermodynamic theory. Einstein's statistical thermodynamic theory was proposed as a statistical mechanics using macroscopic thermodynamic variables as stochastic variables. We generalise Einstein's statistical theory from Finsler geometrical viewpoint, and operating inverse process of variation, path integration, we propose a new statistical theory. Einstein's theory becomes the WKB approximation of our new theory.

1. Geometrisation of Mechanics

We present the Finsler structure in thermodynamics and statistical mechanics. Before that, we will sketch Finsler and contact structure in mechanics.

Conventionally, Lagrange mechanics is defined on configuration space Q. The configuration space have no geometric structure. But there is Finsler structure on an extended configuration space which is time together with configuration space: $M = \mathbb{R} \times Q$. On the extended space M, Lagrangian mechanics can be considered as Finsler geometry. If the conventional Lagrangian $L(x, \dot{x}, t)$ is given, we can define a Finsler function $F(t, x, dt, dx) = L\left(x, \frac{dx}{dt}, t\right)|dt|$ on M. Then using Finsler function, the action $\mathcal{A}[\gamma]$ of curve $\gamma\colon [s_0, s_1] \subset \mathbb{R} \to M$ is defined by

$$\mathcal{A}[\gamma] = \int_\gamma F(x, dx) = \int_{s_0}^{s_1} F\left(x^\mu(s), \frac{dx^\mu(s)}{ds}\right) ds.$$

From homogeneity condition of F, the action $\mathcal{A}[\gamma]$ is reparametrisation invariant, so it defines geometric length. From the principle of least action $\delta\mathcal{A}[\gamma_{\mathrm{c}}] =$

2000 *Mathematics Subject Classification.* 53B40.
Key words and phrases. Finsler structure, contact structure, equilibrium thermodynamic theory.

0, we can get Euler-Lagrange equation of motion $\frac{\delta \mathcal{A}}{\delta \gamma^\mu} = 0$. The important thing is, that this equation of motion is not only time coordinate free, but also space coordinate free. So, equation of motion becomes a geodesic equation. This is the geometrisation of mechanics. For physics from the general principle of relativity, which says that the physical phenomena should be described by geometry, we should consider Lagrangian mechanics as Finsler geometry. So we call this Finsler Lagrangian mechanics.

2. Hamilton formalism

Next we review Hamiltonian formalism for Finsler Lagrangian mechanics. We define conjugate momentum $p_\mu = \frac{\partial F}{\partial y^\mu}$ from Finsler function $F(x,y)$. From homogeneity condition of Finsler function : $y^\mu \frac{\partial F}{\partial y^\mu} = F$, this momentum p_μ have 0-th homogeneity: $p_\mu(x, \lambda y) = p_\mu(x,y)$ or $y^\nu \frac{\partial p_\mu}{\partial y^\nu} = 0$. Therefore, $\det\left(\frac{\partial p_\mu}{\partial y^\nu}\right) = 0$ and there is a constraint equation $G(x,p) = 0$. This function G is a covariant Hamiltonian, or, state equation in thermodynamics. The action given by this Hamiltonian is

$$\mathcal{A}[\gamma] = \int_\gamma \left(p_\mu \frac{dx^\mu}{ds} - \lambda G \right) ds,$$

where λ is a Lagrange multiplier and s is an arbitrary parameter. From the principle of least action, we get Hamilton equation and constraint,

$$\frac{dx^\mu}{ds} = \lambda \frac{\partial G}{\partial p_\mu},\ \frac{dp_\mu}{ds} = -\lambda \frac{\partial G}{\partial x^\mu},\ G = 0.$$

We should call these covariant Hamilton equations. This is the Hamiltonian formalism for Finsler Lagrangian mechanics.

3. Geodesic distance

Here we review geodesic distance in Finsler Lagrangian mechanics. Using solution of equation of motion $\gamma_{\rm c}$, which is a geodesic curve, we can define geodesic distance $W(x_1, x_0) = \mathcal{A}[\gamma_{\rm c}] = \int_{\gamma_{\rm c}} p_\mu dx^\mu$ between two points x_0 and x_1. $W(x_1, x_0)$ is a value of action evaluated on geodesic curve between x_0 and x_1. In language of physics, W corresponds to Hamilton's principal function. By taking variation of this action, we get $\delta W(x_1, x_0) = p_1 \delta x_1 - p_0 \delta x_0$, so $p_\mu = \frac{\partial W}{\partial x^\mu}$. Substituting this into the constraint equation $G(x,p) = 0$, we can get the Hamilton-Jacobi equation $G\left(x^\mu, \frac{\partial W}{\partial x^\mu}\right) = 0$, which is in the covariant form. Here we make a brief statement about the relation to contact geometry, which we will return in the latter section. Let Σ be submanifold of T^*M, defined by the constraint $G(x,p) = 0$. Σ corresponds to extended phase space of mechanics or thermodynamics. With $\Theta = p_\mu dx^\mu$, (Σ, Θ) becomes a contact manifold. This contact manifold can be considered as the stage of covariant Hamiltonian formalism.

4. Variational principle of thermodynamics

We discuss the variational principle in thermodynamics, which is a good example of Finsler structure. That is, thermal configuration space M can be considered simply as the space of extensive variables $\{(E, V)\}$. E, V represent energy and volume of the thermal system respectively. Then the second law of thermodynamics is represented by the following inequality relation.

$$dS \geq \frac{\delta Q}{T_{\mathrm{ex}}} \tag{1}$$

LHS is the infinitesimal entropy difference of thermal points, and RHS is the given amount of heat by the heat bath. If equal, then the thermal process is reversible, which is also called quasi-static change in physics. If not, the thermal process is irreversible. In conventional physics, RHS of (1) could not be written more accurately. But here, we will assume that RHS of (1) can be represented by Finsler function:

$$\frac{\delta Q}{T_{\mathrm{ex}}} = F(E, V, dE, dV).$$

That is to say, the infinitesimal thermal process between (E, V) and $(E+dE, V+dV)$ can be given by Finsler function $F(E, V, dE, dV)$. Then the thermal configuration space becomes Finsler manifold (M, F). For later convenience, We write $F(E, V, dE, dV)$ as

$$F(E, V, dE, dV) = p_e(X, \dot{X})\dot{E}ds + p_v(X, \dot{X})\dot{V}ds$$

using the conjugate momentum of E and V, and considering homogeneity condition of F. Here, X represents E or V and s is an arbitrary parameter. By this assumption, the action of thermodynamic process becomes $\mathcal{A}[\gamma] = \int_\gamma F(X, dX)$, where γ represents the thermodynamic process. Then we can get reversible process or quasi-static change as a maximum change of this action. So quasi-static change γ_{c} is given as a geodesic $\delta\mathcal{A}[\gamma_{\mathrm{c}}] = 0$.

5. Analogy with Mechanics

Table 1 shows the analogy between thermodynamics and mechanics [10]. Entropy S corresponds to the geodesic distance W. First law of reversible thermodynamics $dS = k\beta dE - k\beta p dV$ corresponds to $dW = p_\mu dx^\mu$. State equation of thermodynamics $f(E, V, T, p) = 0$ is in analogy to a constraint $G = 0$. Hamilton-Jacobi equation $G\left(x^\mu, \frac{\partial W}{\partial x^\mu}\right) = 0$ in mechanics corresponds to $f\left(E, V, \frac{\partial S}{\partial E}, \frac{\partial S}{\partial V}\right) = 0$ which is Hamilton-Jacobi equation of thermodynamics. The relations corresponding to Hamilton equation and action functional are unknown. But still we can understand these in a special example, ideal gas.

thermodynamics	analytical mechanics
S	W
$dS = k\beta dE - k\beta p dV$	$dW = p_\mu dx^\mu$
$f(E, V, T, p) = 0$	$G(x^\mu, p_\mu) = 0$
$f\left(E, V, \frac{\partial S}{\partial E}, \frac{\partial S}{\partial V}\right) = 0$	$G\left(x^\mu, \frac{\partial W}{\partial x^\mu}\right) = 0$
$***$	$\frac{dx^\mu}{ds} = \lambda \frac{\partial G}{\partial p_\mu}, \frac{dp_\mu}{ds} = -\lambda \frac{\partial G}{\partial x^\mu}$
$***$	$\mathcal{A}[\gamma] = \int_\gamma F(x, dx)$

TABLE 1. Analogy between thermodynamics and mechanics

6. Geometrisation of thermodynamics

Here we will construct Finsler structure or contact structure of thermodynamics taking a special and simple example, ideal gas. At the beginning, we do not know the Finsler function of thermal configuration space $(M, F) = (\{(E, V)\}, F)$ neither the action $\mathcal{A}[\gamma] = \int_\gamma F(X, dX)$. But we do know the entropy $S = \mathcal{A}[\gamma_c] = \int_{\gamma_c} F(X, dX)$, which is the geodesic distance. So, we can determine the Finsler structure by considering the Hamiltonian formalism. Thermal phase space is a submanifold Σ of T^*M. T^*M is the space of extensive and intensive variables. For example, the coordinates of the space, are E, V, T: temperature, p: pressure. The submanifold Σ, which is the thermal phase space, is determined by thermal state equation $G_1(E, V, T, p) = 0$. For ideal gas, $G_1 = pV - \frac{2}{3}E = 0$. Instead of T and p, we use $p_e = \frac{\partial S}{\partial E} = \frac{1}{kT}$ and $p_v = \frac{\partial S}{\partial V} = \frac{p}{kT}$ that are conjugate momentum of E and V. Then by $(\Sigma, \Theta = p_\mu dX^\mu)$ we can regard this as a contact structure of thermodynamics. But in this case contact form Θ becomes singular by coincidence: $d\Theta \wedge \Theta = 0$. Therefore, we must take further gauge-fixing conditon $G_2(E, V, p_e, p_v) = 0$. Considering these conditions, we obtain the action for Hamiltonian formalism, where λ_1, λ_2 are the Lagrange multipliers;

$$\mathcal{A}[\gamma] = \int_\gamma p_e dE + p_v dV + \lambda_1(p_v V - \tfrac{2}{3}p_e E)ds + \lambda_2 G_2(E, V, p_e, p_v)ds.$$

The Hamilton equations of thermodynamics are

$$\begin{cases} \frac{dE}{ds} = \frac{2}{3}\lambda_1 E - \lambda_2 \frac{\partial G_2}{\partial p_e}, & \frac{dV}{ds} = -\lambda_1 V - \lambda_2 \frac{\partial G_2}{\partial p_v}, \\ \frac{dp_e}{ds} = -\frac{2}{3}\lambda_1 p_e + \lambda_2 \frac{\partial G_2}{\partial E}, & \frac{dp_v}{ds} = \lambda_1 p_v + \lambda_2 \frac{\partial G_2}{\partial V}. \end{cases}$$

And these solution becomes $p_e = rE^{-1}$, $p_v = 3rV^{-1}$ where r is a constant. It is important to notice that usually the ideal gas is defined by the state equation $pV = \frac{2}{3}E$ and $E = rT$. However, we can get the latter relation from

Hamilton equation. Therefore we are able to construct the Finsler structure of thermodynamics if we have the state equation.

7. Einstein's thermal statistics

Here is another story of statistical thermodynamics by Einstein. Usually statistical mechanics is based on microscopic point of view. But Einstein proposed the statistical thermodynamics from macroscopic point of view. In thermodynamics, all the thermal variables are definite values in equilibrium state. But even in this equilibrium state, an accurate observation shows that these variables have fluctuation. Therefore, Einstein considered these variables as stochastic variables. He proposed the measure of these variables as $\mathcal{P}(\alpha) \propto \exp\left[\frac{S(\alpha)}{k}\right]$. The average of these variables α^i are given by $\langle \alpha^i \rangle = \int d\alpha \alpha^i \exp\left[\frac{S(\alpha)}{k}\right]$. However, Einstein's theory is not so useful for physics for its lack of accuracy. In the next section we will generalise Einstein's theory using the previous discussion.

8. Generalise Einstein's theory

We will generalise Einstein's statistical thermodynamics by using the Finsler structure of thermodynamics. We consider the analogy to the construction of quantum mechanics from classical mechanics. By taking the inverse operation of variation, a more fundamental quantum theory can be obtained from classical Lagrangian mechanics.

$$\delta\mathcal{A}[\gamma_c] = 0 \quad \overset{\text{quantisation}}{\Longrightarrow} \quad \Psi = \int \delta\gamma \exp\left(\frac{i}{\hbar}\mathcal{A}[\gamma]\right)$$

This is what Feynman had proposed. Similarly, by inverse operation of variation, we can construct a more fundamental theory, a new statistical theory which is a generalisation of Einstein's theory.

$$\delta\mathcal{A}[\gamma_c] = 0 \quad \overset{\text{statisticalisation}}{\Longrightarrow} \quad \mathcal{P} = \int \delta\gamma \exp\left(\frac{1}{k}\mathcal{A}[\gamma]\right)$$

By using the technique of Finsler path integral proposed by Ootsuka and Tanaka [7], we can define this new statistical theory formally. Evidently, we can regard the Einstein's theory as a WKB approximation of this new theory.

9. Discussion

The geometrisation of thermodynamics in the perspective of contact structure was initiated by Caratheodory, leading to works of Hermann [3] and Mrugała et al's [6]. On the other hand, introduction of a Riemannian structure to thermodynamical phase space was proposed by Ruppeiner [9]. Janyszek-Mrugała [4] makes a research from a standpoint of information geometry [1]. Thermodynamics in curved spacetime or Finsler spacetime was considered by Antonelli-Zastawniak [2] and Vacaru [11]. Our research is based on the Finsler structure, or contact structure, which could be naturally derived from the thermal state equation describing the second law of thermodynamics. As far as we know, the research which points out clearly to the existence of these geometrical structure is done only by Suzuki [10]. Our motivation of geometrisation of thermodynamics and generalisation of Einstein's theory is very similar to one's of Mrugała [5] and Ruppeiner [8], however, it is different in the sense that we utilise *point* Finsler geometry, and also our previous work Finsler pathintegral [7] as a tool to generalise Einstein's statistical mechanics.

Acknowledgments

We thank Lajos Tamássy for helpful advice on point Finsler geometry. We thank Erico Tanaka and astrophysics laboratory of Ochanomizu University for discussions. The work is greatly inspired by late Yasutaka Suzuki.

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